Chapter 2 Outline **Math 235 Fall 2000**

2.1 The Idea of Limit

- **Intuitive Concept of Limit:** Know how to describe the statement $\lim_{x \to c} f(x) = L$ in words (in a sentence involving the word "approaches" as well as in a sentence using the word "close to" and in a sentence using the word "distance"). Be able to draw a picture illustrating this. Know these things for one-sided limits as well, and know the connection between one-sided and two-sided limits.

- **Calculating Limits:** Be able to calculate given limits both graphically (*i.e.* looking at the graph) and algebraically (*i.e.* using algebra and "plugging in" when appropriate). Be able to identify when a given limit does not exist, and if possible, be able to describe why it doesn't exist. Be able to use your calculator to verify your calculations, and to approximate answers for more complicated functions like $x \sin(1/x)$.

- **Application – Secant and Tangent Lines:** Understand how the slope of the tangent line is a limit of the slopes of secant lines; be able to state this fact algebraically using a limit and draw a picture illustrating this fact. Note that there are two ways to express this fact (with $x \to c$ and with $h \to 0$); know both of these, and why they are equivalent. Be able to calculate the slope of the tangent line to a given function at a given point using this fact by calculating the appropriate limit (and then be able to find the equation of this tangent line).

2.2 Definition of Limit

- **The Formal Definition of Limit:** Be able to state the formal definition of the limit statement $\lim_{x \to c} f(x) = L$. Be able to describe what this definition means in words (including the words "delta" and "epsilon"), draw a picture illustrating the definition, and describe how this definition agrees with the intuitive concept of limit we developed in 2.1. Be able to do these things for one-sided as well as two-sided limits.

- **Finding a δ Given a Particular ϵ:** Given a particular limit and a value for ϵ, find a δ that "works" in the definition of limit (by hand or using the calculator, depending on the complexity of the function). Given an ϵ and a working δ, what can you say about smaller and larger values of ϵ or δ? (Think about that True-False quiz/HW question.)

- **Delta-Epsilon Proofs:** Given $\epsilon > 0$, be able to find an expression for δ (most often in terms of ϵ. Be able to write a clear delta-epsilon proof, and explain why it proves that a particular limit statement is true (*i.e.* why it proves that the formal definition of limit is true for that example). Be sure that you can do more complicated delta-epsilon proofs where you have to choose $\delta = \min(1, \underline{\quad})$.

- **Simple Limit Theorems:** Know the four equivalent ways of writing a limit statement (see (2.2.5) on page 76 of the book). Know and be able to prove (using a delta-epsilon proof) the limit statements for the limits of x, $|x|$, and k as $x \to c$.

2.3 Some Limit Theorems

- **Uniqueness of Limits:** You should know and understand the theorem proving the uniqueness of limits (but you do not need to be able to prove it).
- **Limits of Combinations of Functions:** Know, understand, and be able to use the theorems for sums, constant multiples, products, quotients, and differences of limits. Be able to prove all of these except the product and quotient rules for limits (note you can prove the differences of limits rule two different ways). Be able to state these theorems as well as describe what they mean; also know how to apply them in proofs and in particular examples (*e.g.* more complicated limit computations; be able to point out where you use each of these limit rules in your computations).
- **Limits of More Complicated Combinations of Functions:** Know, understand, and be able to use the theorems for the limits of linear combinations and long products of functions. Be able to prove these theorems (using induction). Be able to use these theorems to prove that the method of "plugging in" c for x to solve a limit of a polynomial actually works (first prove that the limit of x^k as $x \to c$ is c^k, and then use the theorem for linear combinations).

2.4 Continuity

- **Definition of Continuity:** Know the formal definition of continuity at a point (this involves a limit) as well as an intuitive description. How does continuity of a function $f(x)$ at a point c relate to being able to "plug in" c to find the limit of $f(x)$ as $x \to c$? Know how to prove formally that a function is continuous at a given point (either by using a δ–ϵ argument for the limit or by using theorems about limits). What does it mean for $f(x)$ to be a continuous function? Is $1/x$ a continuous function? Why or why not? Know what it means (graphically and by definition) for a function to be continuous on an open or closed interval; left-continuous at a point; right-continuous at a point.
- **Determining Continuity:** Be able to tell just by looking at a graph whether or not the function is continuous at various points. Also be able to show algebraically that a function is continuous by using the limit definition of continuity. Know what "types" of functions are continuous (and why), and be able to use those facts to determine continuity of combinations of functions (see below). Be able to determine continuity of piecewise-defined functions.
- **Continuity of Combinations of Functions:** If two functions are continuous at a point c, is their product, sum, difference, or quotient continuous? Is a constant multiple of a continuous function continuous? Know how to prove these facts by using the corresponding rules for limits. What conditions on $f(x)$ and $g(x)$ do we have to have to insure that the composition $f(g(x))$ is continuous? (You don't have to prove that last fact, just know it.)

2.5 The Pinching Theorem; Trigonometric Limits

- **The Pinching Theorem:** Be able to state this theorem precisely, and explain intuitively what it means. Sketch a graph illustrating the theorem and how it works. (You don't need to know how to prove this theorem.)

- **Applying the Pinching Theorem:** Be able to use the Pinching Theorem in limit computations. Watch out for multiplying both sides of an inequality by a variable that may be positive or negative! Be able to use the Pinching Theorem in simple proofs (see the homework).

- **Simple Trigonometric Limits:** You may assume without proof that $\sin x$ and $\cos x$ are continuous at the point $x = 0$. (Can you write what I just said as facts about limits?) Using this, be able to prove that $\sin x$ and $\cos x$ are continuous (everywhere). Given this information, you can do simple trigonometric limits by "plugging in"; make sure you can do enough trig calculuations to effectively do this (*e.g.* $\sin(\pi/3) = \sqrt{3}/2$).

- **Two Very Important Trigonometric Limits:** Memorize the two trig limits in (2.5.5) in your text, page 102. You do not have to know how to prove that these limits are true. Be able to use these two limits to calculate more complicated trigonometric limits (note this often involves extensive algebra and rewriting of expressions).

2.6 Two Basic Properties of Continuous Functions

- **The Intermediate Value Theorem:** Be able to state this theorem precisely, and explain intuitively what it means (including drawing a picture). Be able to explain why the continuity hypothesis of this theorem is necessary. (In other words, if this hypothesis is false, does the conclusion necessarily have to be true? Be able to provide an example of a non-continuous function that does not satisfy the conclusion of the theorem.) You do not have to know how to prove this theorem. Be able to state and use a special case of this theorem to get information about the roots of a continuous function. Be able to state and use the contrapositive of this special case of the theorem to solve inequalities.

- **The Extreme Value Theorem:** Be able to define "boundedness" and "extreme value" in a mathematical way. Be able to state the Extreme Value Theorem, and give an intuitive description of what it means (including drawing a picture). Be able to explain why you need the continuity hypothesis and why you need the closed interval hypothesis for this theorem. Be able to provide examples where the conclusion of the theorem does not hold if either of these hypotheses fail to be true.

- **Using these Theorems:** Be able to use these theorems to sketch graphs of functions with given properties (or to say why a function with a given list of properties couldn't possibly exist). Be able to use the Intermediate Value Theorem to find intermediate values of a function *other* than roots. Be able to apply these theorems in simple proofs (see the starred problems in the homework).