Elementary Logic

Math 235 Fall 2000

Since many proofs require the use and understanding of logic, I've made this handout as an easy reference to the most basic logical constructions. Throughout this handout, A and B will represent statements that could either be true or false.

Implication
$A \Rightarrow B$ means that A *implies* B; in other words, it represents the statement "if A, then B" or "if A is true, then B is true." Note that $A \Rightarrow B$ implies nothing about the truth or falsehood of A; it merely asserts that *if* A is true, then B must also be true. Moreover, if A is false, then the statement $A \Rightarrow B$ does not imply anything about B; *i.e.* if A is false, then B could either be true or false. The following table summarizes the conditions under which the statement $A \Rightarrow B$ is true:

A	B	$A \Rightarrow B$
True	True	True
True	False	False
False	True	True
False	False	True

Therefore, the only way that the statement $A \Rightarrow B$ can be false is if A is true while B is false. In other words,

$$\text{Not}(A \Rightarrow B) \text{ means that } A \text{ is true and } B \text{ is false.}$$

So, for example, if you were trying to prove $A \Rightarrow B$ by contraction (see the handout on proof by contradiction), you would start out by assuming that $A \Rightarrow B$ was false, *i.e.* that A was true but B was false.

Converse
The *converse* of the statement $A \Rightarrow B$ is the statement $B \Rightarrow A$. A common logical mistake is to assume that a statement is equivalent to its converse; however, it is NOT true that if $A \Rightarrow B$ is true, then $B \Rightarrow A$ is true (although sometimes in regular life, implication is used this way). For example, the statement "If you don't pay your taxes, then you will get arrested" is NOT the same as the statement "If you get arrested, then you didn't pay your taxes."

Contrapositive
The *contrapositive* of the statement $A \Rightarrow B$ is the statement (Not B) $\Rightarrow$ (Not A). Unlike what we said above about the converse, it IS true that a statement and its contrapositive are equivalent. One way to see this is to notice that $A \Rightarrow B$ is false only if A is true and B is false (see above), and the contrapositive (Not B) $\Rightarrow$ (Not A) is false only if (Not B) is true and (Not A) is false (*i.e.* if B is false and A is true). Since a statement and its contrapositive are either both true or both false in every possible truth-assignment for A and B, they are equivalent statements. Sometimes proving the contrapositive of a statement is easier than proving the original statement; for example, one way to prove that every differentiable function is continuous is to show that if a function isn't continuous, then it can't be differentiable.

If and Only If

The statement $A \Leftrightarrow B$, or $A \textit{iff} B$, means "A if and only if B". The phrase "if and only if" means that the implication goes both ways: A if B, and A only if B. Said another way, this means: if B, then A; and if A, then B; in other words, A and B are equivalent statements, in the sense that their truth-values must always be the same (if A is true, then B is true, and vice-versa; if A is false, then B is false, and vice-versa). Often when proving a statement of the form $A \Leftrightarrow B$ we first prove that $A \Rightarrow B$, and then that $B \Rightarrow A$.

And & Or

The statement "A and B" is true when A is true and B is true, and false in every other truth-assignment. The statement "A or B" is only false if neither A nor B is true. Note in particular that this means "A or B" is an *inclusive or*; in other words, "A or B" is true if A is true, or if B is true, or if *both* A and B are true. The only tricky thing about "and" and "or" statements is how to negate them:

$$\text{Not}(A \text{ and } B) \text{ is the same as (Not } A\text{) OR (Not } B\text{)}.$$

$$\text{Not}(A \text{ or } B) \text{ is the same as (Not } A\text{) AND (Not } B\text{)}.$$

The following truth tables illustrate why this is the case (note that the statements we are showing are equivalent have the same truth-value for all truth-assignments of A and B):

A	B	A and B	Not(A and B)	(Not A) or (Not B)
True	True	True	False	False
True	False	False	True	True
False	True	False	True	True
False	False	False	True	True

A	B	A or B	Not(A or B)	(Not A) and (Not B)
True	True	True	False	False
True	False	True	False	False
False	True	True	False	False
False	False	False	True	True