General Advice About Proofs

Math 235 Fall 2000

This is the first of a series of handouts that will explain and give examples of various methods of mathematical proof. I'm providing these handouts so you know what I expect of you when I ask you to "prove" something. Think of the example proofs in this series of handouts as templates for "good" proofs; the proofs you write for homeworks, quizzes, and tests should be as clear and as precise as these.

A mathematical proof is a logical argument. Given a set of hypotheses or assumptions, a proof is a series of connected logical statements based on the hypotheses (and perhaps other definitions and theorems that have been proved already) that ends in a conclusion. There are many methods of proof; a simple and straightforward example is given below.

Whatever the method, proofs should be written with great attention to precision and clarity. A good way to start out any proof is to state any hypotheses (things you are assuming to be true; for example you might have the hypothesis that a function f is continuous). Then state what you are trying to prove. Now you're ready to start the actual proof, so write "Proof" so the reader knows that your argument is about to start. If appropriate you can state what method of proof you intend to use (*e.g.* by contradiction, or by induction, *et cetera*) or give a brief outline of your argument. The body of your proof should consist of a clear, concise, logical argument; don't try to make it sound fancy. Make sure to back up any statements that you make using definitions, facts, and theorems that have been proved previously. Use words and phrases like: therefore, then, thus, if ... then ..., since, and because; a person reading your proof needs these words to follow your argument. The end of your proof should clearly state what you have proved.

Don't try write a proof from beginning to end without doing some draft work and example computations first. Convince yourself that you *believe* what you are trying to prove. Then ask yourself *why* you believe it! Often the intuitive reasons you come up with can become the backbone of a proof.

As a simple example we will prove the following statement:

Any real number divisible by 10 is even.

You might start by trying a few examples: 100 is divisible by 10, and indeed it is even; 20 is divisible by 10 and indeed it is even, *etc.* After a while you might be reasonably convinced that the statement is true; but you've been asked to prove it is true for *all* numbers, and no matter how many examples you write down you'll never be able to show that *every* number divisible by 10 is even.

So think: why is this happening? Why is it that being divisible by 10 means that a number must be even? One answer is that if a number is divisible by 10, then it must also be divisible by 2; how can we show that using facts and theorems about numbers that we already know? The proof on the back of the page is one example (there are other ways to do this).

Given that: x is a real number, and x is divisible by 10.

Show that: x is even.

Proof: We will show that if x is divisible by 10, then x must also be divisible by 2 (and thus by definition, x must be even).

If x is divisible by 10, then $x/10$ is an integer (this is what it means for x to be divisible by 10).

Since $x/10$ is an integer, $5(x/10)$ must also be an integer (since the product of any two integers is an integer).

Therefore, $5(x/10) = x/2$ must be an integer; in other words, x must be divisible by 2. Hence x is even. ∎

The box at the end of the proof indicates that the proof has come to an end. Words like "hence", "then", and "since" appear often and help provide the structure of the argument. Note that the proof above contains full sentences; this is not necessary in a proof. You could also write the proof above in the following way:

Given that: $x \in \mathbb{R}$, and $10|x$.

Show that: x is even.

Proof:

$$\begin{array}{rcll} 10|x & \Rightarrow & x/10 \in \mathbb{Z} & \text{(by definition of divisibility)} \\ & \Rightarrow & 5(x/10) \in \mathbb{Z} & \text{(the product of two integers is an integer)} \\ & \Rightarrow & x/2 \in \mathbb{Z} & (5(x/10) = x/2) \\ & \Rightarrow & x \text{ is even} & \text{(definition of even)} \end{array}$$ ∎

Even though there are no sentences in this proof, the $\Rightarrow$ symbols (note $A \Rightarrow B$ means "A implies B", or "if A, then B") provide structure. Reasons for each step are provided so that the implication is clear. However you structure your proof, write only enough words and symbols so that your argument is clear. *Never* add fluff of filler to your proofs; every line in a proof should be completely necessary.

Note that definitions and facts or theorems are used in this proof. For example, the definitions of *divisibility* and *even* are used here. We also use the fact that the product of two integers is an integer (we can accept this fact as an axiom of the integers or treat it as a theorem that has already been proved).

The two most important things to remember are: (1) a proof is a logical argument, and must be clearly structured as such; and (2) your proof should be easily readable to any person with as much mathematical background as you.