MONOMIAL GENERATORS FOR THE NASH SHEAF OF A COMPLETE RESOLUTION

by

Laura Anne Taalman

Department of Mathematics
Duke University

Date: ______________________________

Approved:

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William L. Pardon, Supervisor

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Mark A. Stern

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Chadmark Schoen

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Richard M. Hain

Dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the Department of Mathematics in the Graduate School of Duke University

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ABSTRACT

(Mathematics - Algebraic Geometry)

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Abstract

Given a complex algebraic 3-dimensional variety with isolated singular point, we show that there exists a resolution over which the generators of the Nash sheaf can be written locally as the differentials of certain monomial functions. The exponents of these monomials define a sequence of three divisors (the "resolution data") supported on the exceptional divisor of the resolution. Using these divisors we construct an exact sequence that relates the Nash sheaf to the resolution data.

We show in fact that these results hold for any resolution factoring through the Nash blowup and the blowup of the maximal ideal sheaf over which a certain Fitting ideal is locally principal (we call such resolutions "complete"). The existence of a resolution satisfying the third condition is nontrivial and proved here using a theorem of Hironaka's.

Given a complete resolution and a point in the exceptional divisor, the monomial functions whose differentials locally generate the Nash sheaf are found by extracting distinguished monomial components of so-called "Hsiang-Pati" coordi-

nates on the resolution. The Hsiang-Pati coordinates are constructed by careful choice of linear functionals that satisfy various minimality conditions involving the Nash sheaf and its exterior powers, and the monomial functions are defined as the distinguished monomial parts of these coordinates.

To construct the exact sequence relating the Nash sheaf to the resolution data we require a hyperplane section whose proper transform in (a sufficiently finer resolution) meets the exceptional divisor transversely away from triple points. The existence of such a hyperplane and further resolution is obtained through a second theorem of Hironaka's.

As an application, we use this exact sequence to calculate certain Chern classes of the Nash sheaf, and thus Mather-Chern classes of the variety. Finally, we conjecture and show evidence for general formulae for these Chern classes involving only sheaves of logarithmic forms and the resolution data. These formulae, as well as the results described above, have an obvious generalization to the n-dimensional case; this more general case will appear in a later paper.

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Contents

Chapter 1

Introduction

1.1 Main Results

In this chapter we give a general overview of the main results, their foundations, and their significance. General definitions and results are presented in Chapter 2 and we assume them here.

Given a complex algebraic 3-dimensional variety V with isolated singular point, we show in our Main Proposition that there exists a resolution over which the generators of the Nash sheaf can be written locally as the differentials of certain monomial functions. We state a brief version of the Main Proposition (in the triple point case) here; see Main Propositions 5.2.1, 5.5.1, and 5.6.1 for the full statements.

Main Proposition 1.1.1. *There exists a resolution of V for which we can find "monomial" generators for the Nash sheaf, i.e. generators that can be locally written in the form $d\phi$, $d\psi$, $d\rho$ for some functions ϕ ψ, ρ that are of the form*

$$\begin{aligned} \phi &= u^{m_i} v^{m_j} w^{m_k} \\ \psi &= u^{n_i} v^{n_j} w^{n_k} \\ \rho &= u^{p_i} v^{p_j} w^{p_k}, \end{aligned}$$

where all the exponents are positive and satisfy various ordering and linear independence conditions.

The exponents m_i, m_j, m_k define a sequence of three divisors (the "resolution data") supported on the exceptional divisor E of the resolution. Using these divisors and a generic hypersurface (in a possibly finer resolution), we construct an exact sequence that relates the Nash sheaf to the resolution data.

As an application, we use this exact sequence to calculate certain Chern classes of the Nash sheaf, and thus Mather-Chern classes of the variety. Using MacPherson-Chern classes and Gonzalez-Sprinberg's characterization of the local Euler obstruction we can obtain more information regarding the Chern classes for the Nash sheaf and the Euler characteristic of the exceptional divisor.

Finally, we conjecture and show evidence for general formulae for these Chern classes involving only sheaves of logarithmic forms and the resolution data. These formulae, as well as the results described above, have an obvious generalization to the n-dimensional case; this more general case will appear in a later paper.

In fact, the results above hold for any resolution $\widetilde{V}$ of V factoring through the Nash blowup and the blowup of the maximal ideal sheaf over which a certain Fitting ideal is locally principal (we call such resolutions "complete"). In Chapter 3 we show that such a complete resolution always exists. The existence of a resolution satisfying the third condition is nontrivial and proved here using a theorem from Hironaka's paper [Hir64a].

Given a complete resolution and a point in the exceptional divisor, the monomial functions ϕ, ψ, and ρ (whose differentials locally generate the Nash sheaf) are found by extracting distinguished monomial components of so-called "Hsiang-Pati" coordinates on the resolution $\widetilde{V}$. In fact, the Main Propositions are simple corollaries to the more technical Propositions 5.2.2, 5.5.2, and 5.6.2, which explicitly define and prove the existence of Hsiang-Pati coordinates in an analytic neighborhood of any point in the exceptional divisor (in the associated analytic variety).

Hsiang and Pati showed in [HP85] that, in the case where V is an analytic surface with isolated singular point, these coordinates can be obtained by repeatedly blowing up V and taking appropriate changes of coordinates as necessary. Pati generalizes this result to the 3-dimensional case in [Pat94]. In these papers, however, it is neither clear what resolution $\widetilde{V}$ will eventually be sufficient for these coordinates to be in the correct form, nor how these coordinates relate to the Nash sheaf.

Pardon and Stern give a more conceptual, geometric view of this process (in the 2-dimensional case) in Chapter 3 of [PS97], showing that instead of repeated blowups we can take any resolution that factors through the Nash blowup and the blowup of the maximal ideal, as long as we make a careful choice of linear functions that satisfy certain conditions involving the maximal ideal and the Nash sheaf. The Hsiang-Pati coordinates obtained in this manner then induce monomial generators for the Nash sheaf, which in turn define a sequence of divisors supported on the exceptional divisor of the resolution. Pardon and Stern then use these divisors (and the existence of a certain generic hyperplane H) to obtain an exact sequence expressing the Nash sheaf in terms of the resolution data.

The results here are thus in part a generalization of the results of Pardon and Stern to the 3-dimensional case. In other words, we provide a more conceptual view of Pati's 3-dimensional results (a view which in addition seems to have a clear generalization to the n-dimensional case). In the three-dimensional case the resolution $\widetilde{V}$ must satisfy the same conditions as in Pardon and Stern's two-dimensional case, with the additional property that a particular Fitting ideal must be locally principal over $\widetilde{V}$. The construction of this complete resolution (in particular, a resolution with the last condition) involves a technical argument using [Hir64a] and [Hir64b] and is handled in Chapter 3. The Hsiang-Pati coordinates are then constructed by careful choice of linear functionals that satisfy various minimality conditions involving the Nash sheaf and its exterior powers, and the monomial functions ϕ, ψ, and ρ are defined as the distinguished monomial parts of these coordinates.

To construct the exact sequence relating the Nash sheaf to the resolution data described above we require a hyperplane section H of V whose proper transform in (a sufficiently finer resolution) $\widetilde{V}$ meets the exceptional divisor E transversely away from triple points. The existence of such a hyperplane H and further resolution $\widetilde{V}$ is obtained through a second application of Hironaka's paper [Hir64a] (while in the 2-dimensional case from [PS97] an argument similar to Gonzalez-Sprinberg's results in [GS82] was sufficient).

1.2 Preliminaries

In this section we discuss the conditions (described in detail in Chapter 4) on the resolution $\widetilde{U}$ and the choice of linear functions j, k, and l (in the 3-dimensional case; clearly in the n-dimensional case we will have to make a corresponding choice of n linear functions) that make the Main Proposition possible. Note that in [HP85] the resolution is obtained by repeated blowings-up as become necessary in the process of constructing the Hsiang-Pati coordinates. Here (as in [PS97]) we choose *from the outset* a resolution $\widetilde{U}$ that is already "sufficiently fine". We must then choose our linear functions carefully (in [HP85] they are simply chosen to be the coordinate functions on $\mathbb{C}^n \supset U$) so that they will work in conjunction with this resolution $\widetilde{U}$.

1.2.1 Complete resolutions

A *complete resolution* $\pi\colon \widetilde{U} \to U$ of the singularity $v \in U$ is defined to be a resolution factoring through the Nash blowup and the blowup of the maximal ideal corresponding to the singular point v over which a certain Fitting ideal is locally principal (see Definition 3.1.1). Such a resolution will prove to be sufficiently fine for the construction of Hsiang-Pati coordinates (so no further blowups will be necessary during the construction of the coordinates). Note that in the 2-dimensional case (in [PS97]) it was sufficient to choose the resolution $\widetilde{U}$ so that it factored through the Nash blowup and the blowup of the maximal ideal (*i.e.* no Fitting ideal condition was necessary).

In the 3-dimensional case, a complete resolution satisfies the property that the second Fitting ideal $Fitt_2$ for the inclusion of the Nash sheaf into the sheaf of logarithmic 1-forms (see Section 2.3.4) is locally principal. Obtaining a resolution with this property (see Chapter 3) is a three-step process. First we show that, given a resolution factoring through the Nash blowup and the blowup of the maximal ideal, the Fitting ideal $Fitt_2$ is locally principal at simple points of the exceptional divisor (see Section 3.2). Second, we show that certain further blowups of this resolution are "$Fitt_2$-preserving" in the sense that the inverse image of the Fitting ideal from downstairs is isomorphic to the Fitting ideal in the new resolution (see Section 3.3). Finally, we obtain a complete resolution by a preliminary blowup followed by a sequence of $Fitt_2$-preserving blowups that makes the inverse image of the Fitting ideal locally principal; clearly with such a resolution the new Fitting ideal on the final resolution will also be locally principal. The existence of the sequence of $Fitt_2$-preserving blowups that frees the Fitting ideal is a consequence of a theorem from [Hir64a].

In proving the Main Propositions we will use the Fitting ideal condition to write various wedge products of the generators for the Nash sheaf in a standard form, from which we will be able to define the multiplicities of the Hsiang-Pati coordinates. In the triple point case (*i.e.* at a triple point of the exceptional divisor E where three components of E meet at a point), the condition that the Fitting ideal $Fitt_2$ is locally principal enables us to prove the key Fact 5.3.1, which says that we can write a certain wedge product (namely, the minimal generator of the second exterior power of the Nash sheaf as discussed in the following section) in a standard form. This fact will, in the proof of Proposition 5.2.2 in Section 5.4, be the key that enables us to define the multiplicities n_i that are necessary in the construction of Hsiang-Pati coordinates. Similar procedures are used in the double and simple point cases.

1.2.2 Minimality conditions (the "Flag" Proposition)

Given a complete resolution $\pi\colon (\widetilde{U}, E) \to (U, v)$, consider the associated analytic space $\widetilde{U}^h$ (which we will also denote by $\widetilde{U}$); see Section 2.4. Let $e \in E$ be a point of the exceptional divisor, and choose an analytic neighborhood W of e in $\widetilde{U}$. We will make a careful choice of linear functions j, k, and l which by the Main Proposition 1.1.1 will induce Hsiang-Pati coordinates, and thus monomial generators for the Nash sheaf, when pulled up to $W \subset \widetilde{U}$. Throughout this section we will assume that we have chosen an $e \in E$ as above and that we are working locally in the analytic neighborhood $W \subset \widetilde{U}$.

A set $\{j, k, l\}$ of linear functions that satisfies the conditions that we desire will be called "Nash-minimal" (see Definition 4.1.2). Define $\phi := j \circ \pi$, $\psi := k \circ \pi$, and $\rho := l \circ \pi$. Nash-minimality of $\{j, k, l\}$ means that ϕ, ψ, and ρ satisfy various minimality and generating conditions involving the Nash sheaf, the exterior powers of the Nash sheaf, and the pullback of the maximal ideal sheaf over $W \subset \widetilde{U}$. These conditions will be used extensively in our proof of the Main Proposition. When $\{j, k, l\}$ is Nash-minimal we will also say that $\{\phi, \psi, \rho\}$ is Nash-minimal. That such Nash-minimal functions exist (and in fact are generic) is the main consequence of the "Flag Proposition" (Proposition 4.1.1).

The Flag Proposition basically states that we can find a flag of planes $D_3 \subset D_2 \subset D_1$ (the subscript indicates codimension in $\mathbb{C}^N$) where each D_i is defined as the kernel of an i-tuple of linear functions, and where these linear functions satisfy various minimality and generating conditions when pulled up to W. These conditions involve, as above, the Nash sheaf, its exterior powers, and the maximal ideal sheaf. In fact, the linear functions in the Flag Proposition are chosen so that they induce trivializations (as described in Section 4.2) of these sheaves near e (see the proof of Proposition 4.1.1 in Section 4.3). These linear functions are then "jiggled" in such a way that we get the desired flag of planes. The fact that we can choose these linear functions so that they define such a flag means that we can actually choose the linear function defining D_1 as one of the linear functions defining D_2, and that we can choose the linear functions defining D_2 to be two of the linear functions that define D_3. The three linear functions defining D_3 in this manner are then by definition Nash-minimal; we take these linear functions to be the j, k, and l referred to above.

The proof of the Main Proposition now uses the completeness of the resolution and the Nash-minimality of the linear functions j, k, and l to show that ϕ, ψ, and ρ are Hsiang-Pati coordinates on W. See the proof of Proposition 5.2.2 in Section 5.4 where this is done carefully.

1.3 Consequences

The first consequence of the Main Proposition 1.1.1 is that the exponents m_i, n_i, and p_i of the Hsiang-Pati coordinates give rise to a sequence of divisors supported on E, denoted Z, N, and P, respectively. We refer to these divisors (and the corresponding multiplicities) as "resolution data" (because they are invariants of the resolution) or, in the case of N and P, "higher multiplicities" (since the n_i and p_i are always greater then the m_i). We can use this resolution data to examine the Nash sheaf.

In Section 1.3.2 below we describe an exact sequence of sheaves over $\widetilde{U}$ (constructed in Chapter 8) that relates the Nash sheaf to the resolution data. The sequence is proved to be exact (in Proposition 8.1.1) by proving that the associated analytic sequence is exact and then applying some theorems from [Ser56] (see Theorems 2.4.4 and 2.4.5 and Proposition 2.4.6). We will then use this sequence to determine some of the Chern classes $c_i(\mathcal{N})$ of the Nash sheaf (these are cohomology classes in the resolution $\widetilde{U}$); in low dimensions we can use other results to find the remaining Chern classes (see Section 1.3.3 below for a brief description of how we will do this in Chapter 9).

From the invariants of the resolution described above we can obtain various numerical invariants of the singularity $v \in U$. For example, the Chern number obtained by pushing down (to U) the cup product of $c_2(\mathcal{N})$ and the cohomology class E defined by the exceptional divisor is such an invariant. This number does not depend on the resolution $\widetilde{U}$ (in a similar way, the Mather-Chern class of a variety does not depend on the choice of generalized Nash bundle; see Section 2.6.2).

In the first of the following sections we describe how a generic hyperplane can help us choose some of the linear functions that will induce the Hsiang-Pati coordinates (and thus the local monomial generators of the Nash sheaf). This hyperplane will enable us to construct an exact sequence involving the Nash sheaf and the resolution data (the maps in the sequence will be defined using the linear function that defines this hyperplane; see Section 1.3.2 below). The exact sequence will then enable us to compute some of the Chern classes of the Nash sheaf, which we can use to construct numerical invariants of the singularity v (as briefly described in Section 1.3.3 below).

1.3.1 The generic hyperplane (the "Divisor" Proposition)

Given a complete resolution $\pi\colon (\widetilde{U}, E) \to (U, v)$, an embedding $(U, v) \hookrightarrow (\mathbb{C}^N, 0)$, and a sufficiently "nice" (see Definition 6.1.1) hyperplane $H \subset \mathbb{C}^N$ passing through 0, let $\widetilde{H}$ denote the proper transform of $H \cap U$ in $\widetilde{U}$. We would like to be able to say that $E \cup \widetilde{H}$ is a divisor with normal crossings in $\widetilde{U}$, but this is not the case. However, we can find a finer resolution over which this is true. We prove this is possible (using a theorem of Hironaka's) in Lemma 6.2.1.

Now assume that our complete resolution is fine enough so that $\widetilde{H} \cup E$ is a divisor with normal crossings. The first part of the "Divisor Proposition" (Proposition 6.3.1 in the 3-dimensional case) states that $\mathrm{div}(h \circ \pi) = Z + \widetilde{H}$. (This is a consequence not only of the lemma described above, but also of the fact that H is "nice"; one of the conditions of being "nice" is that the total transform of H in $\widetilde{U}$ must vanish to the minimum order, *i.e.* that of Z, along E.) The Divisor Proposition then states that, for analytic neighborhoods W of points $e \in E$ (in the associated analytic variety $\widetilde{U}^h$) that are not contained in $\widetilde{H}$, we can choose the linear function j to be the function h defining H. In analytic neighborhoods of points e that are contained in $\widetilde{H}$ (it will turn out that this only happens at simple points $e \in E_i$; see Proposition 6.3.1), we will have $m_i = n_i = p_i$, and we can take the linear function k to be h.

Now that we have established the existence and nice properties of such an H, we will use it to define an exact sequence of sheaves on $\widetilde{U}$.

1.3.2 An exact sequence for the Nash sheaf

As mentioned above, Pardon and Stern (in Proposition 3.20 of [PS97]) construct an exact sequence of sheaves over $\widetilde{U}$ that expresses the Nash sheaf $\mathcal{N}_{\widetilde{U}}$ in terms of the resolution data. The 3-dimensional generalization of this sequence (that we prove in Proposition 8.1.1 in the analytic category) is:

$$\begin{aligned} 0 \;\to\; \mathcal{N}(Z-E) \;&\overset{\alpha}{\hookrightarrow}\; \mathcal{I}_E\Omega^1(\log E) \\ &\overset{\beta}{\longrightarrow}\; \left(\Omega^2(\log E)/\textstyle\bigwedge^2\mathcal{N}(2Z)\right)\otimes\mathcal{O}(-Z-E) \\ &\overset{\gamma}{\twoheadrightarrow}\; \Omega^3\otimes\mathcal{O}_{P+N-2Z}(-2Z) \;\to\; 0. \end{aligned}$$

This sequence relates, but does not completely describe, the Nash sheaf in terms of Z, N, and P (the problem is that the sequence involves the second exterior power of the Nash sheaf and is thus self-referential regarding the Nash sheaf). The second and third maps in this sequence are defined to be the map $(\_\_ \wedge \frac{dh}{h})$.

After proving that this sequence is well-defined and exact (in Section 8.2), we put it in a more general context by defining the "weighted Nash" complex and the "weighted log forms" complex over $\widetilde{U}$ (see Section 8.3).

1.3.3 Chern classes and numbers

We can use the exact sequence discussed in the previous section, and various easy formulas involving the Chern classes of tensor products, exterior powers, and torsion sheaves, to calculate the first and last Chern classes of the Nash sheaf over $\widetilde{U}$ (although the first Chern class is also known by other, quite simple, means).

In the 2-dimensional case this will give us all the Chern classes, but in the 3-dimensional case we will be missing the second Chern class $c_2(\mathcal{N}_{\widetilde{U}})$ (and in the n-dimensional case we will of course be missing even more information). Luckily, by combining two formulas for the zeroth MacPherson-Chern class (see Claims 2.6.8 and 2.6.14) of U, and applying Gonzalez-Sprinberg's formula for the local Euler characteristic (from [GS81]; see Claim 2.6.12), we can *almost* solve for this "missing" second Chern class. More precisely, we can solve for the cup product $c_2(\mathcal{N}_{\widetilde{U}})\,E$. In the 2-dimensional case a similar process will result in formulas for the local Euler obstruction of U at v and the Euler characteristic of E. Finally, we conjecture a formula for the Chern classes of the Nash sheaf (involving, of course, the resolution data), show that there is evidence for its being true, and discuss some consequences.

