

Proof by Contradiction

Math 235 Fall 2000

Proof by contradiction is perhaps the strangest method of proof, since you start by assuming that what you want to prove is *false*, and then show that something ridiculous happens - thus your original assumption must have been false! For example, suppose you wish to prove a statement S is true. You start a proof by contradiction by supposing that the statement S is *false*. Then you make logical arguments and conclusions that follow from this supposition that S is false, until you arrive at a *contradiction* (for example, two logically contradictory statements, or a statement that contradicts one of the assumptions of the problem). Since the assumption that S is false led to a contradiction, the statement “ S is false” must itself be false; in other words, S must be true!

One final note: Proof by contradiction should not be used if you can think of a more “constructive” proof, *i.e.* a proof that starts only from the assumptions of the problem and makes a connected logical argument to arrive at the desired conclusion.

Consider the following example from your homework (1.2 #67):

Prove that the sum of a rational number and an irrational number is irrational.

After a few minutes trying to figure out how to prove this statement you’ll probably realize that it’s pretty difficult to prove that a given number is irrational. However, the problem previous to this one in the homework states that the sum of two rational numbers is a rational number; let’s use this fact and the method of proof by contradiction to prove the statement above.

Given that: r is a rational number, and x is an irrational number.

Show that: $r + x$ is irrational.

Proof: Seeking a contradiction, suppose that $r + x$ is rational.

Since r is rational, $-r$ is also rational; thus the sum of $r + x$ and $-r$ must be a rational number (since the sum of two rational numbers is rational).

Thus $(r + x) + (-r) = x$ is rational; this contradicts our original hypothesis that x is irrational.

Therefore it must not be true that $r + x$ is rational, *i.e.* we have shown that $r + x$ must be irrational. ■

Note that in the proof above, we made up some notation; we let r represent an arbitrary rational number, and chose x to represent an arbitrary irrational number. Without doing this the proof would have been very difficult to read and understand. It is also worth noting that if you look up the answer to this problem in the back of the book it says simply “if r and $r + x$ are rational, then $x = (r + x) - r$ is rational.” Keep in mind that this is just a hint of the *idea* of the proof; the snippet in the back of the book is not a complete proof.