Chapter 6 Outline

Math 235 Fall 2000

6.1 More on Area

- **The Limit of a Riemann Sum:** Riemann sums are obtained by partitioning the interval $[a, b]$, choosing values x_i^* whose values in $f(x)$ determine the heights of the rectangles to be used, adding up the areas of these rectangles, and then taking the limit as the size of the partition goes to infinity. Visually, as we take the limit $|P| \to \infty$, the sum becomes an integral (from $x_0 = a$ to $x_n = b$), the heights of the rectangles become the function to be integrated, and the widths Δx_i become the dx in the integral notation. Be sure that you understand the process by which the sum becomes an integral as $|P| \to \infty$.

- **Representative Rectangles:** Since the area of each of these rectangles is given by a function of the same form (namely, $f(x_i^*)\Delta x_i$), it is enough to look at just *one* of the rectangles (a "representative" rectangle) to obtain the general area formula that will apply for each rectangle. Be sure that you can write out a Riemann sum (and the corresponding definite integral) using a representative rectangle. These techniques can also be used to find the area between two functions. What is the representative rectangle in this case?

- **Integration with Respect to y:** The entire process outlined above can be done with respect to the y rather than the x variable. Be sure that you can do this, complete with notation and pictures. Note that one tricky part of doing this is that the functions are of the form $x = F(y)$ rather than $y = f(x)$ in this case.

6.2 Volume by Parallel Cross Sections; Disks and Washers

- **Volumes of Solids with Homogeneous Cross-Sections:** If a solid has the same cross-section at each height, its volume can be expressed as the product of the area of that cross-section and the height of the solid. For example, volumes of cylinders and cubes can be calculated in this way.

- **Volumes of Solids with Varying Cross-Sections:** If a solid has a varying cross-section, it is sometimes possible to calculate the volume of that solid using a definite integral. If the area of the cross-sections can be written as a function $A(x)$ for each value of x (or a function $A(y)$ for each height y), then the volume of the solid is the definite integral of this function $A(x)$ (or $A(y)$) across the appropriate interval. Why? Write down the reason for this amazing fact by writing a Riemann sum. Note that in this case we will have a representative slice rather than a representative rectangle. These slices are often cylinders but could also be fat triangles, squares, or washers.

- **Volumes Where the Cross-Sections are Discs:** If the cross-section of the solid in question are discs, then the definite integral that will calculate the volume of the solid will be the integral of the function $\pi(f(x))^2$. Why? Write down the representative slice and the corresponding Riemann sum to prove this fact (what is $A(x)$ in this case?). Why will this *always* happen when the solid is obtained by revolving a curve around an axis? Do this also for y-slices.

- **Volumes Where the Cross-Sections are Washers:** Another popular cross-section is washer-shaped. In this case we also get a specific formula for the volume, namely the definite integral of the function $\pi((f(x))^2 - (g(x))^2)$. Give a Riemann sum argument involving a representative slice to show that this integral will calculate the volume of the solid. Also give an argument based on the difference between volumes of the "outside" solid and the "inside" solid in such a case (draw a picture where the slices will be washers to see what I mean here). Do this also for y-slices.

- **Calculating Volumes by Parallel Cross-Section:** Be sure you can quickly calculate volumes such as are described above. Note that this will involve graphing and algebra skills as well as anti-differentiation and u-substitution skills. Know how to calculate exact volumes using defininte integrals obtained as above, as well as how to *approximate* these volumes using a given partition and the volumes of the slices on each subinterval.

6.3 Volume by the Shell Method

- **Shells:** Another way to approximate volumes is by adding up the volumes of *shells*. A "shell" is a cylinder with a smaller cylinder taken out of the center. Explain why the volume of such a shell is given by a formula of the form $\pi R^2 h - \pi r^2 h$. Explain why the formula $2\pi r(R - r)h$, while not the exact volume of the shell, will be very close to the volume of the shell when the outside radius R is very close to the inside radius r.

- **Adding Up Shell Volumes:** Be able to sketch a picture and explain how shells can be used to approximate the volume of some solids. Draw a representative shell for a given x or y value and find the formula for its volume. Factor and rewrite this volume so that $\Delta x = x_i - x_{i-1}$ appears, and apply the approximation $x_i + x_{i-1} \approx 2x_i^*$ (why is this not a bad approximation when each Δx_i is very small?). Note that the result is the "inaccurate" volume formula we discussed above. Add up the volumes of these shells to obtain a Riemann sum (why is it a Riemann sum?). What is the integral associated to this Riemann sum?

- **Strange Formula for Shells:** Why are we using the "inaccurate" volume formula here? (Would it be a Riemann sum if we used the true volume formula? Why not?) Is there a case where this "inaccurate" formula is actually accurate? (Hint: write out what happens if x_i^* is chosen to be the midpoint of the interval $[x_{i-1}, x_i]$.) If you were to do an actual approximation (instead of taking the limit) you could intead use the "real" volume formula for a shell. Be sure you can do this too.

- **Formulas for Volumes by Shells:** There are various formulas in the text for definite integrals that will calculate the volumes of various surfaces of revolutions. While memorizing these formulas can be useful, I am more interested in seeing that you understand the process outlined above for each example. In each case (whether one or more functions are involved in the solid of revolution) you should be able to find a a representative shell and calculate its (approximate) volume, then obtain the appropriate definite integral. Note that the volumes of many solids can be obtained either by shells or by slices, with respect to either the x or the y variable.