Chapter 4 Outline

Math 235 Fall 2000

4.1 The Mean-Value Theorem

- **Rolle's Theorem:** State this theorem and sketch an example illustrating the theorem. Sketch examples that show that the hypotheses are necessary; in other words, examples where one of the hypotheses is false (and thus the conclusion may or may not be true). Be able to "argue" why Rolle's Theorem is true (*i.e.* give a fuzzy proof of the theorem).

- **The Mean-Value Theorem:** State this theorem and sketch an illustrating example. Sketch examples illustrating that when one of the hypotheses is false, the conclusion may or may not be true. In what sense is the Mean-Value Theorem a generalization of Rolle's Theorem? Said another way, in what way is Rolle's Theorem a special case of the Mean-Value Theorem?

- **Using the Theorems:** Be able to use the two theorems above in particular examples. Given a function on an interval, does one of the theorems apply? If so, what does the theorem say about that function? Rolle's Theorem is often used to prove that a function has more or less than a certain number of roots; be able to do this. The Mean-Value Theorem is often used to prove that a function was going a certain rate at some point on an interval; be able to do this, especially with real-world functions. Note that both of these theorems only tell you that the point you are looking for *exists* in the interval; they don't tell you where it *is*.

4.2 Increasing and Decreasing Functions

- **Definitions of Increasing and Decreasing:** Be able to write down the formal mathematical definition of what it means for a function to be increasing or decreasing on an interval. Make a sketch illustrating the definition. Use the definition to test a given function to see if it is increasing or decreasing. (Note: none of what I have just said has anything to do with derivatives. That comes next.)

- **Derivatives and Increasing/Decreasing:** What is true about a function f at a point or on an interval where f' is positive? Negative? Zero? Why? Given a particular function, be able to find the intervals on which the function is increasing and the intervals on which the function is decreasing. Be sure you can do this even with piecewise functions.

- **When derivatives are equal:** If $f'(x) = g'(x)$, how are f and g related? Can you explain why using the graphs of f and g? Can you prove it using the fact that when the derivative of a function is zero on an interval, the function must be constant on that interval?

4.3 Local Extreme Values

- **Definition of a Local Maximum or Minimum:** Be able to give the formal mathematical definition of what it means for a function f to have a local maximum or a local mininum at a point $x = c$. Illustrate the definition with an appropriately labeled picture.

- **The Derivative at a Local Extrema:** If $x = c$ is a local extrema of f, what must be true about $f'(c)$? Can you prove it? Draw pictures illustrating various ways that a function can have a local extrema and discuss what the derivative is like at and near those points.

- **Critical Numbers:** Know the definition of (interior) critical numbers, and how to find them for a given function.

- **The First-Derivative Test:** Given a critical number $x = c$, how can you use the first derivative to test if f has a local maximum, minimum, or neither at $x = c$? Why does this test work? Be able to do this given a particular function. Don't forget to think about piecewise-defined functions and functions that involve absolute values. In what way are you using the Intermediate Value Theorem when you do this? Why is continuity required for the First-Derivative Test to work? Draw a picture illustrating that the First-Derivative Test won't always give the right answer if f is discontinuous at the critical point you are testing.

- **The Second-Derivative Test** What is the Second-Derivative Test? What does it do? How does it work? Why does it work? What does the Second-Derivative Test tell you when the second derivative of f at a critical point is zero? Be able to use this test to examine the critical points of particular functions.

4.4 Endpoint and Absolute Extreme Values

- **Absolute (Global) Extreme Values:** What is the formal definition of an absolute/global extreme value? Does a given function on a given interval *have* to have a global maximum? A global minimum? (Hint: if the interval is closed the Extreme Value Theorem tells you that the function has both a global maximum and a global minimum on the interval–what happens if the interval is not closed?) Sketch various examples of functions on intervals that do *not* have a global max, and/or do *not* have a global mininum.

- **Closed intervals:** Endpoints of a closed interval are also considered critical points. When looking for a global maximum on a closed interval, you have to compare the heights of the local maxima to the height of the function at the endpoints. Be able to do this in particular examples. How can you tell (by using the first derivative) if a closed endpoint is a local maximum or a local minimum of the function?