1.3.4 Examples

In Chapter 10 we examine in detail two examples. First, we investigate all patches in a complete resolution of a simple example; we find Nash-minimal linear functions that pull up to Hsiang-Pati coordinates, and thus induce monomial generators, on the resolution. From the multiplicities defined by those monomials, and a generic hyperplane, we can examine the exact sequence of sheaves described in Section 1.3.3 above. From this we can obtain formulas for the first and last Chern classes of the Nash sheaf. In this simple example we can then verify the conjecture for the second Chern class (see Conjecture 9.2.7).

We then examine a non-trivial example; in this example we blow up the maximal ideal and the sheaf of 1-forms, desingularize, and note that the Fitting ideal is locally principal. Once we obtain this complete resolution (in one patch) we find monomial generators for the Nash sheaf in an analytic neighborhood of a point in the exceptional divisor by finding Hsiang-Pati coordinates (which themselves turn out to monomial in this case). The example examined here is a "Case II" double point example; see Main Proposition 5.5.1 and Proposition 5.5.2.

1.3.5 Applications in other fields

The n-dimensional generalization of Hsiang-Pati coordinates presented here may have applications in other fields, in particular the Hodge structure on L^2-cohomology groups (as in [PS97]) and the finiteness of the heat trace (as in [Pat93]). In this section we briefly describe results in these fields that (in the low-dimensional cases where Hsiang-Pati coordinates were already constructed) used Hsiang-Pati coordinates on a resolution of a surface or three-fold with isolated singular point.

Pati uses Hsiang-Pati coordinates in the 3-dimensional case to prove that the trace of the heat operator $\bar{e}^{t\bar{\Delta}}$ is finite (and in fact satisfies a certain bound; see Theorem 1.1 in [Pat93]), where $\bar{\Delta} = \bar{\delta} d$ is the Laplacian acting on L^2 functions with respect to the induced Fubini-Study metric on the smooth part of U. It is worth noting, however, that this theorem has since been proved in greater generality, by different methods, in [LT95].

As discussed above (in Section 1.1), Pardon and Stern describe a conceptualization of Hsiang-Pati coordinates in the 2-dimensional case in Chapter 3 of their paper [PS97]. In Chapter 4 of this paper they use these Hsiang-Pati coordinates to prove two theorems about L^2-cohomology. Their Theorem 4.9 describes the cohomological Hodge structure on the L^2-cohomology of an algebraic surface V in terms of local cohomology groups obtained from a resolution $\widetilde{V}$ of V. Theorem 4.22 in [PS97] uses Hsiang-Pati coordinates (and the resulting expression of the Nash sheaf in terms of the resolution data and the "degeneracy divisor") to compute the $L^2 - \bar{\partial}$ cohomology groups of such a V. Pardon also uses Hsiang-Pati coordinates to examine $L^2 - \bar{\partial}$ cohomology groups of such a V in his paper [Par89].

1.4 Structural Overview

To finish this introductory chapter, we present a quick overview of the structure of the thesis as a whole.

Let V be a 3-dimensional variety with isolated singular point v and neighborhood U of v in V chosen small enough so that we have an embedding $(U, v) \hookrightarrow (\mathbb{C}^N)$. Construct a complete resolution $\pi\colon (\widetilde{U}, E) \to (U, v)$ (see Definition 3.1.1) by first obtaining a resolution that factors through the Nash blowup and the blowup of the maximal ideal sheaf $\mathfrak{m}_v$, and then constructing a further resolution over which the Fitting ideal $Fitt_2$ is locally principal. That such a further resolution can be constructed will be a consequence of theorems from [Hir64a] (see Chapter 3).

Given the complete resolution $\pi\colon (\widetilde{U}, E) \to (U, v)$, pass to the associated analytic space $\widetilde{U}^h$ (which we also denote by $\widetilde{U}$). Choose $e \in E$ in the exceptional divisor with analytic neighborhood $W \subset \widetilde{U}$.

By the Flag Proposition (Proposition 4.1.1) we can choose Nash-minimal linear functions $\{j, k, l\}$ on $\mathbb{C}^N$ (Definition 4.1.2). By the Main Proposition (Proposition 5.2.2), the resulting $\{\phi, \psi, \rho\}$ are Hsiang-Pati coordinates on W (we will assume ψ, ρ are in the form ψ', ρ'). Thus $\{d\phi, d\psi, d\rho\}$ is a set of local monomial generators for the Nash sheaf over W (Corollary 5.2.3), and we have a sequence of divisors Z, N, and P supported on E.

Switch gears for a moment, and given our complete resolution $\widetilde{U}$ (in the algebraic category), choose a generically "nice" H (Definition 6.1.1). By Hironaka there exists a finer resolution (which we also call $\widetilde{U}$) over which $\widetilde{H} \cup E$ is a divisor with normal crossings (Lemma 6.2.1). Passing again to the analytic category, the Divisor Proposition then says that we have $\operatorname{div}(h \circ \pi) = Z + \widetilde{H}$, and that we can use H to choose either j or k in our choice of Nash-minimal linear functions above (Proposition 6.3.1).

Combining the results in the two paragraphs above (still in the analytic category), we construct an exact sequence of sheaves on $\widetilde{U}$ that relates the Nash sheaf to the resolution data (Proposition 8.1.1). The associated algebraic sequence is exact by theorems from Section 2.4. This sequence allows us to compute some of the Chern classes of the Nash sheaf; we can obtain additional Chern classes through other arguments. These Chern classes (together with the resolution data) determine numerical invariants of the singularity (Chapter 9). Finally, we conjecture a general formula (Conjecture 9.2.7) for the Chern classes of the Nash sheaf.

In Chapter 10 we describe two examples, the simplest example of a cone (where we verify the conjecture discussed above), and a non-trivial example (in the double point case).

Chapter 2

Background

We begin this chapter with a discussion of basic objects and constructions: blowups of points, ideals, and sheaves; the Nash blowup, bundle, and sheaf; and Fitting ideals and invariants for modules and sheaves. We then discuss previous results for Hsiang-Pati coordinates (from [HP85], [Pat94], and [PS97]), and how they relate to the results proved here. Finally we discuss some basic results involving Chern classes for singular varieties.

2.1 Blowing Up

Here we define and construct the blowup of a point, extend this construction to the blowup of an ideal, and then further extend the concept to the blowup of a coherent sheaf. We present these constructions in such a way that each is clearly a generalization of the construction before it.

2.1.1 Blowing up a point

We first define the blowup of $\mathbb{C}^N$ at the origin. Let $\{z_1, ..., z_N\}$ be coordinates for $\mathbb{C}^N$ centered at 0. Let $\mathbb{P}^{N-1}$ denote projective space with projective coordinates $\{y_1, ..., y_N\}$. Think of

$$\pi : \mathbb{C}^N \times \mathbb{P}^{N-1} \longrightarrow \mathbb{C}^N$$

as a trivial bundle over $\mathbb{C}^N$, and consider the section

$$\begin{aligned} \sigma : \mathbb{C}^N - \{0\} &\longrightarrow \mathbb{C}^N \times \mathbb{P}^{N-1} \\ z &\longmapsto ((z_1, \ldots, z_N), [z_1, \ldots, z_N]). \end{aligned}$$

over $\mathbb{C}^N - \{0\}$. We define the blowup of $\mathbb{C}^N$ to be the closure (in $\mathbb{C}^N \times \mathbb{P}^{N-1}$) of the image of this section:

Definition 2.1.1. *With notation as above, the* blowup of $\mathbb{C}^N$ at the origin *is defined to be*

$$Bl_0(\mathbb{C}^N) := \mathrm{cl}(\sigma(\mathbb{C}^{\mathrm{N}} - \{0\})).$$

(This is not the standard definition of $Bl_0(\mathbb{C}^N)$, but it is equivalent, and this formulation will provide an obvious generalization in Section 2.1.2 to the blowup of an ideal.)

By definition $Bl_0(\mathbb{C}^N)$ comes equipped with maps

$$\pi\colon Bl_0(\mathbb{C}^N) \longrightarrow \mathbb{C}^N \qquad \text{and} \qquad p\colon Bl_0(\mathbb{C}^N) \longrightarrow \mathbb{P}^{N-1}$$

induced by the restriction of the first projection π and the second projection (to $\mathbb{P}^{N-1}$) of $\mathbb{C}^N \times \mathbb{P}^{N-1}$ to $Bl_0(\mathbb{C}^N)$.

We can also think of $Bl_0(\mathbb{C}^N)$ as the variety in $\mathbb{C}^N \times \mathbb{P}^{N-1}$ (with coordinates $((z_1, \dots, z_N), [y_1, \dots, y_N]))$ defined by the equations

$$z_i y_j - z_j y_i = 0, \quad 1 \leq i, j \leq n.$$

(This is the definition of $Bl_0(\mathbb{C}^N)$ given in [Griffhar] and [Hartshorne].) The fibre above each point $z \in \mathbb{C}^N - \{0\}$ consists of a single point (namely $\sigma(z)$), while the inverse image of the origin is the entire projective space $\mathbb{P}^{N-1}$ (since all the equations $z_i y_j - z_j y_i$ are satisfied, regardless of $[y_1, \dots, y_N]$, above the origin) We define the *exceptional divisor* E to be the fibre of $Bl_0(\mathbb{C}^N)$ over the origin, *i.e.* $E := \pi^{-1}(0) \approx \mathbb{P}^{N-1}$.

To blow up a variety at a point, we will blow up the ambient affine space and then lift the variety up to the blowup. As above we consider a neighborhood U in V of a point v, and an embedding $(U, v) \hookrightarrow (\mathbb{C}^N, 0)$. Since everything here is local we will work entirely in U (or its image $U := i(U)$ in $\mathbb{C}^N$) and define the blowup of U at the point v.

We define the *total transform* of U in the blowup $\pi\colon Bl_0(\mathbb{C}^N) \to \mathbb{C}^N$ to be the total inverse image $\pi^{-1}(U)$ of U. We define the *proper transform* of U in $Bl_0(\mathbb{C}^N)$

to be the closure of the inverse image of $U - \{0\}$ in $Bl_0(\mathbb{C}^N)$. This in fact is what we will define to be the blowup of U at v:

Definition 2.1.2. *The* blowup of U at the point v *is defined to be the proper transform*

$$Bl_0(U) := \mathrm{cl}(\pi^{-1}(U - 0))$$

of U *in the blowup* $\pi \colon Bl_0(\mathbb{C}^N) \to \mathbb{C}^N$ *of* $\mathbb{C}^n$ *at the origin.*

Clearly we can restrict the map π to $Bl_0(U)$ to get the map

$$\pi \,:\, Bl_0(U) \;\longrightarrow\; U.$$

The exceptional divisor E of $Bl_0(U)$ is defined to be the inverse image of v (*i.e.* of the origin) by the restricted map π. Note that this E is *not* the same as the exceptional divisor associated to the blowup of all of $\mathbb{C}^N$ at the origin (which is the inverse image of the non-restricted map π).

See Section 10.1 for an example of blowing up a variety at its singular point.

2.1.2 Blowing up along an ideal

We now extend the concept of blowing up at a point to that of blowing up along an ideal (sometimes referred to as a "monoidal transform"). We will use a formulation similar to that given in Section 1 of [Nob75]. Let V, U, and v be as in the previous section. Let J be an ideal in the ring of regular functions on $\mathbb{C}^N$, and let $\mathcal{V}(J)$ denote the variety in $\mathbb{C}^N$ defined by J. Suppose J is generated by $\{g_1, \dots, g_s\}$, and consider the section

$$\begin{aligned} \sigma : \mathbb{C}^N - \mathcal{V}(J) &\longrightarrow \mathbb{C}^N \times \mathbb{P}^{s-1} \\ z &\longmapsto ((z_1, \dots, z_N), [g_1(z), \dots, g_s(z)]) \end{aligned}$$

of the trivial bundle $\mathbb{C}^N \times \mathbb{P}^{s-1}$. We define the blowup of $\mathbb{C}^N$ along J to be as follows.

Definition 2.1.3. *The* blowup of $\mathbb{C}^N$ along the ideal J *is defined to be the closure*

$$Bl_J(\mathbb{C}^N) := \mathrm{cl}(\sigma(\mathbb{C}^N - \mathcal{V}(J)))$$

of the image of σ *in* $\mathbb{C}^N \times \mathbb{P}^{s-1}$.

By definition $Bl_J(\mathbb{C}^N) \subset \mathbb{C}^N \times \mathbb{P}^{s-1}$ and thus we have maps

$$\pi : Bl_J(\mathbb{C}^N) \longrightarrow \mathbb{C}^N \quad \text{and} \quad \gamma : Bl_J(\mathbb{C}^N) \longrightarrow \mathbb{P}^{s-1}$$

induced from the restrictions of the first and second projections from $\mathbb{C}^N \times \mathbb{P}^{s-1}$. Note that in the special case where $J = \mathfrak{m}_0 = (z_1, \dots, z_N)$ is the maximal ideal of the point 0, we have $\mathcal{V}(J) = \{0\}$, and this definition coincides with the blowup of $\mathbb{C}^N$ at the origin as in Section 2.1.1.

Now let U be a neighborhood of a point v of a variety V with an embedding $(U, v) \hookrightarrow (\mathbb{C}^N, 0)$. Define the *total transform* of U in $Bl_J(\mathbb{C}^N)$ to be the total inverse image of U under the map π. Define the *proper transform* of U in $Bl_J(\mathbb{C}^N)$

to be the closure of the inverse image of $U - (\mathcal{V}(J) \cap U)$ in $Bl_J(\mathbb{C}^N)$. Define the *exceptional set* E to be the total inverse image of $\mathcal{V}(J) \cap U$ in $Bl_J(U)$. As in the case when we blew up a point, we can now define the blowup of U with respect to J.

Definition 2.1.4. *With notation as above, the blowup of U along the ideal J is defined to be the proper transform*

$$Bl_J(U) := \mathrm{cl}(\pi^{-1}(U - (\mathcal{V}(J) \cap U)))$$

of U in $Bl_J(\mathbb{C}^N)$.

Alternately (as in [Nobile]) we could have taken J to be an ideal in the regular functions on U, and then defined the blowup of U along J as we did above for the blowup of $\mathbb{C}^N$.

2.1.3 Blowing up with respect to a sheaf

We now extend the concept of blowing up even further, and define what it means to blow up with respect to a coherent sheaf $\mathcal{F}$. This definition will be analogous to the above definition for blowing up along an ideal in the special case where the sheaf is a sheaf of ideals.

Let V be a complex algebraic variety, and denote the structure sheaf on V as $\mathcal{O}_V$. Let $\mathcal{F}$ be a coherent sheaf of $\mathcal{O}_V$ -modules that is generically of positive rank r. We will locally define the blowup of V relative to $\mathcal{F}$, and then discuss the properties of such a blowup, following the constructions in Section 1 of [NA83] and Section 2 of [Rie71].

We will make use of the following correspondence between locally free sheaves and vector bundles (as in Appendix B.3 of [Ful80] and Exercise 5.18 in [Har77]). Given a vector bundle $p\colon \xi \to U$ of rank r over U, the sheaf of sections $\mathcal{S}(\xi)$ of ξ forms a rank r locally free sheaf over U. On the other hand, given a locally free sheaf $\mathcal{F}$ of rank r on U, there exists a vector bundle $p : \mathbb{V}(\mathcal{F}) \to U$ corresponding to $\mathcal{F}$; this vector bundle is constructed by defining transition functions from the local sections defining the sheaf $\mathcal{F}$. Under this correspondence we have

$$\mathcal{S}(\mathbb{V}(\mathcal{F})) = \mathcal{F} \quad \text{and} \quad \mathbb{V}(\mathcal{S}(\xi)) = \xi^*.$$

Choose a neighborhood U in V small enough so that we have a surjection

$$\mathcal{O}_U^N \twoheadrightarrow \mathcal{F}|_U$$

(this is possible since $\mathcal{F}$ is a coherent sheaf). By hypothesis there exists an open dense set U^0 in U over which $\mathcal{F}$ is locally free of rank r; over this U^0 we have

$$\mathcal{O}_{U^0}^N \twoheadrightarrow \mathcal{F}|_{U^0}. \tag{2.1.1}$$

This surjection of locally free sheaves corresponds to a surjection of vector bundles

$$\mathcal{E}^N|_{U^0} \twoheadrightarrow \mathbb{V}(\mathcal{F}|_{U^0});$$

where $\mathcal{E}^N|_{U^0}$ denotes the trivial bundle of rank N over U^0; thus over each $x \in U^0$ we have a surjection

$$\mathcal{E}^N_x \twoheadrightarrow \mathbb{V}(\mathcal{F}_{U^0})_x;$$

the kernel of this map defines an $(N-r)$-dimensional space

$$K_x := \ker\{\mathcal{E}^N_x \twoheadrightarrow \mathbb{V}(\mathcal{F}_{U^0})_x\} \subset \mathcal{E}^N_x,$$

i.e. an element of the fibre over x $\mathrm{Gr}(N-r,\, \mathcal{E}^N|x)$ of the Grassmannian bundle $\mathrm{Gr}(N-r,\, \mathcal{E}^N|_{U^0})$ of $(N-r)$-planes in $\mathcal{E}^N$ over U^0. Note that by definition we have

$$\mathcal{E}^N_x \big/ K_x = \mathbb{V}(\mathcal{F}_{U^0})_x \tag{2.1.2}$$

for all $x \in U^0$. Using the definition of K_x above we have a section

$$\begin{aligned} \sigma : U^0 &\longrightarrow \mathrm{Gr}(N-r,\, \mathcal{E}^N|_U) \approx U \times \mathrm{Gr}(N-r, N) \\ x &\longmapsto (x, K_x) \end{aligned}$$

of $\mathrm{Gr}(N-r,\, \mathcal{E}^N|_U)$ over U^0.

Now define the blowup of U along $\mathcal{F}$ to be the closure of the image of this section:

Definition 2.1.5. *With notation as above, the* blowup of U with respect to the sheaf $\mathcal{F}$ *is defined to be the closure*

$$\widetilde{U} := \mathrm{cl}(\sigma(U^0))$$

of the image of σ in $U \times \mathrm{Gr}(N-r, N)$.

As when we blew up points and ideals, the blowup of U with respect to $\mathcal{F}$ is embedded in $U \times \mathrm{Gr}(N-r, N)$ and thus comes equipped with maps

$$\pi \colon \widetilde{U} \longrightarrow U \quad \text{and} \quad \gamma \colon \widetilde{U} \longrightarrow \mathrm{Gr}(N-r, N).$$

Define the *exceptional set* E to be the inverse image of $U - U^0$ in $\widetilde{U}$ by the map π. The *total transform* and *proper transform* of a subset $X \subset U$ are defined in the usual way; the total transform of X in $\widetilde{U}$ is defined to be the full inverse image of X by π, and the proper transform is defined as the closure of the inverse image of the intersection $X \cap U^0$ of X with the set U^0 over which $\mathcal{F}$ is locally free.

In the case when $\mathcal{F}$ is a coherent sheaf of ideals, the definition above is clearly analogous to the definition for blowing up an ideal given in Section 2.1.2 (after considering the isomorphism $\mathrm{Gr}(r, N) \approx \mathrm{Gr}(N-r, N)$). The blowup of U along an ideal sheaf $\mathfrak{I}$ satisfies the following properties (see Proposition 7.13 in [Har77]).

