Chapter 5 Outline

Math 235 Fall 2000

5.1 The Definite Integral of a Continuous Function

- **Partitions:** What is a partition of an interval $[a, b]$? Know the notation of partitions, for example $P = \{x_0, x_1, \ldots, x_n\}$ and Δx_i. The partition will determine the widths of the rectangles that will be used to approximate the area under $f(x)$.
- **Upper and Lower Sums:** Be able to write upper and lower sums $U_f(P)$ and $L_f(P)$ in general notation and in specific examples. In particular, be able to describe what the notation M_i and m_i means in this notation. What is x_i^* in these contexts? What does all this mean in terms of rectangles? Sketch various examples of upper and lower sums and be able to calculate these sums. How do these sums compare to the actual area under $f(x)$?
- **Other Sums to Approximate Area:** Be able to sketch, calculate, and write out using notation, other types of sums that could be used to approximate the area under $f(x)$. In other words, be able to work with other ways of choosing the heights of the rectangles in the approximation. For example, one might choose the height of the i^{th} rectangle to be the height of $f(x)$ at the left-hand side of the i^{th} interval (or the right-hand side, or the middle, and so on).
- **General Riemman Sums:** All of the sums above are Riemann sums. What makes a sum a Riemann sum? Can you think of a sum that is *not* a Riemann sum? Again make sure that you understand all of the notation: x_i, x_i^*, Δx_i, and so on.
- **Taking the Limit:** What does it mean when we take the limit as $|P| \to \infty$? What is $|P|$? What happens to a Riemann sum as we take this limit? Be able to write the definite integral $\int_a^b f(x)\, dx$ in terms of a limit of Riemann sums. What does this definite integral represent in terms of area?
- **Finding an *exact* area using a Riemann sum:** You are *not* reponsible for problems like Example 5 in the reading, although I give you permission to read such examples for your own edification.

5.2 The Function $F(x) = \int_a^x f(t)\, dt$

- **Adding Points to a Partition:** First, be able to explain what happens to an upper or lower sum as a partition gets "finer"; in other words, how do $U_f(Q)$ and $U_f(P)$ compare if Q and P are partitions with $P \subseteq Q$? What does it mean to say that $P \subset Q$?
- **Additive Property of the Definite Integral:** The definite integral from a to b can be written as the sum of the definite integral from a to c and the definite integral from c to b, where c is a point in the interval $[a, b]$. Be able to show this in a picture in terms of area and think about a simple Riemann sum argument that would give a "proof" of this.
- **Functions of the form $F(x) = \int_a^x f(t)\, dt$:** Be able to describe what a function $F(x)$ of this form is. What is the independent variable or "input"? What is the "output"? Be able to sketch and calculate values of this function $F(x)$ given a graph of $f(x)$.

- **The Very Important Theorem:** Given that $f(x)$ is continuous, and if $F(x)$ is of the form above, what can you say about $F(x)$? Hint: there are *three* things that you can say about $F(x)$; see Theorem 5.2.5. Know how to prove this theorem (possibly with hints). Note that the proof we did in class is a little easier than the one in the reading. Know how to use the Very Important Theorem to calculate derivatives of functions of the form $F(x)$. In particular, how would you differentiate $F(u(x))$? Why?

5.3 The Fundamental Theorem of Integral Calculus

- **Antiderivatives:** What is the definition of an *antiderivative* of a function $f(x)$? What does this mean about functions of the form $F(x)$ above? Be able to calculate simple antiderivatives. Can a function $f(x)$ have more than one antiderivatives? If so, how are the antiderivatives of $f(x)$ related, and *why*?

- **The Fundamental Theorem:** State the Fundamental Theorem of Calculus (complete with hypotheses). What does this theorem mean about the connection between areas and antiderivatives? Be able to recognize the Fundamental Theorem in different contexts and different notations. Use the Fundamental Theorem to calculate exact values of areas under curves. Prove the Fundamental Theorem of Calculus using the Very Important Theorem (see the class notes; I'd provide you with hints for this one). Why does the Fundamental Theorem work for *any* antiderivative of $f(x)$? Know the "evaluation" notation (the brackets, for example, $[x^2 - 5]_1^2$).

- **Sum and Constant Multiple Rules for Definite Integrals:** The Fundamental Theorem tells us how to calculate definite integrals using antiderivatives. Using this it can be proved that definite integrals obey "sum" and "constant multiple" rules. Be able to state these rules, and prove them using the sum and constant multiple rules for differentiation and the Fundamental Theorem. Calculate more complicated definite integrals using these rules (writing out each step, if asked).

