Delta-Epsilon Proofs

Math 235 Fall 2000

Delta-epsilon proofs are used when we wish to prove a limit statement, such as

$$\lim_{x \to 2}(3x - 1) = 5. \tag{1}$$

Intuitively we would say that this limit statement is true because as x approaches 2, the value of $(3x - 1)$ approaches 5. This is not, however, a *proof* that this limit statement is true. By the formal definition of limit (see the previous handout), to prove (1) is true we must show that:

For all $\epsilon > 0$*, there exists a* $\delta > 0$ *such that:*
If $0 < |x - 2| < \delta$*, then* $|(3x - 1) - 5| < \epsilon$*.*

In other words, given any positive number ϵ, we must show that there exists a positive number δ so that whenever the distance between x and 2 is less than δ, the distance between $(3x - 1)$ and 5 is less than ϵ. Said another way, given $\epsilon > 0$, we need to find a $\delta > 0$ so that if $2 - \delta < x < 2 + \delta$ (but $x \neq 2$), then $5 - \epsilon < 3x - 1 < 5 + \epsilon$.

Finding Delta for a particular Epsilon
For example, if we let ϵ be .5, then $\delta = .16$ will work: if $|x-2| < .16$ (*i.e.* if $1.84 < x < 2.16$), then $|(3x - 1) - 5| < .5$ (*i.e.* then $4.5 < (3x - 1) < 5.5$). Check this and draw a picture! Note that there are other values of δ that would work for $\epsilon = .5$ (can you find one?), so this answer is not unique. Also, if we chose a different ϵ, say $\epsilon = .1$, then we'd need another, smaller value of δ (can you find one?).

To prove statement (1) we must show that for *each* value of ϵ there is *some* value of δ that will work; to do this we will find an expression for δ in terms of ϵ (in other words, a formula that will find a working δ given any choice of ϵ).

Finding Delta in terms of Epsilon
Given $\epsilon > 0$ we have to find a $\delta > 0$ so that if $0 < |x - 2| < \delta$, then $|(3x - 1) - 5| < \epsilon$. In other words, we want to choose a δ so that information about $|x - 2|$ will give us the desired result about $|(3x - 1) - 5|$. It's not clear at the outset what this δ should be, but if we start manipulating $|(3x - 1) - 5|$ we'll soon find a value of δ that will work. Consider the calculation:

$$|(3x - 1) - 5| \;=\; |3x - 1 - 5| \;=\; |3x - 6| \;=\; |3(x - 2)| \;=\; 3|x - 2|.$$

What do we know about $|x - 2|$? We know it is less than δ (whatever we choose δ to be, we'll have $0 < |x - 2| < \delta$ by hypothesis). Using this fact and the calculuation above, we have:

$$|(3x - 1) - 5| \;=\; \cdots \;=\; 3|x - 2| \;<\; 3\delta.$$

What we wanted was to make $|(3x - 1) - 5|$ less than the given ϵ. What we managed to show was that $|(3x - 1) - 5|$ is less than 3δ. Now can you see what we should take δ to be? If we choose $\delta = \epsilon/3$ (note this depends on ϵ, as we suspected), then the calculation above becomes:

$$|(3x - 1) - 5| \;=\; \cdots \;=\; 3|x - 2| \;<\; 3\delta \;=\; \epsilon,$$

and thus we have shown that, if $0 < |x - 2| < \delta = \epsilon/3$, then $|(3x-1) - 5|$ is less than ϵ.

Thus, for example, if we want $(3x - 1)$ to be within .5 of 5 (so $\epsilon = .5$), we can take $\delta = .5/3 = 1/6 = .16\bar{6}$. If we want $(3x - 1)$ to be within .1 of 5 (so $\epsilon = .5$), we can take $\delta = .1/3 = .03\bar{3}$. The formula $\delta = \epsilon/3$ works for any choice of $\epsilon > 0$ in this example.

A Delta-Epsilon Proof
We've now done all the legwork involved in proving that the statement

$$\lim_{x \to 2}(3x - 1) = 5$$

is true. To write it in a proof we just have to write down our ideas clearly and concisely. Remember, we're trying to prove that:

For all $\epsilon > 0$, there exists a $\delta > 0$ such that:
If $0 < |x - 2| < \delta$, then $|(3x - 1) - 5| < \epsilon$.

Thus we must first let ϵ be an arbitrary positive number; then given that ϵ we must find an expression for δ (usually in terms of ϵ). Given this δ we then assume that $0 < |x - 2| < \delta$, and use this to prove that $|(3x - 1) - 5| < \epsilon$. (Recall that if we want to show "If A, Then B", we have to assume that A is true and then prove that B is true.)

Delta-epsilon proofs always follow the same format. The delta-epsilon proof for the example we have been working on is:

Proof: Let $\epsilon > 0$.

Choose $\delta = \underline{\quad \epsilon/3 \quad}$.

If $0 < |x - 2| < \delta$,

$$\begin{aligned} \text{Then } |(3x-1) - 5| &= |3x - 6| \\ &= |3(x-2)| \\ &= 3|x - 2| \\ &< 3\delta && \textit{(by hypothesis)} \\ &= 3(\epsilon/3) && \textit{(since we chose } \delta = \epsilon/3\textit{)} \\ &= \epsilon; \end{aligned}$$

Thus we have $|(3x - 1) - 5| < \epsilon$. ■

Note that we "choose" δ on the second line of the proof, even though if we haven't yet done the calculuation in the proof we don't know what we should choose δ to be. When you're doing one of these proofs, just write "Choose ______" and fill in an appropriate expression for δ when you discover what works. (In the proof above, we don't discover what δ will work until the second-to-last statement in the computation.)

Make sure you understand not only *how* to put one of these kinds of proofs together, but *why* they prove what they do. In other words, compare this proof to the statement it purports to prove (the formal definition of limit in a particular example, as above); make sure you understand why the process outlined above proves that the limit statement is true.