Fact 2.1.6. *If $\pi \colon \widetilde{U} \to U$ is the blowup of U with respect to the coherent sheaf of ideals $\mathfrak{I}$, then*

a. *$\pi^{-1}\mathfrak{I}$ is a locally free sheaf of rank 1 in $\widetilde{U}$; and*

b. *if Y is the closed subscheme corresponding to $\mathfrak{I}$, then the restriction of π to $\widetilde{U} - \pi^{-1}(Y)$ is an isomorphism.*

Fact (a) above allows us to blow up an ideal to ensure that its inverse image will be locally principal in the blowup; we will do this in Chapter 4 to get a certain

Fitting invariant to be locally principal. Fact (b) just states that, in the case where we are blowing up a sheaf of ideals, the U^0 described in the general case above is simply the complement of the scheme defined by the sheaf of ideals $\mathfrak{I}$. Thus, for example, when we blow up the maximal ideal sheaf corresponding to the isolated singular point v, the resulting blowup will be isomorphic to U away from the inverse image of the point v.

In general, the blowup $\widetilde{U}$ of U with respect to a coherent sheaf $\mathcal{F}$, as constructed locally above, is characterized by the following ([NA83], Section 1.1):

Claim 2.1.7. *A map $\pi\colon \widetilde{U} \to U$ is the blowup of U with respect to the coherent sheaf $\mathcal{F}$ if it satisfies the following three conditions:*

- **a.** *π is a birational, proper morphism;*
- **b.** *$\pi^*\mathcal{F}/\mathrm{Torsion}(\pi^*\mathcal{F})$ is a locally free $\mathcal{O}_{\widetilde{U}}$ -module; and*
- **c.** *if $p\colon \bar{U} \to U$ is another map satisfying (a) and (b) above, then there exists a unique factorization $f\colon \bar{U} \to \widetilde{U}$ with $p = \pi \circ f$.*

We will discuss the second of these conditions, as it will be important to our discussion in Section 4.2. Since the pullback of a locally free sheaf is locally free, and the "universal quotient sheaf" is locally free (see below), it suffices to prove the following claim.

Claim 2.1.8. *Given the blowup $\pi\colon \widetilde{U} \to U$ of U with respect to the rank r sheaf $\mathcal{F}$, the pullback of $\mathcal{F}$ (mod torsion) by π is isomorphic to the pullback of the universal quotient sheaf on* $\mathrm{Gr}(r, N)$ *by the map γ:*

$$\pi^*\mathcal{F}/\mathrm{Torsion}(\pi^*\mathcal{F}) \approx \gamma^*\mathcal{Q}.$$

Proof. We first describe the universal quotient sheaf $\mathcal{Q}$. Consider the universal vector bundle γ_N^{N-r} over the grassmannian $\mathrm{Gr}(N-r, N)$ whose fibre over a point $P \in \mathrm{Gr}(N-r, N)$ is simply the $(N-r)$-plane represented by P. Over $\mathrm{Gr}(N-r, N)$ we have (see Section 1.5, [GH78]) an exact sequence of vector bundles:

$$0 \;\to\; \gamma_N^{N-r} \;\hookrightarrow\; \mathrm{Gr}(N-r, N) \times \mathbb{C}^N \;\twoheadrightarrow\; Q_N^{N-r} \;\to\; 0.$$

The fibre of Q_N^{N-r} over a point $P \in \mathrm{Gr}(N-r, N)$ is the r-plane obtained from the quotient of $\mathbb{C}^N$ by the $(N-r)$-plane represented by P. The universal quotient sheaf $\mathcal{Q}$ over $\mathrm{Gr}(N-r, N)$ is defined to be the sheaf of sections of the universal quotient bundle (see the correspondence between locally free sheaves and vector bundles described in Section 2.1.3).

We now show that, to prove the claim, it suffices to show that there is a surjection

$$g\colon \pi^*\mathcal{F} \twoheadrightarrow \gamma^*\mathcal{Q}.$$

Given such a surjection g, we need only show that its kernel is the torsion sheaf $\mathrm{Torsion}(\pi^*\mathcal{F})$. Clearly $\ker g \supseteq \mathrm{Torsion}(\pi^*\mathcal{F})$ (since the pullback $\gamma^*\mathcal{Q}$ of the locally free sheaf $\mathcal{Q}$ is locally free, and thus g must kill all torsion in $\pi^*\mathcal{F}$). To show ($\subseteq$), we need only note that since $\mathcal{F}$ and $\mathcal{Q}$ are both rank r, and pulling back preserves rank, the sheaves $\pi^*\mathcal{F}$ and $\gamma^*\mathcal{Q}$ are of the same rank. Thus only torsion can by killed by g and we have $\ker g = \text{Torsion}(\pi^*\mathcal{F})$ as desired.

By Theorem 3.5 of [Ros68], we do indeed have such a surjection g (whose kernel is rank 0), constructed using the canonical map $\gamma\colon \widetilde{U} \to Gr(N-r, N)$ and the section $\sigma\colon U \to Gr(N-r, N)$ used in the construction of $\widetilde{U}$ (see also Proposition 3.4 in [Ros68]). ■

2.2 Nash Blowups, Bundles, and Sheaves

We first describe MacPherson's characterization of the Nash bundle as the bundle (over an appropriate blowup of U) that is the unique (minimal) extension of the $T(U_{smooth})$ so that the map to $T(\mathbb{C}^n)$ no longer degenerates over fibres.

Given any desingularization $\pi\colon (\widetilde{U}, E) \to (U, v)$ (where U is as usual a neighborhood of an isolated singular point v in a complex algebraic variety V, chosen small enough so that U is embedded in $\mathbb{C}^N$), we can consider the map

$$\widetilde{i} := i \circ \pi\colon \quad \widetilde{U} \xrightarrow{\pi} U \xrightarrow{i} \mathbb{C}^N.$$

This map $\widetilde{i}$ is a map of smooth manifolds, and thus induces a map

$$\widetilde{i}_* : T\widetilde{U} \longrightarrow T\mathbb{C}^N$$

on tangent bundles; this map is injective along fibres away from E (since $\widetilde{U} - E$ is isomorphic to $U - v \subset \mathbb{C} - \{0\}$), but crushes the fibres above E (since π maps all of E to the point $v \in U$). In other words, $\widetilde{i}_*$ drops rank along E and is full rank elsewhere.

Claim 2.2.1. *Given a "suitable" desingularization $(\widetilde{U}, E)$ of (U, v), there exists a (unique) vector bundle $\mathfrak{N}$ over $\widetilde{U}$ and maps n and j for which:*

(a) $n \colon T\widetilde{U} \to \mathfrak{N}$ *is the identity over* $\widetilde{U} - E$*;*

(b) $j \colon \mathfrak{N} \to T\mathbb{C}^N$ *is injective on fibres; and*

(c) $\widetilde{i}_* = j \circ n$*.*

This is MacPherson's characterization of the Nash blowup $\widehat{U}$ (which the "suitable" desingularization must factor through, defined in Section 2.2.1.1) and the Nash bundle $\mathfrak{N}$ (which we define in Section 2.2.2). Note that, given this claim, we have

$$\mathfrak{N}|_{\widetilde{U}-E} \approx T\widetilde{U}|_{\widetilde{U}-E} \approx TU|_{U-v},$$

and since the fibres of $\mathfrak{N}$ inject into $T\mathbb{C}^N$, the Nash bundle in some sense "extends" the tangent bundle of the nonsingular part of U over the singular point (while remaining embedded in the tangent space to $\mathbb{C}^n$). The uniqueness condition in the claim (which we will not prove here) ensures that it is the minimal such extension. In the sections that follow we will construct a space $\widetilde{U}$ and a bundle $\mathfrak{N}$ over $\widetilde{U}$ that satisfy the conditions in Claim 2.2.1.

2.2.1 The Nash blowup

We begin by defining the Nash blowup $\pi\colon \widetilde{U} \to U$ as the space of limits of planes tangent to (the smooth points of) U. We then point out that this construction is equivalent to the blowup of a certain "Jacobian" ideal, and show that it is equivalent to the blowup of the sheaf of 1-forms on U.

2.2.1.1 Definition of the Nash blowup

Let $Gr_n(T\mathbb{C}^N)$ denote the Grassmann bundle of n-planes in $T\mathbb{C}^N$, where n is the dimension of V (hence of U) in the notation above. Over the set U we have

$$Gr_n(T\mathbb{C}^N|_U) \approx U \times \mathrm{Gr}(n, N),$$

where U also denotes the image $i(U)$ of U in $\mathbb{C}^N$. Consider the section

$$\begin{aligned} \sigma : U - v &\longrightarrow U \times \mathrm{Gr}(n, N) \\ z &\longmapsto (z, T_z(U - v)) \end{aligned}$$

over the smooth part of U that sends each point to its tangent space (embedded in the tangent space to $\mathbb{C}^N$ at the point $i(z)$).

Definition 2.2.2. *The* Nash blowup *of U is defined to be the closure*

$$\widehat{U} := \mathrm{cl}(\sigma(\mathrm{U} - \mathrm{v}))$$

of the image of the section σ in $U \times \mathrm{Gr}(n,N)$.

Note that the Nash blowup is embedded in $U \times \mathrm{Gr}(n,N)$ and thus comes automatically equipped with maps

$$\widehat{\pi}\colon \widehat{U} \longrightarrow U \qquad \text{and} \qquad \gamma\colon \widehat{U} \longrightarrow \mathrm{Gr}(n,N)$$

given by the restriction of, respectively, the first and second projections from $U \times \mathrm{Gr}(n,N)$ to $\widehat{U}$.

By definition, in the case when U has isolated singular point v, the inverse image $\widehat{\pi}^{-1}(z)$ of a nonsingular point $z \in U - v$ consists of a single point in $U \times \mathrm{Gr}(n,N)$, namely $(z, T_z(U-v))$. The inverse image $\widehat{\pi}^{-1}(v) =: E$ of the singular point of U consists of limits of tangent planes; more precisely, $(v,P) \in U \times \mathrm{Gr}(n,N)$ is a point of $\widetilde{U}$ if and only if P is an n-plane in $\mathbb{C}^N$ for which there exists a sequence $\{v_i\}$ of points in U converging to v so that the sequence $\{T_{v_i}(U-v)\}$ of tangent planes to the v_i converges to P in $\mathrm{Gr}(n,N)$.

In general, the Nash blowup is an isomorphism away from the singular points of U; as in the Main Theorem (Theorem 2) from [Nob75], we have:

Claim 2.2.3. *The Nash blowup $\widehat{\pi}\colon \widehat{X} \to X$ of a complex algebraic variety X is an isomorphism if and only if X is nonsingular.*

Since the Nash blowup is a completely local construction, this means that over the nonsingular points of U, the Nash blowup is an isomorphism. In the case where U

has isolated singular point the Nash blowup is an isomorphism away from v and we have $\widehat{U} - E \approx U - v$.

We will say that a blowup $\pi\colon \widetilde{U} \to U$ of U is a *generalized Nash blowup* if it factors through the Nash blowup of U, *i.e.* if there is a map

$$\widetilde{\pi} : \widetilde{U} \longrightarrow \widehat{U}$$

so that $\pi = \widehat{\pi} \circ \widetilde{\pi}$. A generalized Nash blowup is in fact characterized by a universal property involving the Nash bundle (or the Nash sheaf); see Sections 2.2.2 and 2.2.3.

2.2.1.2 The Nash blowup as a monoidal transform

We can also express the Nash blowup as the blowup of a certain ideal; Nobile proves this (see Theorem 1 in [Nob75]) using an embedding of $\mathrm{Gr}(n, N)$ into a large projective space; we merely state his results here.

Claim 2.2.4. *The Nash blowup is locally a monoidal transform, with center a suitable ideal.*

In fact, if V is a hypersurface, with local defining equation $f(z_1, \dots, z_N)$, the Nash blowup of V is isomorphic to the blowup of V along the Jacobian ideal

$$Jac_f := \left(\frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_N}\right)$$

corresponding to f.

In the case where V is a local complete intersection of dimension $N-r$, with local defining equations $f_1, ...f_r$, the Nash blowup of V is the blowup of V along the ideal J generated by all of the $(n-r)\times(n-r)$ subdeterminants of the Jacobian matrix

$$\begin{pmatrix} \frac{\partial f_1}{\partial z_1} & \cdots & \frac{\partial f_1}{\partial z_N} \\ \vdots & & \vdots \\ \frac{\partial f_r}{\partial z_1} & \cdots & \frac{\partial f_r}{\partial z_N} \end{pmatrix}.$$

In Section 2.3 we will see that this ideal J is the r^{th} Fitting invariant $Fitt_r$ of the derivative map

$$df : T\mathbb{C}^N \longrightarrow T\mathbb{C}^r$$

induced by the map $f = (f_1, \dots, f_r)\colon \mathbb{C}^N \to \mathbb{C}^r$.

Finally, for general V of dimension $n-r$ (but with possibly more than r local defining functions), the Nash blowup is equivalent to the blowup of V along an ideal J generated by a (nonvanishing) subset of the $(n-r)\times(n-r)$ subdeterminants of the Jacobian ideal.

2.2.1.3 The Nash blowup as the blowup of the sheaf of 1-forms

Our final characterization of the Nash blowup is as the blowup of the sheaf of 1-forms (as done in detail in Section 1.1 of [LT81] and Section 2 of [GS81]). We will blow up the sheaf of 1-forms as in Section 2.1.3 and then show that the resulting blowup is equivalent to the Nash blowup.

Let $\mathcal{F} = \Omega^1_U$ be the (coherent) sheaf of 1-forms on U. As in Section 2.1.3, we have chosen U small enough to be embedded in $\mathcal{C}^N$; thus we have a surjection

$$\mathcal{O}_U^N \twoheadrightarrow \Omega^1_U \longrightarrow 0$$

of Ω^1_U. Since U has only one singular point v, the sheaf of 1-forms on U is locally free over all of $U - v$; thus the open dense subset U^0 over which Ω^1_U is locally free is simply $U - v$. The surjection above now corresponds to an inclusion

$$T(U - v) \hookrightarrow T\mathbb{C}^N|_U,$$

since Ω^1_{U-v} is the sheaf of sections of the cotangent bundle $T^*(U - v)$ to $U - v$, and $T\mathbb{C}^N|_U$ is the rank N trivial bundle over U. Thus we have a section

$$\sigma : U - v \longrightarrow Gr_r(T\mathbb{C}^N|_U) \approx U \times \mathrm{Gr}(r, N),$$

given by $z \mapsto (z, T_z(U - v))$. The blowup $\widehat{U}$ of U with respect to the sheaf Ω^1_U of 1-forms on U is given by the closure of the image of this section:

$$\widehat{U} := \mathrm{cl}(\sigma(U - v)).$$

Looking back at Section 2.2.1.1, in particular the definition of the section σ there, this is clearly the same construction as the Nash blowup.

2.2.2 The Nash bundle

Let $\widehat{\pi} : \widehat{U} \to U$ be the Nash blowup of U as defined above. We will define the Nash bundle as the pullback to $\widehat{U}$ of the universal subbundle over the grassmannian, and then show that this bundle is the bundle in MacPherson's characterization of the Nash bundle (see Claim 2.2.1).

By the construction of the Nash bundle $\widehat{U}$ we have a map

$$\gamma : \widehat{U} \longrightarrow \mathrm{Gr}(r, N);$$

let γ_N^r denote the universal subbundle over the grassmannian $\mathrm{Gr}(r, N)$, *i.e.* the bundle whose fibre over an r-plane $P \in \mathrm{Gr}(r, N)$ consists of the vectors in that r-plane.

Definition 2.2.5. *With the notation above, define the* Nash bundle $\nu\colon \mathfrak{N} \to \widehat{U}$ *over the Nash blowup* $\widehat{U}$ *of* U *to be the pullback by* γ

$$\mathfrak{N} := \gamma^*(\gamma_N^r)$$

of the universal subbundle of the grassmannian $\mathrm{Gr}(r, N)$.

By definition, the fibre $\mathfrak{N}_{\widehat{z}}$ of $\mathfrak{N}$ over a point $\widehat{z}$ in $\widehat{U} - E$ is the r-plane $T_z(U - v)$ in $T_z(\mathbb{C}^N)$. Thus we have

$$\mathfrak{N}_{\widehat{U}-E} \approx T(U - v).$$

Moreover, since $\widehat{\pi} : \widehat{U} \to U$ is an isomorphism away from E we have

$$T(U - v) \approx T(\widehat{U} - E);$$

thus away from the exceptional set E, the Nash bundle is isomorphic to the tangent bundle of the Nash blowup (*i.e.* satisfies part (a) of Claim 2.2.1).

Since the fibre of the Nash bundle over a point $\widehat{z} \in E$ in the exceptional set of $\widehat{U}$ is a limit of tangent planes (see Section 2.2.1.1), it injects naturally into $T\mathbb{C}^N$, *i.e.* we have a well-defined map

$$n : \mathfrak{N} \longrightarrow T\mathbb{C}^N$$

that is injective on fibres. Thus the Nash bundle as defined above satisfies part (b) of Claim 2.2.1, and is the vector bundle characterized by MacPherson that "extends" the tangent bundle of the smooth part of U over the singular point.

We will say that a bundle over a generalized Nash blowup $\pi\colon \widetilde{U} \to U$ (*i.e.* a blowup factoring through the Nash blowup via some map $\widetilde{\pi}\colon \widetilde{U} \to \widehat{U}$) is a *generalized Nash bundle* on $\widetilde{U}$ if it is the pullback of the Nash bundle $\mathfrak{N}$ of the Nash blowup $\widehat{U}$ by the factoring map $\widetilde{\pi}$. In fact, the existence of such a bundle over a resolution $\widetilde{U}$ of U characterizes generalized Nash blowups (see (A.3.1) in the Appendix of [PS97]):

Claim 2.2.6. *A resolution $\pi\colon \widetilde{U} \to \widehat{U}$ is a generalized Nash blowup of U if and only if there exists a unique bundle $\mathfrak{N}_{\widetilde{U}}$ over $\widetilde{U}$ and maps n, j satisfying the conditions in Claim 2.2.1.*

2.2.3 The Nash sheaf

One way to define the Nash sheaf is via the correspondence between vector bundles and locally free sheaves (see Section 2.1.3), as in the definition below.

Definition 2.2.7. *The* Nash sheaf $\mathcal{N}$ *is the (locally free) sheaf of sections of the dual of the Nash bundle* $\nu\colon \mathfrak{N} \to \widehat{U}$.

Note that, since as vector bundles we have $\mathfrak{N}|_{\widehat{U}-E} \approx T(\widehat{U}-E)$, as locally free sheaves we have

$$\mathcal{N}|_{\widehat{U}-E} \approx \Omega^1_{\widehat{U}-E}.$$

We can also define the Nash sheaf to be the pullback of the universal quotient sheaf on $\mathrm{Gr}(N-r, N)$ by the canonical map γ; if we think of the Nash blowup as the blowup $\pi\colon \widetilde{U} \to U$ of the sheaf of 1-forms Ω^1_U as in Section 2.2.1.3, we can define $\mathcal{N}$ to be the locally sheaf

$$\mathcal{N} := {}^{\pi^*\Omega^1_U}\big/_{\mathrm{Torsion}(\pi^*\Omega^1_U)} \approx \gamma^*\mathcal{Q}$$

as in Section 2.1.3 and Claim 2.1.8. Since the universal quotient sheaf over $\mathrm{Gr}(N-r, N)$ corresponds to the dual of the universal sheaf over $\mathrm{Gr}(r, N)$, it is clear by going back through the definitions that the Nash sheaf as defined here is indeed the dual of the sheaf of sections of the Nash bundle.

We will call a sheaf $\mathcal{N}$ on a blowup $\widetilde{U}$ of U a *generalized Nash sheaf* if $\widetilde{U}$ is a generalized Nash blowup (so factors through the Nash blowup $\widehat{U}$) and $\mathcal{N}$ is the pullback of the Nash sheaf on $\widehat{U}$. As in the case of the Nash bundle, the existence of such a sheaf on a resolution $\widetilde{U}$ characterizes generalized Nash blowups. In other words, we have a sheaf version of Claim 2.2.6 (and thus of MacPherson's characterization in Claim 2.2.1); Pardon and Stern prove Claim 2.2.6 in this sheaf version form (from (A3.4) in the Appendix to [PS97]).