- **Open intervals:** If the interval you are working on is open (on at least one side), how can you test to see if there is a global maximum on the interval? Sketch a graph on an open interval where none of the local maximums are global maximums, and a graph where one of the local maximums is in fact a global maximums. How can you test this without a graph? Be able to do this for specific functions. Can a function ever have a local minimum or maximum at an open endpoint?

- **Infinite intervals:** Sketch a graph on an infinite interval that has local minimums but no global minimum. Sketch a graph on an infinite interval that has both a global maximum and a global minimum. Given a specific function, how can you test (without the graph) to see if any of your local maximums are global maximums?

4.5 Some Max-Min Problems

- **Setting Up Max-Min Problems:** Be able to sketch and label an appropriate picture or diagram for the word problem. All variables should be labelled on the diagram. What is it that you know? What is it that you are trying to find? What function are you trying to maximize or minimize, and *on what interval*? Note that you can't maximize or minimize a function without knowing the interval first–why?

- **Solving a Max-Min Problem:** Don't forget that you are really looking for the *global* maximum or minimum of the given function on the given interval. Finding the critical points is easy; you must test to see if they are local minimums or local maximums, and *then* you must test to see if any of them are global minimums or maximums. This means you will have to consider the interval that you are working over, and test what is happening on the "ends" of the interval (whether the "end" is closed, open, or infinite, as above). Be able to do this, showing all work, for specific problems. Don't forget that places where the derivative does not exist are also critical points!

4.6 Concavity and Points of Inflection

- **What does f'' say about f'?:** What does it mean about $f'(c)$ if $f''(c)$ is positive? Negative? Zero? Why?

- **Concavity:** What does f look like if f' is increasing? Decreasing? Why does a function have that "concave up" shape when the derivative is increasing? Given that the definition of "concave up" is that the derivative is increasing, why does this imply that the second derivative is positive whenever the function has that concave up shape? Same questions for downward concavity. Draw some pictures. Given a particular function, be able to find the intervals on which it is concave down, and the intervals on which it is concave up.

- **Inflection Points:** What is the definition of an inflection point? What must be true about $f'(c)$ if c is an inflection point of f? Give an example where the second derivative is zero at a point but the function does not have an inflection point there. How can you tell from the second derivative nearby whether or not this is happening? Draw a picture of a function f where the derivative has a maximum. A minimum. Given a particular function, be able to find all of its inflection points.

4.7 Vertical and Horizontal Asymptotes; Vertical Tangents and Cusps

- **Places Where the Derivative Does Not Exist:** If $f'(c)$ does not exist, what could be happening to f at $x = c$? Be able to give conditions involving f and/or f' and/or appropriate limits that characterize: corners, vertical cusps, vertical tangents, vertical asymptotes, jump discontinuities, and removable discontinuities. Given a particular function where the derivative does not exist at a point be able to determine which of these things is happening.

- **Horizontal Asymptotes of Rational Functions:** What is a rational function? Where are its zeros, holes, or vertical asymptotes? State a theorem that explains when and where a rational function has a horizontal asymptote. Why does this theorem work? Be able to argue (1) why a rational function is determined by the ratio of its leading terms "near infinity", and (2) why this implies the theorem you just stated. Given a particular rational function, be able to determine if and where it has a horizontal asymptote.

4.8 Some Curve Sketching

- **Putting it all Together:** Use all of the tools above to sketch the graph of a given function (without your calculator, of course). Make sure that you show all your work clearly and show all your computations. You'll have to find the domain of the function, find any obvious roots and the y-intercept, find the critical points, test for local and global extrema (including examining any open, closed, or infinite "ends"), find intervals where the function is increasing, decreasing, concave up, and concave down, find inflection points, find any horizontal or vertical asymptotes, examine the behavior of the function at non-differentiable and non-continuous points, use any symmetry or periodicity, find the values of the function at any interesting points, and put it all together into a graph that clearly shows all of these properties. Be sure you do plenty of examples, including piecewise functions, trigonometric functions, and functions involving absolute values.

- **Sketching f Without an Equation:** Be able to sketch the graph of a function (or say why such a function cannot exist) given only a list of information like the above (without the equation of the function). Given a list of such information, be able to state what extra information you would need to sketch an accurate graph of the function. Be able to sketch a graph of a fucntion f given the graph of its first and/or second derivatives.