Proof by Induction

Math 235 Fall 2000

Inductive proofs are a clever way of proving that a statement holds for all positive integers. Consider the following statement P concerning the integers:

$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}. \tag{1}$$

For which integers n is this statement true? By trying a few examples you might start to believe that it is true for all positive integers. However, we can't prove this statement just by using algebra; in other words, we can't just simplify or rewrite the left hand side of the equality and obtain the right hand side of the equality. Instead we will use an *inductive* proof.
Suppose P is a statement concerning the integers. Proofs by induction use the following axiom, called the *induction axiom*:

If P is true for $k = 1$, and if whenever P is true for an integer k it is also true for $k + 1$, then P is true for all positive integers.

In other words, if we know that P is true for $k = 1$, and if we can show that whenever P is true for k it is also true for $k + 1$, then P true for $k = 1$ implies that P is true for $k = 2$, which implies that P is true for $k = 3$, and so on. One way to think of this is in terms of climbing a ladder; if you can climb onto the first rung of the ladder, and you know how to climb up from one rung to another, then you can climb all the way up the ladder. Note that it is important that we know that P is true for $k = 1$ here; if we can't step onto the first rung of the ladder, we can't climb up the ladder even if we know how to get from one rung to another.

Actually, we don't have to start with $k = 1$ in the boxed axiom above; we could start with, say $k = 5$, and then the conclusion would be that P is true for all integers greater than or equal to five. Note that inductive proofs can only prove statements concerning the integers. Don't use a proof by induction if you can use a direct proof instead; direct proofs are much more illuminating (not to mention less complicated).

To write an inductive proof, first show that the statement is true for $k = 1$ (or some other starting integer). The meat of the proof is showing that:

If the statement is true for an integer k, then it is also true for the next integer $k + 1$.

To do this, you must first assume that the statement is true for k; this assumption is called the *inductive hypothesis*. (Note we've been using k intead of n here; you could use n throughout the proof if you prefer.) Given this assumption you must then show that the statement is true for $k + 1$. This last step usually involves some pretty messy algebra, and it *always* involves using the inductive hypothesis.

We will now use induction to prove that the statement (1) above is true for all positive integers.

Proof: (By induction.)

True for $k = 1$: $1 = \frac{(1)(2)}{2}$.

Assume true for k, *i.e.* assume: $1 + 2 + 3 + \ldots + k = \frac{k(k+1)}{2}$.

Given this assumption, we will show that the statement is true for $k + 1$, *i.e.* we will show that:

$$1 + 2 + 3 + \ldots + (k + 1) = \frac{(k + 1)((k + 1) + 1)}{2}.$$

$$\begin{aligned}
1 + 2 + 3 + \ldots + (k + 1) &= (1 + 2 + 3 + \ldots + k) + (k + 1) \\
&= \frac{k(k + 1)}{2} + (k + 1) \qquad \textit{(Using ind. hyp.)} \\
&= \frac{k(k + 1) + 2(k + 1)}{2} \\
&= \frac{(k + 2)(k + 1)}{2} \\
&= \frac{(k + 1)((k + 1) + 1)}{2}.
\end{aligned}$$

Thus by induction, statement (1) is true for all positive integers. ■

Although it takes a while to understand why and how inductive proofs work, one good thing about proof by induction is that the structure of the proof is always the same. Inductive proofs should *always* follow the template below:

Proof: (By induction.)

True for $k =$____: ____________________________

Assume true for k, *i.e.* assume: ____________________________

Prove true for $k + 1$, *i.e.* prove: ____________________________

(Then prove the $k + 1$ statement using algebra and the inductive hypothesis.)

Therefore, by induction, the statement is true for all integers greater than or equal to ____. ■