Claim 2.2.8. *A resolution $\pi\colon \widetilde{U} \to \widehat{U}$ is a generalized Nash blowup of U if and only if there exists a (unique) sheaf $\mathcal{N}_{\widetilde{U}}$ over $\widetilde{U}$, and maps $\nu\colon \mathcal{N}_{\widetilde{U}} \hookrightarrow \Omega^1_{\widetilde{U}}$ and $\mu\colon \pi^*\Omega^1_U \to \mathcal{N}$ so that*

(i) $\mathcal{N}_{\widetilde{U}}$ is locally free of rank $n = \dim(U)$;

(ii) ν is an isomorphism over $\widetilde{U} - \pi^{-1}(U_{\text{sing}})$;

(iii) μ is a surjective morphism over $\widetilde{U}$; and

(iv) the canonical map $\delta\colon \pi^*\Omega^1_U \to \Omega^1_{\widetilde{U}}$ factors as $\delta = \nu \circ \mu$.

2.3 Fitting Ideals and Invariants

In this section we first define (following the discussion in Section 20.2 of [Eis95]) the Fitting ideals associated to a morphism of modules and the corresponding Fitting invariants. We then generalize this construction to define (as in Chapter 2 of [Kwi87]) the Fitting invariants of a morphism of sheaves. Finally, we look at a particular morphism (the inclusion of the Nash sheaf into the sheaf of 1-forms with logarithmic poles along E) and its Fitting ideals.

2.3.1 Fitting ideals for maps of modules

Given a map $\phi\colon F \to G$ of finitely generated free R-modules, we have an obvious induced map $\Lambda^j\phi\colon \Lambda^j F \to \Lambda^j G$ given by

$$\Lambda^j\phi(f_1 \wedge \ldots \wedge f_j) = \phi(f_1) \wedge \ldots \wedge \phi(f_j).$$

This induces an adjoint map $(\Lambda^j\phi)^\star\colon \Lambda^j F \otimes \Lambda^j G^* \to R$ defined by

$$\begin{aligned}
(\Lambda^j\phi)^\star((f_1 \wedge \ldots \wedge f_j) \otimes (g_1^* \wedge \ldots \wedge g_j^*)) & \\
&= (g_1^* \wedge \ldots \wedge g_j^*)(\phi(f_1) \wedge \ldots \phi(f_j)) \\
&= \sum_\sigma (-1)^{\mathrm{sgn}(\sigma)} g_1^*(\phi(f_{\sigma_1})) \cdots g_j^*(\phi(f_{\sigma_j})) \\
&= \det[g_i^*(\phi(f_l))], \qquad (2.3.1)
\end{aligned}$$

where σ is a permutation of $[1, \ldots, j]$ with sign $\mathrm{sgn}(\sigma)$ and $[g_i^*(\phi(f_l))]$ denotes the $j \times j$ matrix with (i, l) entry $g_i^*(\phi(f_l))$.

Definition 2.3.1. *Given the notation above, we define the j^{th}* Fitting ideal of ϕ *to be*

$$Fitt_j(\phi) := \mathrm{im}((\Lambda^{\mathrm{j}}\phi)^\star) \subset \mathrm{R}.$$

Given bases $\{\alpha_1, \ldots, \alpha_f\}$ and $\{\beta_1, \ldots, \beta_g\}$ for F and G respectively we can represent ϕ by a matrix (which we also denote as ϕ). Then by (2.3.1) the image of the map $(\Lambda^j\phi)^\star$ is generated by elements

$$\det[\beta_i^*(\phi(\alpha_l))] = \det[\beta_i^*(\phi_l)] = \det[\phi_{il}],$$

where i and l run through a j-length subset of $[1, \ldots, f]$ and $[1, \ldots, g]$ respectively, and ϕ_l denotes the l^{th} column of the matrix ϕ. In other words, we have the following remark.

Remark 2.3.2. *Given bases for modules F and G, the j^{th} Fitting ideal $Fitt_j(\phi)$ of the map $\phi\colon F \to G$ is generated by the $j \times j$ subdeterminants of the matrix for ϕ.*

By convention we say that the determinant of a 0×0 matrix is 1, and thus $Fitt_0(\phi) = R$. We also define $Fitt_j(\phi) = R$ for $j \leq 0$. The Fitting ideal $Fitt_j(\phi)$ is of course independent of the choice of bases for F and G.

2.3.2 Fitting Invariants of Modules

Let M be a finitely presented R-module, and let

$$F \xrightarrow{\phi} G \longrightarrow M \longrightarrow 0$$

be any presentation of M (so G is free of rank, say, r).

Definition 2.3.3. *Given the notation above, the j^{th}* Fitting invariant of M *is defined to be the ideal*

$$F_j(M) := Fitt_{r-j}(\phi).$$

This ideal is an invariant of the module M, *i.e.* does not depend on the choice of presentation (see [Eisenbud]).

2.3.3 Fitting Invariants of Sheaves

Given a coherent sheaf $\mathcal{G}$ on a complex algebraic variety V, and an open affine set $U \subset V$, let

$$\mathcal{O}_U^m \xrightarrow{\phi} \mathcal{O}_U^n \longrightarrow \mathcal{G}|_U \longrightarrow 0$$

be any local finite presentation of $\mathcal{G}$ over U. Choose bases for the free modules $\mathcal{O}_U^m$ and $\mathcal{O}_U^n$, and consider the matrix of the morphism ϕ with respect to those bases. We define the Fitting invariants of $\mathcal{G}$ locally in the same fashion as we did for modules in the section above:

Definition 2.3.4. *With notation as above, the j^{th}* Fitting invariant *$\mathcal{F}_j(\mathcal{G})|_U$ of $\mathcal{G}$ over U is defined to be the sheaf of ideals generated by the $(n-j)$ subdeterminants of the matrix of ϕ.*

The $\mathcal{F}_j(\mathcal{G})|_U$ patch together to form a coherent sheaf of ideals $\mathcal{F}_j(\mathcal{G})$ on $\mathcal{O}_V$ (see [Kwiecinski]). As with modules, we will use $\mathcal{F}_j(\phi)$ to denote the Fitting invariant for the cokernel sheaf of the morphism ϕ; $Fitt_j(\phi)$ will denote the j-th Fitting ideal (*i.e.* the $n-j$-th Fitting invariant) of ϕ.

2.3.4 Fitting Ideals and the Nash Sheaf

We will be concerned primarily with the Fitting ideals corresponding to the inclusion $\alpha_1 : \mathcal{N}_{\widetilde{U}} \hookrightarrow \Omega^1_{\widetilde{U}}(\log E)$ of the Nash sheaf into the sheaf of logarithmic 1-

forms (*i.e.* the sheaf of 1-forms with logarithmic singularities along the exceptional set E) over a generalized Nash blowup $(\widetilde{U}, E)$ of (U, v). Note that the inclusion α_1 is the composition of the inclusion $\mathcal{N}_{\widetilde{U}} \hookrightarrow \Omega^1_{\widetilde{U}}$ arising from the construction of $\mathcal{N}_{\widetilde{U}}$ (see Claim 2.2.8) with the inclusion $\Omega^1_{\widetilde{U}} \hookrightarrow \Omega^1_{\widetilde{U}}(\log E)$ of the sheaf of 1-forms into the sheaf of logarithmic 1-forms. Consider the presentation

$$\mathcal{N}_{\widetilde{U}} \overset{\alpha_1}{\hookrightarrow} \Omega^1_{\widetilde{U}}(\log E) \longrightarrow Cok(\alpha_1) \longrightarrow 0;$$

of the cokernel sheaf

$$Cok(\alpha_1) = \Omega^1_{\widetilde{U}}(\log E)/\mathcal{N}_{\widetilde{U}}.$$

By the definitions above, $Fitt_j(\alpha_1) = Fitt_j(\Omega^1_{\widetilde{U}}(\log E)/\mathcal{N}_{\widetilde{U}})$ is the sheaf of ideals generated by the $j \times j$ subdeterminants of the matrix for α_1 (given some bases for $\mathcal{N}_{\widetilde{U}}$ and $\Omega^1_{\widetilde{U}}(\log E)$; see Section 5.3 where we choose particular bases), where n is the rank of $\Omega^1_{\widetilde{U}}(\log E)$. in particular, in the case where $n = 3$, $Fitt_2 = Fitt_2(\alpha_1)$ is generated by 2×2 subdeterminants of the matrix for α_1, and $Fitt_1(\alpha_1)$ is generated by the 1×1 subdeterminants (*i.e.* the entries) of the matrix for α_1.

Note that if we define α_2 to be the map

$$\alpha_2\colon \quad \Lambda^2\mathcal{N}_{\widetilde{U}} \longrightarrow \Omega^2_{\widetilde{U}}(\log E),$$

then, in the case where $n = 3$ (and thus the rank of $\Omega^2_{\widetilde{U}}(\log E)$ is also 3), we have

$$Fitt_2(\alpha_1) = Fitt_1(\alpha_2),$$

since the entries in the matrix for α_2 are simply the 2×2 subdeterminants of the entries in the matrix for α_1. In the general (n-dimensional) case, we have

$$\begin{aligned} F_j(\alpha_1) &= Fitt_{n-j}(\alpha_1) \\ &= \mathrm{im}((\Lambda^{n-j}\alpha_1)^\star) \\ &= \mathrm{im}((\Lambda^1\alpha_{n-j})^\star) \\ &= Fitt_1(\alpha_{n-j}) \\ &= F_{n-1}(\alpha_{n-j}). \end{aligned} \tag{2.3.2}$$

2.4 Algebraic and Analytic Varieties

The results in this work involve both the algebraic and the analytic categories of varieties. In this section we make precise the passage from an algebraic variety to its associated analytic variety, and from a coherent algebraic sheaf to the associated coherent analytic sheaf. Many of the proofs in the chapters that follow will be done on the level of this associated analytic variety (and sheaves); in this section we also catalogue the lemmas we need to know to get back to the algebraic category from the analytic category. For example, we will see that, given a sequence of coherent algebraic sheaves, if the associated sequence of coherent analytic sheaves is exact then the original sequence of coherent algebraic sheaves is also exact.

Given a complex algebraic variety X we can consider an associated analytic variety, according to the following proposition (see Proposition 2 and the discussion at the end of Section 2.5 in Serre's article [Ser56]):

Proposition 2.4.1. *Every complex algebraic variety X can be given the structure of a complex analytic variety X^h; every regular map $f\colon X \to Y$ between algebraic varieties is a holomorphic map $X^h \to Y^h$ between the associated analytic varieties.*

Since every regular function is holomorphic, given a point $x \in X$ we can compare the local ring of regular functions to the local ring of holomorphic functions at x ([Ser56], Section 2.6):

Proposition 2.4.2. *There is a homomorphism $\theta\colon \mathcal{O}_x \to \mathcal{H}_x$ from the ring $\mathcal{O}_x$ of regular functions at x to to the ring $\mathcal{H}_x$ of holomorphic functions on X^h in a neighborhood of x.*

Given a coherent algebraic sheaf $\mathcal{F}$ on an algebraic variety X we can define an associated coherent analytic sheaf (*i.e.* sheaf of $\mathcal{H}_X$-modules) on the associated analytic variety X^h, as follows (see Section 3.9 of [Ser56]). Let $\mathcal{F}'$ denote the sheaf over X^h defined by the inverse image of $\mathcal{F}$ under the map $X \to X^h$ taking X to its associated analytic variety. Note that for every $x \in X$ we have $\mathcal{F}' = \mathcal{F}$; the sheaves $\mathcal{F}$ and $\mathcal{F}'$ differ only in their topologies. Applying this process to the sheaf $\mathcal{O}$ of regular functions on X we obtain a subsheaf $\mathcal{O}'$ of the sheaf $\mathcal{H}$ of germs of holomorphic functions on X^h. We can now define a coherent analytic sheaf on X^h associated to the sheaf $\mathcal{F}$ (as in Definition 2 of [Ser56]).

Definition 2.4.3. *Given X, X^h, $\mathcal{F}$, and $\mathcal{F}'$ as above, we define the* coherent analytic sheaf associated to $\mathcal{F}$ *to be the sheaf on the associated analytic variety X^h defined by:*

$$\mathcal{F}^h := \mathcal{F}' \otimes_{\mathcal{O}'} \mathcal{H}.$$

The study of coherent analytic sheaves on X^h is essentially the study of coherent algebraic sheaves on X (see Remark 1 in [Ser56]); in the rest of this section we try to make that remark a little more precise.

We will be working with associated analytic varieties and sheaves in the following chapters and sections: Sections 3.2 and 3.3; Chapters 4 and 5; Section 6.3; and Chapters 7 and 8. To get back to the algebraic category from the analytic results obtained in these sections we will use the following two theorems (see Theorems 2 and 3 in [Ser56]).

Theorem 2.4.4. *Given coherent algebraic sheaves $\mathcal{F}$ and $\mathcal{G}$ on a complex algebraic variety X, every analytic homomorphism $\mathcal{F}^h \to \mathcal{G}^h$ on the associated analytic sheaves induces a unique algebraic map $\mathcal{F} \to \mathcal{G}$.*

Theorem 2.4.5. *Let X be a complex algebraic variety with associated analytic variety X^h. Given a coherent analytic sheaf $\mathcal{M}$ on X^h, there exists a unique (up* *to isomorphism) coherent algebraic sheaf* $\mathcal{F}$ *on* X *for which we have* $\mathcal{F}^h \approx \mathcal{M}$*.*

In Section 3.2 we will show that a certain Fitting ideal sheaf is locally principal by showing that its associated coherent analytic ideal sheaf is locally principal. In Section 3.3 we show that under certain types of blowups we have an isomorphism between two Fitting ideal sheaves; this result is obtained by examining the associated coherent analytic ideal sheaves. The two theorems above enable us to turn these analytic proofs into algebraic results.

In Chapters 4, 5, and 7 as well as Section 6.3, we use the associated analytic variety $\widetilde{U}^h$ to a complex algebraic space in order to be able to choose local analytic coordinates and absorb local units into those coordinates. The Main Propositions 5.2.2, 5.5.2, and 5.6.2 are in fact local analytic results. In Chapter 8.1.1 we construct a sequence of coherent algebraic sheaves and show it is exact by showing that the associated sequence of coherent analytic sheaves is exact. To make this last passage from the analytic proof to the algebraic result we need to be able to say that the sheaf $\mathcal{H}$ of germs of holomorphic functions is *faithfully flat.* A sheaf $\mathcal{J}$ is "faithfully flat" if, given any sequence of sheaves

$$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0,$$

we have the following property concerning $\mathcal{J}$: if the sequence

$$0 \to \mathcal{A} \otimes \mathcal{J} \to \mathcal{B} \otimes \mathcal{J} \to \mathcal{C} \otimes \mathcal{J} \to 0$$

is exact, then the original sequence is exact. By Proposition 10a in [Ser56], the functor $\mathcal{F} \mapsto \mathcal{F}^h$ is an exact functor: any exact sequence of coherent algebraic sheaves will be exact after passing to the associated sequence of coherent analytic sheaves. Thus, by the definition of $\mathcal{F}^h$, the ring $\mathcal{H}_x$ of germs of holomorphic functions at $x \in X^h$ is *flat* (any exact sequence remains exact after tensoring with $\mathcal{H}_x$). Recall from 2.4.2 that we have a local ring homomorphism $\theta \colon \mathcal{O}_x \to \mathcal{H}_x$. In Section 4 of Matsumura's book [Mat70] it is proved that, given a local homomorphism of local rings $A \to B$, B is flat over A if and only if B is faithfully flat over A. Thus, if a sequence of coherent analytic sheaves associated to a sequence of coherent algebraic sheaves is exact, then the sequence of coherent algebraic sheaves must also be exact. This gives us the following proposition:

Proposition 2.4.6. *Suppose X is an algebraic variety, and we are given a sequence of sheaves*

$$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$$

over X. If the associated sequence of sheaves

$$0 \to \mathcal{A}^h \to \mathcal{B}^h \to \mathcal{C}^h \to 0$$

over the associated analytic variety X^h is exact, then the original algebraic sequence is exact.

2.5 Hsiang-Pati Coordinates

In this section we briefly state and discuss the results of Hsiang, Pati, Pardon, and Stern concerning the existence and construction of Hsiang-Pati coordinates. The interested reader should consult [HP85], [Pat94], and [PS97] for a more detailed discussion of these results.

2.5.1 Hsiang and Pati

Let V be a 2-dimensional analytic variety with isolated singular point v; locally (near v) V is embedded in some $\mathbb{C}^N$ with coordinates $\{z_1, \dots, z_N\}$ centered at v. Hsiang and Pati begin by blowing up V at the point v and then blowing up enough times so that the singularity is resolved and the exceptional set is a divisor with normal crossings. A local parameterization for V pulled up to this blowup, in a coordinate chart $\{u, v\}$ about a point in the exceptional divisor, takes the form:

$$z_i = u^{n_i} v^{m_i} \phi_i(u, v) \quad \text{for } 1 \leq i \leq N$$

for some integers n_i, m_i (although this is a different notation than we will be using later, we keep the notation in [HP85] since it does not readily translate in its entirety to our notation) and smooth (possibly after further blowings-up) curves ϕ_i. Hsiang and Pati then "arrange" the powers n_i, m_i by a series of successive blowups, reorderings, and coordinate changes until the z_i take the final form (note that the n_i and m_i here may be different exponents than the ones above):

$$\begin{aligned} z_1 &= u^{n_1}v^{m_1} \\ z_2 &= f_2(z_1) + u^{n_2}v^{m_2} \\ z_3 &= f_3(z_1) + u^{n_3}v^{m_3}g_3 \\ &\vdots \\ z_N &= f_N(z_1) + u^{n_N}v^{m_N}g_N \end{aligned}$$

where for each i,

1. $f_i = \sum_j \alpha_{ij} z_1^{\epsilon_j}$ for rational $\epsilon_j \geq 1$;
2. $\dfrac{\partial f}{\partial z_1}$ is a holomorphic function of u and v;
3. g_i is a local unit (or zero); and
4. $n_i \geq n_2$, $m_i \geq m_2$ if $i \geq 2$.

Finally (as far as we are concerned), Hsiang and Pati show that these coordinate functions induce a metric quasi-isometric to the pullback of the standard metric g on $V - v$ (the one induced from the local embedding into $\mathbb{C}^N$). Define

$$\xi_1 := u^{n_1}v^{m_1} \quad \text{and} \quad \xi_2 := u^{n_2}v^{m_2};$$

and let $\pi\colon \widetilde{U} \to U$ denote the final resolution obtained from the blowings-up described above. Then (see [HP85]):

Claim 2.5.1. *The metric*

$$d\xi_1 \wedge d\bar{\xi}_1 + d\xi_2 \wedge d\bar{\xi}_2,$$

wherever it is nonsingular, is locally quasi-isometric to the pullback $\pi^* g$ of the standard metric on U to $\widetilde{U}$.

2.5.2 Pardon and Stern

Let (V, v) be an algebraic surface with neighborhood U of v as above. Pardon and Stern (in Chapter 3 of [PS97]) put Hsiang and Pati's results in a more abstract setting, and instead of using repeated blowups to get Hsiang-Pati coordinates in standard form, they show that a certain type of resolution $\widetilde{U}$ is sufficient. However, Pardon and Stern choose the coordinates carefully using particular conditions involving transversality, genericity, minimality, and the Nash sheaf. The resolution $\pi\colon \widetilde{U} \to U$ used in [PS97] is taken to be any resolution of U that factors through the (normalization of) the Nash blowup and the blowup of the maximal ideal sheaf $\mathfrak{m}_v$ of the singular point v so that the exceptional set $E = \sum_i E_i$ is a divisor with normal crossings. Given a point $e \in E$ we can choose coordinates $\{u, v\}$ on an analytic neighborhood W in $\widetilde{U}$ centered at e so that, in W, E is either of the form $\{u = 0\} \cap \{v = 0\} = E_i \cap E_j$ (if e is a "double point" of the exceptional divisor where two components meet) or simply of the form $\{u = 0\} = E_i$ (if e is a "simple point").