5.4 Some Area Problems

- **Signed *vs.* "True" Area:** Explain why the definite integral will count area under the x-axis negatively (use Riemann sums). What definite integral will calculate such a "negative" area positively? Be able to calculate the "true" area between a curve and the x-axis by splitting up the interval $[a, b]$ and calculating the appropriate definite integrals.

- **Areas Bounded Above and Below by Functions:** How can you calculate an area bounded above by a function $f(x)$ and below by a function $g(x)$? Can you explain why this works using rectangles and Riemann sums? What happens if one or both of f and g are negative on an interval?

- **Various Calculations:** Note that you will have to know the roots of $f(x)$ and the intervals on which it is positive and negative in order to calculate the "true" area between $f(x)$ and the x-axis (so you can split up the definite integrals accordingly). If you are dealing with an area bounded by two functions $f(x)$ and $g(x)$ then clearly you'll need to find the intersection points of these functions and when one is above or below the other.

5.5 Indefinite Integrals

- **Definition of an Indefinite Integral:** What is an indefinite integral? Does the definite integral *a priori* have any obvious relation to area? What key theorem tells you that in fact the indefinite integral *does* have a connection to area?
- **Rules for Indefinite Integrals:** Since indefinite integrals are merely new notation for writing anti-derivatives, they obey sum and constant multiple rules. Be able to state, use, and prove these rules. When using these rules be able to write them out step by step if asked.
- **Calculating Indefinite Integrals:** Be sure that you can quickly calculate basic indefinite integrals. Be especially sure that you can recognize which functions we do *not* immediately know how to anti-differentiate.
- **Position, Velocity, and Acceleration:** Since position, velocity, and acceleration are related by differentiation, they are also related by anti-differentiation. Given any of these three formulae be able to calculate the remaining two formulae. Make sure that you know how to take into account any initial conditions when you do this. What does the Fundamental Theorem say in this context? In other words, how can you calculate the change in distance using a definite integral? How can you calculate the *total* distance travelled (in any direction) using a definite integral? Be sure you can do word problems involving these concepts.

5.6 The u-Substitution; Change of Variables

- **Doing the Chain Rule "Backwards":** Write out the chain rule for differentiation for a function $f(u(x))$. Now write this out "backwards" using an indefinite integral. The method of u-substitution helps you to solve any integral with this pattern. Be able to recognize and use this pattern without explicitly going through the steps of u-substitution.
- **The Method of u-Substitution:** Although the "algorithm" of u-substitution is not necessary to integrate functions that have the pattern discussed above, it does simplify the process of recognizing and using the pattern. Know what to choose for u, how to relate dx to du, and if necessary, how to do other algebraic manipulations to get a "new" integral entirely in terms of the variable u. Make sure that you show all of your work clearly so that it is clear exactly what it is that you are doing. Know also when u-substitution does *not* apply!
- **u-Substitution and Definite Integrals:** As we saw in class there are two ways to deal with definite integrals using u-substitution. Know both of these ways and be able to use them.

5.7 Additional Properties of the Definite Integral

- **Various Properties:** Many properties of definite integrals are listed in this section. Be able to state them, use them, and describe why they are true both in terms of area pictures and by using a Riemman sum argument. (Do this for each property up to Property 5.7.6; don't worry about the Riemann sum arguments for the other properties.) Usage of these properties can be specific (for example, for a particular function) or abstract (for example, questions of the form: if a certain fact is true, must another fact be true?)

5.8 Mean-Value Theorems for Integrals; Average Values

- **Average Value of a Function:** Define the average value of a function using a definite integral. Draw a picture of this average value and what it represents in terms of areas. Be able to calculate this value in particular examples and use average values to solve various word problems, in particular problems about position, velocity, and acceleration.
- **The Mean-Value Theorem for Integrals:** This theorem basically says that there is some point on the interval where $f(x)$ actually takes on its average value. Be able to explain this and to state it precisely. The proof of this theorem is an application of both the Fundamental Theorem of Calculus and the Very Important Theorem; be able to prove this theorem by applying the Mean-Value Theorem to the function $F(x) = \int_a^b f(t)\,dt$ on the interval $[a, b]$.
- **Other Stuff in this Section:** You are not responsible for any material in this section after and including Theorem 5.8.3.