Formal Limits

Math 235 Fall 2000

In this handout we will develop the formal defnition of a limit. Consider the limit statement:

$$\lim_{x\to 2}(3x-1) = 5. \tag{1}$$

When computing limits in practice, we've said that you can take this limit by "plugging in" $x = 2$ to $3x - 1$ and (since nothing "bad" happens), that the answer $3(2) - 1 = 6 - 1 = 5$ is the desired limit. So far this is just a computational tool; we have not shown that doing limit calculuations this way actually works or makes sense! *Proving* that the limit of $3x - 1$ as x goes to 2 is actually equal to 5 involves a long and complicated process called a "delta-epsilon proof" that uses a formal definition of limit.

The limit statement in (1) can be intuitively described as saying:

As x approaches 2, $(3x - 1)$ approaches 5.

But what do we mean by "approaches?" The reason this intuitive description of limit isn't a precise mathematical definition is that we haven't given a rigorous mathematical definition of "approaches." Another way to say the statement above is to say that:

We can get $3x - 1$ as close as we like to 5 by making x sufficiently close to 2.

What do we mean by "close?" Mathematically, $3x - 1$ will be "close" to 5 if the distance between the number $3x - 1$ (for a given x) and the number 5 is small. Similarly, x is sufficiently close to 2 if the distance between x and 2 is sufficiently small. Since the distance between two real numbers a and b is given by $|a - b|$, the statement above can be written:

We can make $|(3x - 1) - 5|$ as small as we like by making $|x - 2|$ sufficiently small.

In other words, we can make $|(3x - 1) - 5|$ smaller than any number we choose simply by making $|x - 2|$ small enough. Suppose we choose a positive number that we would like $|(3x - 1) - 5|$ to be smaller than, and call it ϵ (this is the traditional name for such a number). The statement above then says that we can make $|(3x - 1) - 5| < \epsilon$ by making $|x - 2|$ sufficiently small; in other words, by making $|x - 2|$ smaller than some positive number we'll call δ (again, the traditional name for a number in this role). Each ϵ that we choose may require a different δ, but the statement above asserts that for *every* ϵ, there exists *some* δ that will work. (Since we can make $|(3x - 1) - 5|$ smaller than any ϵ just by making δ, and thus $|x - 2|$, sufficiently small.) Using this notation, the statement above can now be written:

Given any $\epsilon > 0$, we can make $|(3x - 1) - 5| < \epsilon$
by requiring that $0 < |x - 2| < \delta$ for some sufficiently small δ.

The reason that we specify that $0 < |x - 2|$ in the statement above is that we are only interested in choosing a value of x that is *near* 2; we don't care what happens when x *is* 2. (The inequality $0 < |x - 2|$ simply means that $x \neq 2$.) Finally, rewording the statement above we arrive at the rigourous mathematical definition of limit (for this example):

$$\textit{For all } \epsilon > 0, \textit{ there exists a } \delta > 0 \textit{ such that:}$$
$$\textit{If } 0 < |x - 2| < \delta, \textit{ then } |(3x - 1) - 5| < \epsilon.$$

This is the precise mathematical definition of the limit in (1) above, and it is what we mean when we intuitively say that "as x approaches 2, $(3x - 1)$ approaches 5."

In general, let c be any real number, and let $f(x)$ be a function defined "near," but not necessarily at, c. In other words, let $f(x)$ be a function defined on at least an interval of the form

$$(c - p, c) \cup (c, c + p)$$

for some positive number p. Replacing the specific function $(3x - 1)$ from the discussion above with such an arbitrary function $f(x)$, replacing the 2 by an arbitrary c, and replacing 5 with a real number L, we arrive at the formal definition of limit:

$$\lim_{x \to c} f(x) = L \textit{ if and only if:}$$

$$\textit{For all } \epsilon > 0, \textit{ there exists a } \delta > 0 \textit{ such that:}$$
$$\textit{If } 0 < |x - c| < \delta, \textit{ then } |f(x) - L| < \epsilon.$$

Make sure you read this handout a few times until you *really* understand what this definition means. You will need to know how to do the following things with this definition:

- State this definition.
- Draw a picture illustrating what this definition means.
- Explain in your own words what this definition means and how it is related to the "intuitive" definition of limit.
- Given a particular limit (for example, $\lim_{x\to 1}(x^2 + 1) = 2$), and a particular value of ϵ (for example, $\epsilon = .01$), find a value for δ that works.
- Given a particular limit, find an expression for δ in terms of ϵ. (Then for *any* given ϵ you will have a formula that gives you a working δ.)
- Given a particular limit, prove using the definition that it is true. (This is basically a rewriting of the item listed above into proof form.)

The handout on delta-epsilon proofs will explain the last two items on this list. The fourth item is covered in lecture and in your textbook (see especially Project 2.2 on page 80).