A generic hyperplane $\widetilde{H}$ obtained by lifting the zero set $H := \{h = 0\}$ of a linear function $h\colon \mathbb{C}^N \to \mathbb{C}$ to $\widetilde{U}$ (more precisely, $\widetilde{H}$ is the proper transform of the intersection $H \cap U$ in $\widetilde{U}$) will satisfy the condition that $\operatorname{div}(h \circ \pi) = Z + \widetilde{H}$ (and that H meets E transversely at simple points; Pardon and Stern prove this in Proposition 3.6 of [PS97] by an arguments similar to the one given in the proof of Lemma 2.1 of [GS82]). Pardon and Stern use this generic hyperplane to choose linear functionals k and l that lift to functions

$$\phi := l \circ \pi \quad \text{and} \quad \psi := k \circ \pi$$

on $\widetilde{U}$. In particular, at points e that are contained in $\widetilde{H}$ (these are necessarily simple points), l is chosen to be h; at points $e \notin \widetilde{H}$, k is chosen to be h. After choosing l or k in this fashion, the remaining linear functional is chosen so that (after possibly perturbing the chosen l or k):

1. Over W, ϕ generates the inverse image $\pi^{-1}\mathfrak{m}_v(W)$ of the maximal ideal sheaf for v.

2. $\{d\phi,\, d\psi\}$ generates the Nash sheaf $\mathcal{N}_{\widetilde{U}}(W)$ over W.

It is possible to choose ϕ and ψ in this fashion due to a genericity argument (see the proof of Proposition 3.6 in [PS97]); we will be using an extension of this argument in Section 5.4).

The ϕ and ψ chosen as above turn out to be, with appropriate choice of local coordinates $\{u, v\}$ in W about e, Hsiang-Pati coordinates in W; in other words, we have the following.

Claim 2.5.2. *With $(\widetilde{U}, E)$, ϕ, and ψ chosen as described above, and a point $e \in E$ with analytic neighborhood W in $\widetilde{U}$, there exist integers m_i, m_j and n_i, n_j so that (in the case where e is a double point of E) near e (i.e. in W):*

- **(a)** $\phi = u^{m_i} v^{m_j}$, *where $Z := \sum_i m_i E_i$ is the divisor corresponding to $\pi^{-1}(\mathfrak{m}_v)$;*
- **(b)** $\psi = f(\phi) + \psi'$, *where $f(\phi) := \sum s_l \phi^{\epsilon_l}$ is a rational series in ϕ where each rational power ϵ_l is ≥ 1, and $\psi' = u^{n_i} v^{n_j}$;*
- **(d)** $m_i \leq n_i$, $m_j \leq n_j$, *and* $\left| \begin{smallmatrix} m_i & n_i \\ m_j & n_j \end{smallmatrix} \right| \neq 0$;
- **(e)** *near e, the metric $d\phi \wedge d\bar{\phi} + d\psi' \wedge d\bar{\psi}'$ is quasi-isometric to the pullback $\pi^* g$ of the standard metric g on $U - v$.*

Moreover, the n_i and n_j are minimal; if a linear function $\widehat{k}\colon \mathbb{C}^N \to \mathbb{C}$ lifts to a function $\widehat{\psi} := \widehat{k} \circ \pi$ of the form $\widehat{f}(\phi) + \widehat{\psi}'$ where $\widehat{f}$ is a rational series with powers ≥ 1 and $\widehat{\psi}' = u^{\widehat{n}_i} v^{\widehat{n}_j}$, and $\widehat{n}_i$, $\widehat{n}_j$ satisfy (d) above, then we have $\widehat{n}_i \geq n_i$ and $\widehat{n}_j \geq n_j$.

The case when e is a simple point of E is similar, with the m_j taken to be 0 and the n_j taken to be 1; in other words, $\phi = u^{m_i}$ and $\psi' = u^{n_i} v$.

Pardon and Stern then use these Hsiang-Pati coordinates (and the corresponding generators for the Nash sheaf) to define divisors $Z := \sum m_i E_i$, $N := \sum n_i E_i$, and $P := \sum p_i E_i$ supported on the exceptional divisor E. These divisors, and the monomial form of the generators for the Nash sheaf, enable them to construct an exact sequence that describes the Nash sheaf in terms of the resolution data Z, N, and P, as follows (see Proposition 3.20 and Remark 3.21 in [PS97]).

Proposition 2.5.3. *We have the following exact sequence of sheaves on $\widetilde{U}$:*

$$0 \rightarrow \mathcal{N} \rightarrow \mathcal{I}_E \Omega^1_{\widetilde{U}}(\log E) \otimes \mathcal{O}_{\widetilde{U}}(E - Z) \rightarrow \Omega^2_{\widetilde{U}} \otimes \mathcal{O}_{N-Z}(E - 2Z) \rightarrow 0.$$

In the above sequence, $N - Z$ is an effective divisor, and thus $\mathcal{O}(Z - N)$ is a subsheaf of $\mathcal{O}$; therefore the sheaf

$$\mathcal{O}_{N-Z} := \mathcal{O}/\mathcal{O}(Z - N)$$

is well-defined. Note that we are abusing notation by denoting the scheme associated to the effective divisor $N - Z$ simply by $N - Z$. We will use similar notation in Chapter 8.

2.5.3 Pati

Pati generalized (in [Pat94] and [Pat85]) the results in Hsiang and Pati's paper [HP85] to the case where V is a 3-dimensional variety with isolated singular point v. As in [HP85], Pati begins with a resolution map $\pi\colon \widetilde{U} \rightarrow U$ that blows up the singular point v and ensures that the exceptional set E is a divisor with normal crossings. The coordinate functions $z_1 \dots z_N$ pull up to functions

$$z_i = u^{a_i} v^{b_i} w^{c_i} \phi_i(u, v, w) \quad \text{for } 1 \leq i \leq N \tag{2.5.1}$$

on the resolution, where $\{u, v, w\}$ are coordinates in an analytic neighborhood of a point e of the exceptional divisor chosen so that, if e is a triple point of E, then $E = \{u = 0\} \cap \{v = 0\} \cap \{w = 0\}$; if e is a double point, $E = \{u = 0\} \cap \{v = 0\}$ (and the c_i are 0); and if e is a simple point, $E = \{u = 0\}$ (and the b_i, c_i above are all zero).

To put these coordinates in "standard" form (i.e. to make them "Hsiang-Pati" coordinates), Pati obtains a further resolution (which we will also call $\widetilde{U}$) through repeated blowups of "lists" of exponents, using what he calls "Type A" and "Type B" operations (blowups of points and lines, respectively, in various charts). After a series of such blowups, and various rescalings, reorderings, and coordinate changes, the coordinates can be put in the final form (in the triple point case):

$$\begin{aligned}
z_1 &= \xi_1 \\
z_2 &= f_2(\xi_1) + \xi_2 \\
z_3 &= f_3(\xi_1, \xi_2) + \xi_3 \\
z_4 &= f_4(\xi_1, \xi_2, \xi_3) \\
&\vdots \\
z_N &= f_N(\xi_1, \xi_2, \xi_3),
\end{aligned}$$

where $\xi_i = u^{a_i} v^{b_i} w^{c_i}$ and the determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

is nonzero (where the a_i, b_i, and c_i may be different exponents than the ones in equation 2.5.1). The functions f_i above are holomorphic functions of u, v, and w, and (see Proposition 2.2.11 in [Pat94]) consist of monomials $u^a v^b w^c$ whose exponents (a, b, c)

1. always dominate (a_1, b_1, c_1);

2. dominate (a_2, b_2, c_2) if (a, b, c) is linearly independent of (a_1, b_1, c_1);

3. dominate (a_3, b_3, c_3) if (a, b, c) is linearly independent of the set $\{(a_1, b_1, c_1), (a_2, b_2, c_3)\}$.

Moreover, these conditions ensure that the ξ_i-derivatives of the f_i are bounded holomorphic functions near e.

The double and simple point cases are similar to the above, with the following exceptions. If e is a double point of E then we take $c_1 = c_2 = 0$ and $c_3 = 1$ in the final form of the z_i; the determinant condition then becomes simply $a_1b_2 - a_2b_1 \neq 0$. The exponents (a, b, c) appearing in the monomials of the f_i satisfy similar conditions to those above, but with slightly different linear independence conditions (which can be obtained by considering the new c_i mentioned above). If e is a simple point then we likewise take $b_1 = b_3 = 0$, $b_2 = 1$ and $c_1 = c_2 = 0$,

$c_3 = 1$ in the above.

In Chapter 5 the notation above will become

$$\begin{aligned} \{m_i, m_j, m_k\} &= \{a_1, b_1, c_1\}, \\ \{n_i, n_j, n_k\} &= \{a_2, b_2, c_2\}, \\ \{p_i, p_j, p_k\} &= \{a_3, b_3, c_3\}; \end{aligned}$$

this new notation will make it easier for us to discuss these integers as sets of multiplicities on the components of the exceptional divisor, and thus define divisors $Z := \sum m_i E_i$, $N := \sum n_i E_i$, and $P := \sum p_i E_i$ on E.

The Main Proposition in this paper (see Sections 5.2, 5.5, and 5.6) will show that we can choose, from the outset, a resolution over which we can define Hsiang-Pati coordinates (the kind of resolution for which this is true will be termed "complete", and will be defined using the Nash blowup, the blowup of the maximal ideal sheaf, and the blowup of a certain Fitting ideal; see Section 3.1), and that we can find linear functionals that will pull up to Hsiang-Pati coordinates by choosing them so that they induce certain "minimal" generators of the Nash sheaf and its exterior powers.

2.6 Chern Classes for Singular Varieties

2.6.1 Chern classes for smooth complex manifolds

Given a complex vector bundle

$$\pi\colon \xi \longrightarrow M^n$$

of rank r over a smooth complex manifold M of dimension n, we can define Chern classes for ξ as follows (see Chapter 14 of [MS74]).

Definition 2.6.1. *The* Chern classes *of ξ are cohomology classes*

$$c_k(\xi) \in \mathrm{H}^{2k}(M)$$

satisfying the following four axioms:

1. *(Degree)* $c_0(\xi) = 1$ *and* $c_k(\xi) = 0$ *for* $k > r$*; define the* total Chern class *to be the class*

   $$c(\xi) := c_0(\xi) + c_1(\xi) + \ldots + c_r(\xi)$$

   in the cohomology ring $\mathrm{H}^*(M, \mathbb{Z})$*.*

2. *(Naturality)* $c(f^*\xi) = f^*c(\xi)$*, where* $f\colon N \to M$ *and* $f^*\xi$ *is the pullback of* ξ *to* N *by* f*.*

3. *(Whitney Sum)* $c(\xi \oplus \nu) = c(\xi)c(\nu)$*, where*

   $$c_i(\xi \oplus \nu) = \sum_{k=0}^{i} c_k(\xi) \cup c_{k-i}(\nu).$$

4. *(Normalization)* $c(\gamma^1) = 1 + a$, *where* a *is a generator of* $\mathrm{H}^2(\mathbb{CP}^1;\mathbb{Z})$ *and* γ^1 *is the canonical line bundle over* $\mathbb{CP}^1$.

The Chern class of a smooth complex manifold M *is defined to be the Chern class* $c(M) := c(TM)$ *of its tangent bundle.*

We wish to extend the notion of Chern classes to (possibly singular) complex algebraic varieties. Mather and MacPherson do this, in different ways, by defining classes in homology that in the smooth case are the duals of the usual Chern classes defined above. MacPherson's extension also satisfies axioms analogous to the naturality and Whitney sum axioms above. In this section we define and explore these extensions, and give two formulas for the MacPherson-Chern classes of a variety with isolated singular point.

2.6.2 Mather-Chern Classes

Let V be a (possibly singular) complex algebraic variety with Nash blowup $\pi\colon \widehat{V} \to V$ and Nash bundle $\mathfrak{N}_{\widehat{V}}$ as defined in Sections 2.2.1.1 and 2.2.2. Mather's extension of Chern classes to such a V is defined as follows (see Chapter 2 of [Mac74]).

Definition 2.6.2. *The* Mather-Chern class *of* V *is defined to be the pullback of the dual (in* $\widehat{V}$*) of the Chern class of the Nash bundle* $\mathfrak{N}_{\widehat{V}}$*:*

$$c^{\mathrm{M}}(V) := \pi_* \mathrm{Dual}\, c(\mathfrak{N}_{\widehat{V}}).$$

Note that in the case where V is smooth, we have $\widehat{V} \approx V$ and $\mathfrak{N}_{\widehat{V}} \approx TV$; thus in this case the Mather-Chern class is dual to the "usual" Chern class of V. We can extend Definition 2.6.2 to algebraic cycles (*i.e.* weighted sums of irreducible subspaces of V) $\sum_j n_j V_j$ by linearity, *i.e.* we define:

$$c^{\mathrm{M}}(\sum_j n_j V_j) := \sum_j n_j (i_j)_* c^{\mathrm{M}}(V_j),$$

where $i_j \colon V_j \hookrightarrow V$. This extension of Mather-Chern classes will be utilized in Section 2.6.6 below.

We can also use a generalized Nash blowup and bundle to define the Mather-Chern classes: if $\pi' \colon \widetilde{V} \to V$ is a generalized Nash blowup with generalized Nash bundle $\mathfrak{N}_{\widetilde{V}}$ (as defined in Sections 2.2.1.1 and 2.2.2), with factoring map $\widetilde{\pi} \colon \widetilde{V} \to \widehat{V}$, then we have:

$$\begin{aligned} \pi'_* \mathrm{Dual}\, c(\mathfrak{N}_{\widetilde{V}}) &= \pi_* \widetilde{\pi}_* \mathrm{Dual}\, c(\widetilde{\pi}^* \mathfrak{N}_{\widehat{V}}) \\ &= \pi_* \mathrm{Dual}\, c(\mathfrak{N}_{\widehat{V}}) \\ &= c^{\mathrm{M}}(V). \end{aligned}$$

Although they seem a natural extension of Chern classes to singular varieties (since the Nash bundle is the natural extension of the tangent bundle for such varieties), the Mather-Chern classes do not satisfy nice enough properties; for example, they don't satisfy an axiom analogous to naturality in Definition 2.6.1. The MacPherson-Chern classes defined in the next section have a more subtle definition (which can be expressed in terms of Mather-Chern classes) and will satisfy more of the desired axioms (in particular, naturality).

2.6.3 Definition of the MacPherson-Chern class

Let V be a compact complex algebraic variety (the material in this section also holds for non-compact varieties, with Borel-Moore (locally finite support) homology replacing homology, cohomology with compact support replacing cohomology, and all maps taken to be proper). We begin with a few definitions (taken from Chapter 1 of [Mac74]).

Definition 2.6.3. *A* constructible set *in V is a set obtained from the subvarieties of V by finitely many set-theoretic operations.*

Definition 2.6.4. *A* constructible function *on V is a function $\alpha\colon V \to \mathbb{Z}$ for which there exists a finite partition of V into constructible sets so that α is constant on each set in the partition.*

Given a subvariety $W \subset V$, we define $\mathbf{1}_W$ to be the constructible function whose value is 1 everywhere in W and 0 elsewhere, so for $p \in V$,

$$\mathbf{1}_W(p) := \begin{cases} 1, & p \in W \\ 0, & p \notin W \end{cases}.$$

MacPherson uses constructible functions to define a unique functor from the category $\mathfrak{Var}$ of compact complex algebraic varieties to the category of $\mathfrak{Ab}$ of abelian groups as follows (see Proposition 1 of [Mac74]).

Claim 2.6.5. *There is a unique covariant functor* $F\colon \mathfrak{Var} \to \mathfrak{Ab}$ *for which*

- **a.** *given* $V \in \mathfrak{Var}$, $F(V)$ *is the set of constructible functions on* V*;*
- **b.** *given a map* $f\colon V \to V'$, *the map* $F(f) =: f_*$ *satisfies*

$$f_*(\mathbf{1}_W)(p) = \chi(f^{-1}(p) \cap W),$$

for all subvarieties $W \subset V$, *where* χ *denotes the Euler characteristic.*

Given an arbitrary constructible function α on V we can define $f_*(\alpha)$ by linearly extending the definition above as follows. The constructible functions $\mathbf{1}_{W_i}$, where the $W_i \subset V$ range over all irreducible subvarieties of V, form a basis for the set of constructible functions $F(V)$ on V. Thus we can write α uniquely as

$$\alpha = \sum a_i W_i,$$

where the a_i are in $\mathbb{Z}$. We then can write, for $p \in V$,

$$\begin{aligned} f_*(\alpha)(p) &= f_*(\sum a_i W_i)(p) \\ &= \sum a_i f_*(\mathbf{1}_{W_i})(p) \\ &= \sum a_i \chi(f^{-1}(p) \cap W_i). \end{aligned}$$

In fact, given a sufficiently nice stratification $\mathfrak{S}$ of V consisting of constructible sets S_i, we have (see the proof of Proposition 1 in [Mac74])

$$f_*(\alpha)(p) = \sum_{S_i \in \mathfrak{S}} \chi_c(f^{-1}(p) \cap S_i),$$

where χ_c denotes Euler characteristic with compact support. The value of $f_*(\alpha)(p)$ is independent of the choice of stratification $\mathfrak{S}$.

Deligne and Grothendieck conjectured the following theorem, which MacPherson proved (see Theorem 1 in [Mac74]).

Theorem 2.6.6. *There exists a unique natural transformation $c_*\colon F \to \mathrm{H}_*$ from the constructible functions functor F to homology such that $c_*(\mathbf{1}_V) = \mathrm{Dual}\, c(V)$ for every smooth variety V.*

Using this theorem we can define the MacPherson-Chern class.

Definition 2.6.7. *Given $V \in \mathfrak{Var}$, we define the* MacPherson-Chern class *of V to be the image under the natural transformation c_* (given in Theorem 2.6.6) of the constructible function identifying V:*

$$c^{\mathrm{MP}} := c_*(\mathbf{1}_V).$$

The fact that c_* is a "natural transformation" means that, for each $V \in \mathfrak{Var}$, we have:

1. (Naturality) Given $\alpha \in F(V)$ and $f\colon V \to V'$, $f_* c_*(\alpha) = c_* f_*(\alpha)$; and
2. ("Whitney Sum") Given α, β in $F(V)$, $c_*(\alpha + \beta) = c_*(\alpha) + c_*(\beta)$.

Thus the MacPherson-Chern classes as defined above satisfy axioms analogous to (1) and (2) in 2.6.1; moreover, the MacPherson Chern class of a smooth variety V is by definition the dual of the "usual" Chern class of V.

2.6.4 A formula when V has an isolated singular point

We can immediately prove the following formula for the MacPherson-Chern class of a variety with isolated singular point.

Claim 2.6.8. *Let V be a variety with isolated singular point v and resolution $\pi\colon (\widetilde{V}, E) \to (V, v)$. Then:*

$$c^{\mathrm{MP}}(V) = \pi_* \mathrm{Dual}\, c(\widetilde{V}) + (1 - \chi(E)).$$

Proof. By the definition of π_* in Claim 2.6.5, we have for $p \in V$ (since the inverse image under π of a point $p \neq v$ is a single point in $\widetilde{V}$):

$$\pi_*(\mathbf{1}_{\widetilde{V}})(p) \;=\; \chi(\pi^{-1}(p) \cap \widetilde{V}) \;=\; \begin{cases} \chi(E), & p = v \\ \chi(\pi^{-1}(p)) = 1, & p \in V - v. \end{cases}$$

Thus $\pi_*(\mathbf{1}_{\widetilde{V}}) = \chi(E)\mathbf{1}_v + \mathbf{1}_{V-v}$; since $\mathbf{1}_V = \mathbf{1}_{V-v} + \mathbf{1}_V$, we thus have

$$\mathbf{1}_V = \pi_*(\mathbf{1}_{\widetilde{V}}) + (1 - \chi(E))\mathbf{1}_v.$$

Therefore, using Theorem 2.6.6 and the fact that the resolution $\widetilde{V}$ and point v are smooth varieties,

$$\begin{aligned} c^{\mathrm{MP}}(V) &= c_*(\mathbf{1}_V) \\ &= c_*\pi_*(\mathbf{1}_{\widetilde{V}}) + c_*((1 - \chi(E))\mathbf{1}_v) \\ &= \pi_*\mathrm{Dual}\, c(\widetilde{V}) + (1 - \chi(E))\mathrm{Dual}\, c(v) \\ &= \pi_*\mathrm{Dual}\, c(\widetilde{V}) + (1 - \chi(E))[v]. \end{aligned}$$

■

In Chapter 9 we will need the following easy corollary to Claim 2.6.8.

Corollary 2.6.9. *The zeroth MacPherson-Chern class of a variety V with isolated singular point v and resolution $\pi\colon (\widetilde{V}, E) \to (V, v)$ is:*

$$c_0^{\mathrm{MP}}(V) = \pi_* \mathrm{Dual}\, c_n(\widetilde{V}) + (1 - \chi(E)).$$

2.6.5 The local Euler obstruction

In order to obtain another formula for the MacPherson-Chern classes (for use in Chapter 9), we must first discuss how the MacPherson-Chern classes (and hence the transformation c_*) can be constructed from the Mather-Chern classes. To do this we must define a numerical invariant called the "local Euler obstruction" of V at p. Since the construction will be entirely local, we will consider a (possibly singular) variety V and a neighborhood U of a point p in V chosen small enough so as to have an embedding $i\colon (U, p) \hookrightarrow (\mathbb{C}^N, 0)$ into some affine space with coordinates $\{z_1, \ldots, z_N\}$ centered at $0 = i(p)$. (Clearly U is not compact here; as stated earlier, assume all homology is Borel-Moore and all cohomology has compact support. Alternatively we could find an embedding of all of V into some $\mathbb{C}^N$ and go from there.) We take most of this discussion from Chapter 4 of [GS81] and Chapter 3 of [Mac74].

Suppose $\pi\colon \widetilde{U} \to U$ is a resolution of U factoring through the Nash blowup, and let $\mathfrak{N}_{\widetilde{U}}$ be the (generalized) Nash bundle on $\widetilde{U}$. Define the function

$$\begin{aligned} \phi : \mathbb{C}^N &\longrightarrow \mathbb{R} \\ z &\longmapsto \|z\|^2 = \sum_{i=1}^{N} z_i \bar{z}_i \end{aligned}$$

from the distance function centered at the origin. Since ϕ is real-valued, $d\phi$ is a real differential form on $\mathbb{C}^N$, *i.e.* a section of the (real) cotangent bundle $T^*\mathbb{C}^N$. Let $r := \pi^* i^* d\phi$ be the pullback of this section to a section of the dual Nash bundle $\mathfrak{N}^*_{\widetilde{U}}$ over $\widetilde{U}$.

Now let B_ϵ be a ball of radius ϵ about 0 in $\mathbb{C}^N$, and let S_ϵ be the sphere bounding B_ϵ. We will need the following lemma:

Lemma 2.6.10. *For all small enough ϵ, r is a nowhere-vanishing section over $(i \circ \pi)^{-1}(B_\epsilon - \{0\}) \subset \widetilde{U}$.*

See Section 4.1 of [GS81] for a proof of the lemma; basically it involves using a stratification of U satisfying Whitney Condition A (see Section 1.2 of [GM80]) which insures that we can make sufficiently small spheres about p transverse to the strata (thus making the distance function on a small punctured neighborhood of p transverse to the strata of U). For the remainder of this section we will denote the composition $i \circ \pi$ simply as π. and denote the dimension of V by n.

Define the cohomology class

$$Eu(\mathfrak{N}^*_{\widetilde{U}}, r) \in \mathrm{H}^{2n}(\pi^{-1}(B_\epsilon), \pi^{-1}(S_\epsilon); \mathbb{Z})$$

to be the obstruction to extending the nowhere-vanishing section r on $\pi^{-1}(S_\epsilon)$ (insured by the lemma above) to a nowhere-vanishing section on all of $\pi^{-1}(B_\epsilon)$. (See Chapters 12 and 14 of [MS74]; we are in the $2n$ level of cohomology because we are measuring the obstruction to extending a section over an real $(2n-1)$-dimensional space to a section over an real $2n$-dimensional (complex n-dimensional) space, as is the case with the Euler class.) We are now in a position to define the local Euler obstruction of U at p.

Definition 2.6.11. *With notation as above, the* local Euler obstruction *of U at p is defined to be*

$$Eu_p(U) := Eu(\mathfrak{N}^*_{\widetilde{U}}, r) \cap [\pi^{-1}(B_\epsilon), \pi^{-1}(S_\epsilon)],$$

i.e. the evaluation of the obstruction class defined above on the corresponding fundamental orientation class in $\mathrm{H}_{2n}(\pi^{-1}(B_\epsilon), \pi^{-1}(S_\epsilon); \mathbb{Z})$.

It is important to note that (as MacPherson points out; see Section 3 of [Mac74]) if U is smooth at p, then $Eu_p(U) = 1$. Moreover (in general), $Eu_p(U)$ defines a constructible function on U (see Lemma 4 of [Ken90]).

To finish this discussion we present an algebraic formula for the local Euler obstruction (at $p = v$) in the case where U has isolated singular point v (see [GS81], Section 4.3 where this is done for general U). We will use this formula in Chapter 9.

Let V be a variety with isolated singular point v (and neighborhood U of v in V), and choose a resolution $\pi\colon (\widetilde{U}, E) \to U$ factoring through the Nash blowup and the blowup of the maximal ideal $\mathfrak{m}_v$. Let ξ denote the line bundle over E corresponding to the divisor obtained from the inverse image of the maximal ideal, and let $\mathfrak{N}$ denote the (generalized) Nash bundle on $\widetilde{U}$.

Claim 2.6.12. *With notation as above, the local Euler obstruction at the singular point v of U is given by the formula*

$$Eu_v(U) = \pi_*(c_{n-1}(\mathfrak{N} - \xi) \cap [E]),$$

where $c_{n-1}(\mathfrak{N} - \xi) = (c(\mathfrak{N})/c(\xi))_{n-1}$ and $\dim(U) = n$.

2.6.6 MacPherson classes from Mather classes

MacPherson-Chern classes are built out of Mather-Chern classes; indeed, we will use the Mather-Chern classes to construct a natural transformation c_* that satisfies the properties required by Theorem 2.6.6. To this end we first define an isomorphism between algebraic cycles and constructible functions on a variety V, namely (see Lemma 2 in [Mac74]):

$$\begin{aligned} T:\ \text{Cycles(V)} &\longrightarrow F(V) \\ \sum \alpha_i V_i &\longmapsto T(\sum \alpha_i V_i), \end{aligned}$$

where we define, for $p \in V$,

$$T(\sum \alpha_i V_i)(p) := \sum \alpha_i Eu_p(V_i).$$

We now use this isomorphism and the extension of Mather-Chern classes to algebraic cycles to construct the desired natural transformation c_* (see Theorem 2 in [Mac74]).

Theorem 2.6.13. *The transformation*

$$c^{\mathrm{M}} \circ T^{-1} \colon F(V) \to \mathrm{Cycles}(V) \to \mathrm{H}_*(V)$$

satisfies the conditions in Theorem 2.6.6, and thus is equal to the transformation c_**. Hence the MacPherson-Chern class of a variety* V *is given by*

$$c^{\mathrm{MP}}(V) = c_*(\mathbf{1}_V) = c^{\mathrm{M}}(T^{-1}(\mathbf{1}_V)).$$

To effectively use this theorem to compute the MacPherson-Chern classes of a variety V in terms of the Mather-Chern classes, we must first write $\mathbf{1}_V$ as a linear combination of local Euler obstructions

$$\mathbf{1}_V = \sum \alpha_i Eu_p(V_i)$$

on the irreducible subvarieties V_i of V. Once we do this we have

$$c^{\mathrm{MP}}(V) = c^{\mathrm{M}}(\sum \alpha_i V_i) = \sum \alpha_i c^{\mathrm{M}}(V_i)$$

(where we omit the inclusion maps $V_i \hookrightarrow V$ for simplicity). In fact, we can write $c^{\mathrm{MP}}(V)$ as a weighted sum of Mather-Chern classes with respect to some stratification $\mathfrak{S}$ of V:

$$c_j^{\mathrm{MP}}(V) = \sum_{\dim S \geq j} \theta_S(V)\, c_j^{\mathrm{M}}(\bar{S}), \tag{2.6.1}$$

where the θ_s are constructible functions with respect to the strata in $\mathfrak{S}$ (*i.e.* are locally constant along the strata). Note that this formula illustrates that c^{MP} and c^{M} can be very different when V is a non-smooth variety (and the strata are thus nontrivial).

2.6.7 Another isolated singular point formula

We now use the local Euler obstruction and the Mather-Chern class to prove another formula for the MacPherson-Chern classes of a variety V with isolated singular point v.

Claim 2.6.14. *Let V be a variety with isolated singular point v and resolution $\pi\colon (\widetilde{V}, E) \to (V, v)$. Then:*

$$c^{\mathrm{MP}}(V) = c^{\mathrm{M}}(V) + (1 - Eu_v(V)).$$

Proof. First we write $\mathbf{1}_V$ in terms of local Euler obstructions; since V has isolated singular point v we can stratify V by $\mathfrak{S} = \{v,\, V - v\}$. The local Euler obstructions $Eu_p(v)$ and $Eu_p(V - v)$ on these strata are constant since v and $V - v$ are smooth (and thus each of these local Euler obstructions are constantly 1 on v, $V - v$

respectively). At $p \in V$ we have

$$\begin{aligned}
\mathbf{1}_V(p) &= Eu_p(V) + Eu_p(V - v) \\
&= Eu_p(v) + Eu_p(V) - Eu_v(V)\, Eu_p(v) \\
&= (1 - Eu_v(V))\, Eu_p(v) + Eu_p(V),
\end{aligned}$$

and thus

$$\begin{aligned}
c^{\mathrm{MP}}(V) &= c_*(\mathbf{1}_V) \\
&= c^{\mathrm{M}}(T^{-1}(\mathbf{1}_V)) \\
&= c^{\mathrm{M}}((1 - Eu_v(V)) \cdot [v] + 1 \cdot V) \\
&= (1 - Eu_v(V))c^{\mathrm{M}}(v) + c^{\mathrm{M}}(V) \\
&= c^{\mathrm{M}}(V) + (1 - Eu_v(V)),
\end{aligned}$$

where the last line follows because v and $V - v$ are smooth. ■

Comparing this formula with the one in (2.6.1) we have

$$\theta_{V-v}(V) = 1 \quad \text{and} \quad \theta_v(V) = 1 - Eu_v(V),$$

which are clearly locally constant along strata.

Finally, we present the following simple corollary to Claim 2.6.14 (for later use in Chapter 9).

Corollary 2.6.15. *The zeroth MacPherson-Chern class of a variety V with isolated singular point v and resolution $\pi\colon (\widetilde{V}, E) \to (V, v)$ with (generalized) Nash* *bundle* $\mathfrak{N}$ *on* $\widetilde{V}$ *is:*

$$c_0^{\mathrm{MP}}(V) = \pi_* \mathrm{Dual}\, c_n(\mathfrak{N}) + (1 - Eu_v(V)).$$

Proof. By definition we have $c_0^{\mathrm{M}}(V) = \pi_* \mathrm{Dual}\, c_n(\mathfrak{N})$. ■

Chapter 3

Complete Resolutions

3.1 First Steps to a Complete Resolution

Let (U, v) be a neighborhood of an isolated singular point v in a complex 3-dimensional algebraic variety V, small enough so that we have an embedding $(U, v) \subset (\mathbb{C}^N, 0)$. We are interested in finding a resolution of U over which a careful choice of linear functions will produce Hsiang-Pati coordinates. To this end we define the notion of a *complete* resolution.

Definition 3.1.1. *We will call a resolution* $\pi\colon (\widetilde{U}, E) \to (U, v)$ *of an isolated singular point* $v \in U$ complete *if it satisfies the following three conditions:*

(i) *The sheaf-theoretic inverse image* $\pi^{-1}(\mathfrak{m}_v)$ *of the maximal ideal sheaf* $\mathfrak{m}_v$ *by the map* π *is locally principal on* $\widetilde{U}$*;*

(ii) *The generalized Nash sheaf* $\mathcal{N}_{\widetilde{U}}$ *is locally free on* $\widetilde{U}$*; and*

(iii) *The second Fitting ideal* $Fitt_2(\alpha)$ *corresponding to the inclusion* $\alpha\colon \mathcal{N}_{\widetilde{U}} \hookrightarrow \Omega^1_{\widetilde{U}}(\log E)$ *is a locally principal sheaf of ideals on* $\widetilde{U}$*.*

Recall from Section 2.3.3 that, given bases for $\mathcal{N}_{\widetilde{U}}$ and $\Omega^1_{\widetilde{U}}(\log E)$, the sheaf $Fitt_2(\alpha)$ is locally generated by the 2×2 subdeterminants of the matrix representing the inclusion α. When convenient we will denote the Fitting invariant $Fitt_2(\alpha)$ simply by $Fitt_2$. We could also think of $Fitt_2$ as the ideal sheaf generated by the entries of the matrix representing the inclusion $\Lambda^2(\alpha)\colon \Lambda^2\mathcal{N}_{\widetilde{U}} \to \Omega^2_{\widetilde{U}}(\log E)$ (see the computation in Section 2.3.2).

To construct a complete resolution of a variety with isolated singular point (U, v), we first construct a resolution that factors through the blowups of the maximal ideal and the Nash sheaf. Consider the diagram:

$$\begin{array}{ccccc} (\breve{U}, \breve{E}) & \xrightarrow{\breve{\pi}} & (\check{U}, \check{E}) & \xrightarrow{\check{\pi}_1} & (\widehat{U}, \widehat{E}) \\ & & {\scriptstyle \check{\pi}_0} \downarrow & & {\scriptstyle \pi_1} \downarrow \\ & & (Bl(U), C) & \xrightarrow{\pi_0} & (U, v) \end{array}$$

In this diagram, $\pi_0\colon Bl(U) \to U$ is the blowup of the maximal ideal $\mathfrak{m}_v$ corresponding to the isolated singular point $v \in U$; thus $\mathfrak{m}_v\mathcal{O}_{Bl(U)}$ is locally principal, and will remain locally principal under the further blowups $\check{\pi}_0$, and $\breve{\pi}$, so $\mathfrak{m}_v\mathcal{O}_{\widetilde{U}}$ is locally principal.

The map $\pi_1\colon \widehat{U} \to U$ denotes the blowup of the sheaf Ω^1_U of 1-forms on U; in other words, $\widehat{U}$ is the Nash blowup of U, and thus the Nash sheaf $\mathcal{N}_{\widehat{U}}$ is locally free on $\widehat{U}$ (and remains so under the further blowups). Thus $\mathcal{N}_{\widetilde{U}}$ is a locally free sheaf on $\widetilde{U}$.

We complete the square with the blowup $\breve{\pi}_1 \colon \breve{U} \to \widehat{U}$ of $\pi_1^{-1}(v)$ and the unique map $\breve{\pi}_0 \colon \breve{U} \to Bl(U)$ that makes the diagram commute (see Section 2.1 in [Tra79]). We then desingularize $(\breve{U}, \breve{E})$ by the map $\breve{\pi} \colon (\breve{U}, \breve{E}) \to (\breve{U}, \breve{E})$.

Now consider the resolution $\pi \colon (\breve{U}, \breve{E}) \to (U, v)$, where

$$\pi := \pi_0 \circ \breve{\pi}_0 \circ \breve{\pi} = \pi_1 \circ \breve{\pi}_1 \circ \breve{\pi}.$$

If $Fitt_2(\breve{\alpha})$ is a locally principal sheaf of ideals on $\breve{U}$, then $\breve{U}$ is a complete resolution and we are done. If $Fitt_2(\breve{\alpha})$ is not locally principal, we must find a further resolution $\widetilde{\pi} \colon (\widetilde{U}, E) \to (\breve{U}, \breve{E})$ for which $Fitt_2(\widetilde{\alpha})$ is locally principal. Clearly we can find a resolution $\widetilde{\pi} \colon \widetilde{U} \to \breve{U}$ for which the inverse image ideal sheaf $\widetilde{\pi}^{-1}(Fitt_2(\breve{\alpha}))$ is locally principal (for example we could take any resolution factoring through the blowup of the Fitting ideal $Fitt_2(\breve{\alpha})$). However it may not necessarily be true that this inverse image of the Fitting ideal is isomorphic to the Fitting ideal for $\widetilde{U}$; in other words, it is not immediately obvious that we would have:

$$\widetilde{\pi}^{-1}(Fitt_2(\breve{\alpha})) \approx Fitt_2(\widetilde{\alpha}),$$

and thus not obvious that the Fitting ideal $Fitt_2(\widetilde{\alpha})$ would be locally principal on $\widetilde{U}$. Obtaining a resolution $\widetilde{U}$ for which the Fitting ideal is locally principal will consist of three steps. First, we will first show that the Fitting ideal of any resolution factoring through the Nash blowup and the blowup of the maximal ideal must be locally principal at simple points of the exceptional divisor (Section 3.2). Second, we will then show that blowups of certain curves and points that do not intersect the simple point set "preserve" the Fitting ideal, in the sense that for blowups of these types, the inverse image of the "downstairs" Fitting ideal is locally equal to the "upstairs" Fitting ideal (see Section 3.3). Finally, we will show that, after a preliminary blowup, there exists a finite sequence of such Fitting ideal preserving blowups that results in a resolution $\widetilde{U}$ of $\breve{U}$ that factors through the blowup of the Fitting ideal $Fitt_2(\breve{\alpha})$ (see Section 3.4). With such a resolution $\widetilde{U}$, the inverse image $\widetilde{\pi}^{-1}(Fitt_2(\breve{U}))$ will be both locally principal and equal to the desired Fitting ideal $Fitt_2(\widetilde{\alpha})$.

3.2 Simple Points

Let $\pi\colon (\breve{U}, \breve{E}) \to (U, v)$ be any resolution of (U, v) that factors through both the blowup of the maximal ideal sheaf and the Nash blowup. Let $Fitt_2 = Fitt_2(\breve{\alpha})$ denote the Fitting ideal that is locally generated by the 2×2 subdeterminants of the matrix locally representing the inclusion $\breve{\alpha}\colon \mathcal{N}_{\breve{U}} \to \Omega^1_{\breve{U}}(\log \breve{E})$ of the Nash sheaf into the sheaf of logarithmic 1-forms on $\breve{U}$. The remainder of this section will be dedicated to proving the following lemma:

Lemma 3.2.1. *Let $(\breve{U}, \breve{E})$ and $Fitt_2$ be as described above. Given a simple point $e \in \breve{E}$, and an analytic neighborhood W of e in $\breve{U}$, the Fitting ideal $Fitt_2(W)$ is principal.*

Proof. Given a simple point e with analytic neighborhood W, we begin by choosing bases for $\mathcal{N}_{\breve{U}}(W)$ and $\Omega^1_{\breve{U}}(\log \breve{E})(W)$. When convenient we will suppress the W in the notation that follows (although we will always be working locally, in W). With local coordinates $\{u, v, w\}$ about e in $\breve{U}$, where $\{u = 0\}$ defines the exceptional divisor E in W, we take the standard basis $\{\frac{du}{u}, dv, dw\}$ for $\Omega^1_{\breve{U}}(\log E)$.

Let ϕ, ψ, and ρ be functions in $\mathcal{O}_{\breve{U}}(W)$ with the following two properties: $\{d\phi, d\psi, d\rho\}$ generates $\mathcal{N}_{\breve{U}}(W)$, and ϕ generates $\pi^{-1}(\mathfrak{m}_v)$. Functions ϕ, ψ, and ρ satisfying the first property are ensured by the arguments given in the proof of part (b) of Proposition 4.1.1; in fact, we can ensure that ϕ, ψ, and ρ are the pullbacks of linear functions j, k, and l, respectively, on U (so $\phi = \pi \circ j$, *et cetera*). By a similar argument (see the proof of part (a) of Proposition 4.1.1) we can find a function ϕ' satisfying the second property, where $\phi' = \pi \circ h$ for some linear function h on U. Since such choices for ϕ, ψ, and ρ, and for ϕ', are generic (again, see Proposition 4.1.1), we can choose these functions so that $\phi' = \phi$ (see in particular Lemma 4.3.4). Given such ϕ, ψ, and ρ, we take the basis of $\mathcal{N}_{\breve{U}}$ to be $\{d\phi, d\psi, d\rho\}$.

Under these bases, the Fitting ideal $Fitt_2 = Fitt_2(\breve{\alpha})$ is given by the 2×2

determinants of the matrix:

$$\begin{pmatrix} u\phi_u & u\psi_u & u\rho_u \\ \phi_v & \psi_v & \rho_v \\ \phi_w & \psi_w & \rho_w \end{pmatrix}.$$

Note that since we are at a simple point $e \in \breve{E}_i = \{u = 0\}$, ϕ vanishes only along $u = 0$. In other words, $\phi = u^{m_i}\,\mu$ for some local unit μ. We can absorb this unit into the coordinate u (by the coordinate change $u \mapsto u\,\mu^{-1/m_i}$); under the new coordinates we have $\phi = u^{m_i}$. Thus $u\phi_u = m_i u^{m_i}$, $\phi_v = 0$, and $\phi_w = 0$. Using this and the definition of $Fitt_2$ we thus have:

$$\begin{aligned} Fitt_2 &= \langle m_i u^{m_i}\psi_v,\, m_i u^{m_i}\psi_w,\, 0,\, m_i u^{m_i}\rho_v,\, m_i u^{m_i}\rho_w,\, 0, && (3.2.1)\\ &\qquad u\psi_u\rho_v - u\rho_u\psi_v,\, \psi_v\rho_w - \rho_v\psi_w,\, u\psi_u\rho_w - u\rho_u\psi_w\rangle. && (3.2.2) \end{aligned}$$

We will show this Fitting ideal is principal by putting the functions ψ and ρ in a "good form", and then showing that such "good" functions produce a principal Fitting ideal. This proof is basically equivalent to showing that we can find coordinates $\{u, v, w\}$ on $\breve{U}$ so that ϕ, ψ, and ρ are Hsiang-Pati coordinates at the simple point e, as in Proposition 5.6.2 (although the method of proof is slightly different). Specifically, we will show that we can choose coordinates $\{u, v, w\}$ on W, with $\breve{E}_i = \{u = 0\}$, so that ϕ, ψ, and ρ are in the form:

$$\begin{aligned} \phi &= u^{m_i} \\ \psi &= S + u^{n_i} v \\ \rho &= T + u^{n_i} w, \end{aligned}$$

for some S and T that will end up not being involved in the final form of the Fitting ideal.

We begin by showing that ψ can be written in the form $S + u^{n_i}R$ for some S with $d\phi\, dS = 0$ and some function R that can be taken to be a coordinate independent of the coordinate u. *A priori* we can write ψ as a series

$$\psi = \sum_{(a,b,c)} r_{(a,b,c)} u^a v^b w^c.$$

We first separate out the terms $u^a v^b w^c$ for which (a, b, c) is dependent on $(m_i, 0, 0)$; in other words we will collect all the terms in which $b = 0$ and $c = 0$. Denote the sum of such terms by S; clearly we have $d\phi\, dS = 0$. The remaining terms now have exponents (a, b, c) that are linearly independent of $(m_i, 0, 0)$ (a property we will now denote by $\star_m$). Define the integer n_i to be the minimum of all the a's with $(\star_m)$, *i.e.* :

$$n_i := \min_{\substack{(a,b,c) \\ r \neq 0, \star_m}} (a).$$

Note that $n_i \geq m_i$ since ϕ generates $\pi^{-1}\Updownarrow_v$ and thus ϕ divides ψ. Using this we can write:

$$\begin{aligned} \psi &= S + \sum_{\substack{(a,b,c) \\ \star_m}} r_{(a,b,c)} u^a v^b w^c \\ &= S + u^{n_i} \sum_{\substack{(a,b,c) \\ \star_m}} r_{(a,b,c)} u^{a-n_i} v^b w^c \\ &=: S + u^{n_i} R, \end{aligned}$$

where R is defined to be the sum left after factoring out the u^{n_i}.

We must now show that R is a coordinate independent of u; it suffices to prove that $du\,dR$ is a nowhere-vanishing 2-form, and we prove that now. We begin by calculating:

$$\begin{aligned} d\phi\,d\psi &= d\phi\,dS + u^{n_i}\,d\phi\,dR + R\,d\phi\,d(u^{n_i}) \\ &= 0 + u^{n_i}\,d\phi\,dR + 0 \\ &= m_i u^{m_i+n_i-1} du\,dR. \end{aligned}$$

Since $d\phi$ and $d\psi$ are two of the generators of $\mathcal{N}_{\breve{U}}$, the wedge product $d\phi\,d\psi$ is a generator of $\Lambda^2\mathcal{N}_{\breve{U}}$. As above, there is an inclusion $\alpha\colon \mathcal{N}_{\breve{U}} \to \Omega^1_{\breve{U}}(\log \breve{E})$ of the Nash sheaf into the sheaf of logarithmic 1-forms on $\breve{U}$; outside of $\breve{E}$ this inclusion is in fact an isomorphism. The analogous fact is also true for the second exterior power of the Nash sheaf. Thus the 2-form $d\phi\,d\psi$ vanishes only along $\breve{E}_i = \{u = 0\}$, and by the computation above we see that $du\,dR$ must be of the form $u^{l_i}\omega$ for some integer l_i and nowhere-vanishing 2-form ω. It now suffices to prove that this exponent l_i is in fact zero, *i.e.* that u does not divide $du\,dR$.

Since $du\,dR = R_v\,du\,dv + R_w\,du\,dw$, it will suffice to prove that u cannot divide both R_v and R_w. Seeking a contradiction, suppose that u divides both R_v and R_w. By the definition of R we have

$$R_v = \sum_{\substack{(a,b,c)\\ \star_m}} r_{(a,b,c)}(b)u^{a-n_i}v^{b-1}w^c \quad \text{and} \quad R_w = \sum_{\substack{(a,b,c)\\ \star_m}} r_{(a,b,c)}(c)u^{a-n_i}v^{b}w^{c-1}.$$

Thus the fact that u divides R_v implies that $a - n_i > 0$ for all (a, b, c) with $b > 0$

(and $r_{(a,b,c)} \neq 0$). Similarly, u dividing R_w implies that $a - n_i > 0$ for all (a, b, c) with $c > 0$. But by definition R cannot have any pure u terms (else those terms, times u^{n_i}, would have been in S); hence all triples (a, b, c) appearing in R have either $b > 0$ or $c > 0$. Thus the implications above show that we must have $a - n_i > 0$ for all such a; in other words, u must divide R (which contradicts the definitions of R and n_i).

We have now successfully shown that R must be a coordinate independent of the coordinate u. Thus we can change coordinates by setting $v = R$ (note that this will not affect ϕ, since ϕ involves only the u coordinate); with these new coordinates we now have $\phi = u^{m_i}$ and $\psi = S + u^{n_i}v$, as desired.

To get the functions ϕ, ψ, and ρ in "good form" we need only fix the function ρ. We begin by separating out the terms of ρ that are dependent on ϕ and $u^{n_i}v$, and factoring out the highest possible power of u from the remaining terms, as follows:

$$\begin{aligned}
\rho &= \sum_{(\alpha,\beta,\gamma)} r_{(\alpha,\beta,\gamma)} u^{\alpha} v^{\beta} w^{\gamma} \\
&= T + \sum_{\substack{(\alpha,\beta,\gamma) \\ \star\{m,n\}}} r_{(\alpha,\beta,\gamma)} u^{\alpha} v^{\beta} w^{\gamma} \\
&= T + u^{p_i} \sum_{\substack{(\alpha,\beta,\gamma) \\ \star\{m,n\}}} r_{(\alpha,\beta,\gamma)} u^{\alpha - p_i} v^{\beta} w^{\gamma} \\
&=: T + u^{p_i} \widehat{R},
\end{aligned}$$

where T collects all the terms in ρ where (α, β, γ) is linearly dependent on the set

$\{(m_i, 0, 0), (n_i, 1, 0)\}$ (and we say that the remaining terms satisfy the condition $\star_{\{m,n\}}$). Note that u^{m_i} divides both T and u^{p_i}. The integer p_i is defined here as

$$p_i := \min_{\substack{(\alpha,\beta,\gamma) \\ r\neq 0, \star_{\{m,n\}}}} (\alpha),$$

and $\widehat{R}$ is defined to be what remains from the $(\star_{\{m,n\}})$ terms of ρ after factoring out u^{p_i}.

With this notation, it now suffices to show that $\widehat{R}$ is a local coordinate independent of u and v; in other words, we must show that $du\, dv\, dR$ is a nowhere-vanishing 3-form on $\breve{U}$. The argument will be similar to the argument given above for R. We first note that if we define

$$\psi' := u^{n_i} v \qquad \text{and} \qquad \rho' := u^{p_i} \widehat{R},$$

then $d\phi\, d\psi\, d\rho = d\phi\, d\psi'\, d\rho'$, by the definitions of S and T. Thus

$$\begin{aligned} d\phi\, d\psi\, d\rho &= d\phi\, d\psi'\, d\rho' \\ &= d\phi(u^{n_i} dv + v d(u^{n_i}))(u^{p_i} d\widehat{R} + \widehat{R} d(u^{p_i})) \\ &= d\phi(u^{n_i} dv)(u^{p_i} d\widehat{R}) \\ &= m_i u^{m_i+n_i+p_i-1} du\, dv\, d\widehat{R}, \end{aligned}$$

since $d\phi\, d(u^l) = 0$ for any power l. Because $d\phi$, $d\psi$, and $d\rho$ generate the Nash sheaf, the triple wedge $d\phi\, d\psi\, d\rho$ vanishes only along $\breve{E}_i = \{u = 0\}$. Thus by the calculation above, $du\, dv\, d\widehat{R}$ is of the form $u^{l_i}\omega$ for some integer l_i and nowhere-vanishing 3-form ω. To show that $\widehat{R}$ is a local coordinate (independent of u and

v), it now suffices to show that this l_i is zero, *i.e.* that $du\,dv\,d\widehat{R}$ is not divisible by u.

Seeking a contradiction, suppose that u divides $du\,dv\,d\widehat{R}$. Since $du\,dv\,d\widehat{R} = \widehat{R}_w du\,dv\,dw$, we thus have u dividing $\widehat{R}$. By the definition of $\widehat{R}$ we have

$$\widehat{R}_w = \sum_{\substack{(\alpha,\beta,\gamma)\\ \star_{\{m,n\}}}} r_{(\alpha,\beta,\gamma)}(\gamma) u^{\alpha-p_i} v^{\beta} w^{\gamma-1},$$

so if u divides $\widehat{R}_w$ then $\alpha - p_i > 0$ for all (α,β,γ) with $(\star_{\{m,n\}})$, $r_{(\alpha,\beta,\gamma)} \not\equiv 0$, and $\gamma > 0$. However, by the definition of T, all of the γ in triples satisfying $(\star_{\{m,n\}})$ are strictly greater than zero (since any triple $(\alpha,\beta,0)$ is linearly dependent on $\{(m_i,0,0),(n_i,1,0)\}$). Thus all of the $\alpha - p_i$ must be strictly greater than zero, and hence u divides $\widehat{R}$ (which contradicts the definitions of $\widehat{R}$ and p_i).

We have now shown that $\widehat{R}$ must be a local coordinate independent of u and v; thus we can change coordinates by setting $w = \widehat{R}$ (note this will not affect ϕ and ψ). We now have coordinates $\{u,v,w\}$ on W so that the functions ϕ, ψ, and ρ are of the form:

$$\begin{aligned} \phi &= u^{m_i} \\ \psi &= S + u^{n_i} v \\ \rho &= T + u^{n_i} w \end{aligned}$$

Note that by the definitions of S and T we can write

$$S = \sum_l s_l \phi^l \qquad \text{and} \qquad T = \sum_{l,k} t_{(l,k)} \phi^l (\psi')^k,$$

and that $S_v = 0$, $S_w = 0$, and $T_w = 0$.

To complete the proof of Lemma 3.2.1 we now show that with ϕ, ψ, and ρ as above, the Fitting ideal (see 3.2.1) is principal in W. With bases chosen as at the start of this proof, the inclusion $\breve{\alpha}\colon \mathcal{N}_{\breve{U}} \to \Omega^1_{\breve{U}}(\log \breve{E})$ is locally represented by the logarithmic Jacobian matrix

$$\begin{pmatrix} m_i u^{m_i} & uS_u + n_i u^{n_i} v & uT_u + p_i u^{p_i} w \\ 0 & u^{n_i} & T_v \\ 0 & 0 & u^{p_i} \end{pmatrix}.$$

The Fitting ideal $Fitt_2 = Fitt_2(\alpha)$ is thus generated by the 2×2 determinants of this matrix. By the definitions of S and T, and the fact that $m_i \leq n_i$ and $m_i \leq p_i$ (since ϕ generates the maximal ideal sheaf $\pi^{-1}\mathfrak{m}_v$), we obtain the following simple facts concerning divisibility:

$$u^{m_i} \mid uS_u; \qquad u^{m_i} \mid uT_u; \qquad u^{n_i} \mid T_v.$$

We briefly discuss the last of the three facts above. Note that T can be split into pure u terms (terms with $k = 0$) and terms that involve v (*i.e.* terms with $k > 0$). The former set of terms all vanish in T_v, and the latter set of terms (as well as their v-derivatives) are all divisible by u^{n_i}. Armed with these facts we are ready to show that the Fitting ideal is locally principal. We start by calculating:

$$\begin{aligned} Fitt_2 &= \langle m_i u^{m_i+n_i},\, 0,\, 0,\, m_i u^{m_i} T_v,\, m_i u^{m_i+p_i},\, 0, \\ &\qquad (uS_u + n_i u^{n_i} v)T_v - (uT_u + p_i u^{p_i} w)u^{n_i}, \\ &\qquad (uS_u + n_i u^{n_i} v)u^{p_i},\, u^{n_i+p_i}\rangle \\ &= \langle u^{m_i+n_i},\, u^{m_i} T_v,\, u^{m_i+p_i}, \\ &\qquad uS_u T_v + n_i u^{n_i} v T_v - uT_u u^{n_i},\, uS_u u^{p_i},\, u^{n_i+p_i}\rangle \\ &= \langle u^{m_i+n_i},\, u^{m_i} T_v,\, u^{m_i+p_i},\, n_i u^{n_i} v T_v - uT_u u^{n_i}\rangle \\ &= \langle u^{m_i+n_i},\, u^{m_i+p_i}\rangle. \end{aligned}$$

Since we must have either $n_i \leq p_i$ or $p_i \leq n_i$, this ideal is principal (generated by either $u^{m_i+n_i}$ or by $u^{m_i+p_i}$, respectively). This completes the proof of Lemma 3.2.1. ■

It is worth noting that later on we will wish to have chosen ψ and ρ so that $n_i \leq p_i$. If the opposite is the case here, *i.e.* if we have chosen ψ and ρ so that $n_i > p_i$, then we can switch ψ and ρ and do a simple change of coordinates under which the new n_i will be less than or equal to the new p_i.

3.3 $Fitt_2$-preserving blowups

By Lemma 3.2.1 we now know that the Fitting ideal (which we will now denote by $Fitt_2(\breve{\alpha})$ or simply $Fitt_2$) will always be locally principal at simple points. In this section we show that blowups $\widetilde{\pi}\colon (\widetilde{U}, E) \to (\breve{U}, \breve{E})$ along double lines or triple points are $Fitt_2$-preserving, in the sense that, near double and triple points of E, $\widetilde{\pi}^{-1}(Fitt_2(\breve{\alpha})) = Fitt_2(\widetilde{\alpha})$. We also show that the blowup of a single double point is "almost" $Fitt_2$-preserving, in the sense that the pulled-up Fitting ideal is locally equal to the upstairs Fitting ideal near double points of E; this result will be useful in the next section. We thus prove the following lemma:

Lemma 3.3.1. *If $\widetilde{\pi}\colon (\widetilde{U}, E) \to (\breve{U}, \breve{E})$ is the blowup along a double line $\breve{E}_i \cap \breve{E}_j$ or a triple point $\breve{E}_i \cap \breve{E}_j \cap \breve{E}_k$, then near double or triple points of E we have $\widetilde{\pi}^{-1}(Fitt_2(\breve{\alpha})) = Fitt_2(\widetilde{\alpha})$. Moreover, if $\widetilde{\pi}$ is the blowup of a single double point of $\breve{E}$, then we have $\widetilde{\pi}^{-1}(Fitt_2(\breve{\alpha})) = Fitt_2(\widetilde{\alpha})$ near double (but not necessarily triple) points of E.*

Proof. We begin by examining the blowup of a double line $E_i \cap E_j$. Suppose $e \in \breve{E}_i \cap \breve{E}_j$ is a double point of $\breve{E}$, with analytic neighborhood W in $\breve{U}$. Choose coordinates $\{u, v, w\}$ on W so that $\breve{E}_i \cap \breve{E}_j = \{u = 0\} \cap \{v = 0\}$. We will show that blowing up the curve $u = v = 0$ is a $Fitt_2$-preserving operation.

Choose ϕ, ψ, and ρ so that ϕ generates $\pi^{-1}\mathfrak{m}_v$ and $\{d\phi, d\psi, d\rho\}$ generates $\mathcal{N}_{\breve{U}}$ (as in Section 3.2). *A priori* near a double point e in the double line we have

$$\begin{aligned}
\phi &= u^{m_i} v^{m_j} \\
\psi &= \sum r u^a v^b w^c \\
\rho &= \sum \widehat{r} u^\alpha v^\beta w^\gamma;
\end{aligned}$$

note that we write $r = r_{(a,b,c)}$ and $\widehat{r} = \widehat{r}_{(\alpha,\beta,\gamma)}$ for notational simplicity. With the standard basis for $\Omega^1_{\breve{U}}(log\breve{E})$, the matrix for the inclusion map $\breve{\alpha}$ is:

$$[\breve{\alpha}] = \begin{pmatrix} u\phi_u & u\psi_u & u\rho_u \\ v\phi_v & v\psi_v & v\rho_v \\ \phi_w & \psi_w & \rho_w \end{pmatrix} = \begin{pmatrix} m_i u^{m_i} v^{m_j} & \sum rau^a v^b w^c & \sum \widehat{r}\alpha u^\alpha v^\beta w^\gamma \\ m_j u^{m_i} v^{m_j} & \sum rbu^a v^b w^c & \sum \widehat{r}\beta u^\alpha v^\beta w^\gamma \\ 0 & \sum rcu^a v^b w^{c-1} & \sum \widehat{r}\gamma u^\alpha v^\beta w^{\gamma-1} \end{pmatrix}$$

Let $\widetilde{\pi}\colon (\widetilde{U}, E) \to (\breve{U}, \breve{E})$ be the blowup of $\breve{U}$ along the line $u = v = 0$ of double points. Consider the patch of this blowup given by the map

$$\widetilde{\pi}(\widetilde{u}, \widetilde{v}, \widetilde{w}) = (\widetilde{u}, \widetilde{u}\widetilde{v}, \widetilde{w}),$$

where $\{\widetilde{u}, \widetilde{v}, \widetilde{w}\}$ are local coordinates about a point $\widetilde{e}$ in the inverse image $\widetilde{\pi}(e)$ of our original double point e (see Section 2.1.2). Note that, under this map, the exceptional divisor $\breve{E} = \{u = 0\} \cap \{v = 0\}$ of $\breve{U}$ pulls up to the exceptional divisor $E = \{\widetilde{u} = 0\} \cap \{\widetilde{v} = 0\}$; thus each point $\widetilde{e}$ in the inverse image of e must be either a double or a simple point of E. We do not need to consider the case where $\widetilde{e}$ is a simple point, since we know by Section 3.2 that the Fitting ideal must already be locally principal at such points. We therefore examine the case when $\widetilde{e}$ is a double point of E.

From now on we will drop the "tildes" from our coordinate notation, as it will be clear from the context whether we are in $\widetilde{U}$ or $\breve{U}$. Note that a monomial $u^a v^b w^c$ in $\breve{U}$ pulls up via $\widetilde{\pi}$ to the monomial $u^{a+b} v^b w^c$ in $\widetilde{U}$. The inclusion $\breve{\alpha}$ pulls up to a map $\widetilde{\pi}^* \breve{\alpha}$ (mapping from $\widetilde{\pi}^* \mathcal{N}_{\breve{U}} = \mathcal{N}_{\widetilde{U}}$ to $\widetilde{\pi}^* \Omega^1_{\breve{U}}(\log \breve{E})$). This map is locally represented by the following matrix (obtained by pulling up each of the entries in

$[\breve{\alpha}]$ by $\widetilde{\pi}$):

$$[\widetilde{\pi}^*\breve{\alpha}] \;=\; \begin{pmatrix} m_i u^{m_i+m_j}v^{m_j} & \sum rau^{a+b}v^b w^c & \sum \widehat{r}\alpha u^{\alpha+\beta}v^\beta w^\gamma \\ m_j u^{m_i+m_j}v^{m_j} & \sum rbu^{a+b}v^b w^c & \sum \widehat{r}\beta u^{\alpha+\beta}v^\beta w^\gamma \\ 0 & \sum rcu^{a+b}v^b w^{c-1} & \sum \widehat{r}\gamma u^{\alpha+\beta}v^\beta w^{\gamma-1} \end{pmatrix} \quad (3.3.1)$$

On the other hand, the functions ϕ, ψ, and ρ pull up via $\widetilde{\pi}$ to the functions

$$\begin{aligned} \widetilde{\phi} &= u^{m_i+m_j}v^{m_j} \\ \widetilde{\psi} &= \sum ru^{a+b}v^b w^c \\ \widetilde{\rho} &= \sum \widehat{r}u^{\alpha+\beta}v^\beta w^\gamma. \end{aligned}$$

Since the (generalized) Nash sheaf $\mathcal{N}_{\widetilde{U}}$ is by definition the pullback of the Nash sheaf $\mathcal{N}_{\breve{U}}$ from $\breve{U}$, the 1-forms $\{d\widetilde{\phi}, d\widetilde{\psi}, d\widetilde{\rho}\}$ generate the Nash sheaf $\mathcal{N}_{\widetilde{U}}$. Thus the inclusion map $\widetilde{\alpha}\colon \mathcal{N}_{\widetilde{U}} \to \Omega^1_{\widetilde{U}}(\log E)$ is locally (about the double point $\widetilde{e}$) represented by the matrix

$$[\widetilde{\alpha}] \;=\; \begin{pmatrix} u\widetilde{\phi}_u & u\widetilde{\psi}_u & u\widetilde{\rho}_u \\ v\widetilde{\phi}_v & v\widetilde{\psi}_v & v\widetilde{\rho}_v \\ \widetilde{\phi}_w & \widetilde{\psi}_w & \widetilde{\rho}_w \end{pmatrix} \quad (3.3.2)$$

$$= \begin{pmatrix} (m_i+m_j)u^{m_i+m_j}v^{m_j} & \sum r(a+b)u^{a+b}v^b w^c & \sum \widehat{r}(\alpha+\beta)u^{\alpha+\beta}v^\beta w^\gamma \\ m_j u^{m_i+m_j}v^{m_j} & \sum rbu^{a+b}v^b w^c & \sum \widehat{r}\beta u^{\alpha+\beta}v^\beta w^\gamma \\ 0 & \sum rcu^{a+b}v^b w^{c-1} & \sum \widehat{r}\gamma u^{\alpha+\beta}v^\beta w^{\gamma-1} \end{pmatrix}$$

To show that $\widetilde{\pi}(Fitt_2(\breve{\alpha})) = Fitt_2(\widetilde{\alpha})$ we must now show that the ideal generated by the 2×2 subdeterminants of the matrix $[\widetilde{\pi}^*\breve{\alpha}]$ is equal to the ideal generated by the 2×2 subdeterminants of the matrix $[\widetilde{\alpha}]$. To simplify the situation somewhat, denote the entries of the matrix $[\widetilde{\pi}^*\breve{\alpha}]$ given in (3.3.1) by the letters A through I, as follows:

$$[\widetilde{\pi}^*\breve{\alpha}] = \begin{pmatrix} A & B & C \\ D & E & F \\ 0 & H & I \end{pmatrix}.$$

In this notation, the matrix $[\widetilde{\alpha}]$ (see 3.3.2) is given by the entries:

$$[\widetilde{\alpha}] = \begin{pmatrix} A+D & B+E & C+F \\ D & E & F \\ 0 & H & I \end{pmatrix}.$$

The 2×2 subdeterminants of the matrix for $\widetilde{\pi}^*\breve{\alpha}$ are the generators for the pulled-up Fitting ideal:

$$\begin{aligned} \widetilde{\pi}(Fitt_2(\breve{\alpha})) &= \langle AE - BD,\, AH,\, DH,\, AF - CD,\, AI,\, DI, \\ &\qquad BF - CE,\, BI - CH,\, EI - FH\rangle. \end{aligned} \tag{3.3.3}$$

It is now just a simple computation to show that the Fitting ideal $Fitt_2(\widetilde{\alpha})$ equals the ideal above:

$$\begin{aligned} Fitt_2(\widetilde{\alpha}) &= \langle (A+D)E - (B+E)D,\, (A+D)H,\, DH, \\ &\qquad (A+D)F - (C+F)D,\, (A+D)I,\, DI, \\ &\qquad (B+E)F - (C+F)E,\, (B+E)I - (C+F)H,\, EI - FH\rangle \\ &= \langle AE - BD,\, AH + DH,\, DH,\, AF - CD,\, AI + DI, \\ &\qquad DI,\, BF - CE,\, BI + EI - CH - FH,\, EI - FH\rangle \\ &= \langle AE - BD,\, AH,\, DH,\, AF - CD,\, AI,\, DI, \\ &\qquad BF - CE,\, BI - CH,\, EI - FH\rangle \\ &= \widetilde{\pi}(Fitt_2(\breve{\alpha})). \end{aligned}$$

In fact, we can tell simply by looking at the matrices for $\widetilde{\alpha}$ and $\widetilde{\pi}^*\breve{\alpha}$ that the 2×2 subdeterminants will generate the same ideal, since the matrix $[\widetilde{\alpha}]$ is simply the matrix $[\widetilde{\pi}^*\breve{\alpha}]$ with the second row added to the first.

Clearly the proof for the other patch

$$\widetilde{\pi}(\widetilde{u},\widetilde{v},\widetilde{w}) = (\widetilde{u},\widetilde{u}\widetilde{v},\widetilde{w})$$

is entirely analogous to the above, with the roles of u and v interchanged. This completes the proof of Lemma 3.3.1 in the case when we blow up a double line in $\breve{E}$ and are in the neighborhood of a double point in that double line.

Now suppose again that we are blowing up a double line $\breve{E}_i \cap \breve{E}_j = \{u = 0\} \cap \{v = 0\}$, but that this time we are in the neighborhood of a triple point e in that double line (such points may or may not exist along the double line). The proof in this case is similar to that of the case above, as follows. In this case the functions ϕ, ψ, and ρ are of the form:

$$\begin{aligned} \phi &= u^{m_i}v^{m_j}w^{m_k} \\ \psi &= \sum r u^a v^b w^c \\ \rho &= \sum \widehat{r} u^\alpha v^\beta w^\gamma, \end{aligned}$$

and the matrix for the inclusion map $\breve{\alpha}$ is:

$$[\breve{\alpha}] = \begin{pmatrix} u\phi_u & u\psi_u & u\rho_u \\ v\phi_v & v\psi_v & v\rho_v \\ w\phi_w & w\psi_w & w\rho_w \end{pmatrix} = \begin{pmatrix} m_i u^{m_i}v^{m_j}w^{m_k} & \sum rau^a v^b w^c & \sum \widehat{r}\alpha u^\alpha v^\beta w^\gamma \\ m_j u^{m_i}v^{m_j}w^{m_k} & \sum rbu^a v^b w^c & \sum \widehat{r}\beta u^\alpha v^\beta w^\gamma \\ m_k u^{m_i}v^{m_j}w^{m_k} & \sum rcu^a v^b w^c & \sum \widehat{r}\gamma u^\alpha v^\beta w^\gamma \end{pmatrix}.$$

Once again we will consider the patch of the blowup given by $\widetilde{\pi}(\widetilde{u},\widetilde{v},\widetilde{w}) = (\widetilde{u},\widetilde{u}\widetilde{v},\widetilde{w})$ (the proof for the other patch is entirely analogous). The map $\widetilde{\pi}^*\breve{\alpha}$ is locally represented by the matrix

$$[\widetilde{\pi}^*\breve{\alpha}] \quad = \quad \begin{pmatrix} m_i u^{m_i+m_j} v^{m_j} w^{m_k} & \sum rau^{a+b}v^b w^c & \sum \widehat{r}\alpha u^{\alpha+\beta}v^\beta w^\gamma \\ m_j u^{m_i+m_j} v^{m_j} w^{m_k} & \sum rbu^{a+b}v^b w^c & \sum \widehat{r}\beta u^{\alpha+\beta}v^\beta w^\gamma \\ m_k u^{m_i+m_j} v^{m_j} w^{m_k} & \sum rcu^{a+b}v^b w^c & \sum \widehat{r}\gamma u^{\alpha+\beta}v^\beta w^\gamma \end{pmatrix}.$$

First suppose that $\widetilde{e}$ is a triple point of E. Then the inclusion map $\widetilde{\alpha}\colon \mathcal{N}_{\widetilde{U}} \to \Omega^1_{\widetilde{U}}(\log E)$ is locally represented (near $\widetilde{e}$) by the matrix

$$[\widetilde{\alpha}] \quad = \quad \begin{pmatrix} u\widetilde{\phi}_u & u\widetilde{\psi}_u & u\widetilde{\rho}_u \\ v\widetilde{\phi}_v & v\widetilde{\psi}_v & v\widetilde{\rho}_v \\ w\widetilde{\phi}_w & w\widetilde{\psi}_w & w\widetilde{\rho}_w \end{pmatrix},$$

which has the form

$$\begin{pmatrix} (m_i+m_j)u^{m_i+m_j} v^{m_j} w^{m_k} & \sum r(a+b)u^{a+b}v^b w^c & \sum \widehat{r}(\alpha+\beta) u^{\alpha+\beta}v^\beta w^\gamma \\ m_j u^{m_i+m_j} v^{m_j} w^{m_k} & \sum rbu^{a+b}v^b w^c & \sum \widehat{r}\beta u^{\alpha+\beta}v^\beta w^\gamma \\ m_k u^{m_i+m_j} v^{m_j} w^{m_k} & \sum rcu^{a+b}v^b w^c & \sum \widehat{r}\gamma u^{\alpha+\beta}v^\beta w^\gamma \end{pmatrix}.$$

Once again, if we write the matrix $[\widetilde{\pi}^*\breve{\alpha}]$ in the form

$$[\widetilde{\pi}^*\breve{\alpha}] = \begin{pmatrix} A & B & C \\ D & E & F \\ 0 & H & I \end{pmatrix},$$

then $[\widetilde{\alpha}]$ is given by

$$[\widetilde{\alpha}] = \begin{pmatrix} A+D & B+E & C+F \\ D & E & F \\ 0 & H & I \end{pmatrix}.$$

As above, these two matrices differ only by the elementary row operation of adding one row to another, and thus their 2×2 subdeterminants generate the same Fitting ideal.

If $\widetilde{e}$ is instead a double point of E (for example, with the map $(u,v,w) \mapsto (u,uv,w)$ we might have $\widetilde{e}$ on the double line $u = w = 0$), then the matrices $[\breve{\alpha}]$ and $\widetilde{\pi}^*\breve{\alpha}]$ are the same, but the matrix $[\widetilde{\alpha}]$ is given by

$$[\widetilde{\alpha}] \;=\; \begin{pmatrix} u\widetilde{\phi}_u & u\widetilde{\psi}_u & u\widetilde{\rho}_u \\ \widetilde{\phi}_v & \widetilde{\psi}_v & \widetilde{\rho}_v \\ w\widetilde{\phi}_w & w\widetilde{\psi}_w & w\widetilde{\rho}_w \end{pmatrix},$$

which then has the form

$$\begin{pmatrix} (m_i+m_j)u^{m_i+m_j}v^{m_j}w^{m_k} & \sum r(a+b)u^{a+b}v^bw^c & \sum \widehat{r}(\alpha+\beta)u^{\alpha+\beta}v^\beta w^\gamma \\ m_ju^{m_i+m_j}v^{m_j-1}w^{m_k} & \sum rbu^{a+b}v^{b-1}w^c & \sum \widehat{r}\beta u^{\alpha+\beta}v^{\beta-1}w^\gamma \\ m_ku^{m_i+m_j}v^{m_j}w^{m_k} & \sum rcu^{a+b}v^bw^c & \sum \widehat{r}\gamma u^{\alpha+\beta}v^\beta w^\gamma \end{pmatrix}.$$

With respect to the matrix $[\widetilde{\pi}^*\breve{\alpha}]$, this matrix is of the form

$$[\widetilde{\alpha}] = \begin{pmatrix} A+D & B+E & C+F \\ D/v & E/v & F/v \\ 0 & H & I \end{pmatrix}.$$

Since at $\widetilde{e}$ we are away from $v=0$, $1/v$ is a local unit; thus once again, $[\widetilde{\alpha}]$ and $[\widetilde{\pi}^*\breve{\alpha}]$ define the same Fitting ideal.

We now consider the blowup of a triple point $e \in \breve{E}_i \cap \breve{E}_j \cap \breve{E}_k$. Choose analytic neighborhood W of e in $\breve{U}$ and coordinates $\{u, v, w\}$ on W so that $\breve{E}_i = \{u = 0\}$, $\breve{E}_j = \{v = 0\}$, and $\breve{E}_k = \{w = 0\}$. Choose ϕ, ψ, and ρ as we did earlier in this proof. Let $\widetilde{\pi}\colon (\widetilde{U}, E) \to (\breve{U}, \breve{E})$ be the blowup of $\breve{U}$ at this triple point e. Consider the patch of this blowup given by the map

$$\widetilde{\pi}(\widetilde{u}, \widetilde{v}, \widetilde{w}) = (\widetilde{u}, \widetilde{u}\widetilde{v}, \widetilde{u}\widetilde{w}),$$

where $\{\widetilde{u}, \widetilde{v}, \widetilde{w}\}$ are local coordinates about a point $\widetilde{e}$ in the inverse image of e under $\widetilde{\pi}$. Again we will drop the "tildes" from our notation, as it will be clear from the context whether we are in $\widetilde{U}$ or in $\breve{U}$. Since a monomial $u^av^bw^c$ in $\breve{U}$ is pulled up by $\widetilde{\pi}$ to a monomial $u^{a+b+c}v^b w^c$, a computation similar to that above shows that the matrix for $\widetilde{\alpha}$ near a triple point $\widetilde{e}$ will be of the form:

$$[\widetilde{\alpha}] = \begin{pmatrix} A+D+G & B+E+H & C+F+I \\ D & E & F \\ G & H & I \end{pmatrix},$$

where the matrix for $\widetilde{\pi}^*\breve{\alpha}$ is of the form

$$[\widetilde{\pi}^*\breve{\alpha}] = \begin{pmatrix} A & B & C \\ D & E & F \\ G & H & I \end{pmatrix}.$$

Once again, the matrix $[\widetilde{\alpha}]$ differs from the matrix $[\widetilde{\pi}^*\breve{\alpha}]$ only by the elementary row operation of adding one row to another, and thus the Fitting ideals obtained from these two matrices will be the same. The other two patches for this blowup are similar, with the roles of u, v, and w suitably permuted. Thus we have shown that the blowup of a triple point of $\breve{E}$ is $Fitt$-preserving, *i.e.* that under such a blowup we have $Fitt_2(\widetilde{\alpha}) = \widetilde{\pi}^*(Fitt_2(\breve{\alpha}))$.

In the case when $\widetilde{e}$ is a double point of E (say along $u = v = 0$ but $w \neq 0$), the matrix $[\widetilde{\alpha}]$ is of the form

$$[\widetilde{\alpha}] = \begin{pmatrix} A+D+G & B+E+H & C+F+I \\ D & E & F \\ G/w & H/w & I/w \end{pmatrix}.$$

As we saw in a previous case, since we are away from $w = 0$, this matrix defines the same Fitting ideal as the matrix for $[\widetilde{\alpha}]$.

To prove the final part of Lemma 3.3.1 we must show that the blowup of a single double point $e \in \breve{E}_i \cap \breve{E}_j$ is $Fitt_2$-preserving at double points of E. Choose coordinates $\{u, v, w\}$ in an analytic neighborhood so that $\breve{E}_i = \{u = 0\}$ and

$\breve{E}_j = \{v = 0\}$, and choose ϕ, ψ, and ρ as above. Consider the patch of this blowup given by the map

$$\widetilde{\pi}(\widetilde{u}, \widetilde{v}, \widetilde{w}) = (\widetilde{u}, \widetilde{u}\widetilde{v}, \widetilde{u}\widetilde{w}).$$

Under this map, $\breve{E} = \{u = 0\} \cap \{v = 0\}$ pulls up to the exceptional divisor $E = \{u = 0\} \cap \{v = 0\}$. Using methods as above we can reduce this case to the question of whether the 2×2 subdeterminants of the matrices

$$[\widetilde{\pi}^*\breve{\alpha}] = \begin{pmatrix} A & B & C \\ D & E & F \\ 0 & H & I \end{pmatrix} \quad \text{and} \quad [\widetilde{\alpha}] = \begin{pmatrix} A + D & B + E + wH & C + F + wI \\ D & E & F \\ 0 & H & I \end{pmatrix}$$

generate the same ideal. Once again the matrix $[\widetilde{\alpha}]$ is simply the matrix $[\widetilde{\pi}^*\breve{\alpha}]$ with multiples of the second and third rows added to the first. Of course the case for the patch given by the map

$$\widetilde{\pi}(\widetilde{u}, \widetilde{v}, \widetilde{w}) = (\widetilde{u}, \widetilde{u}\widetilde{v}, \widetilde{u}\widetilde{w})$$

is entirely similar, since u and v can be interchanged at this double point $e \in \breve{E}$.

Finally, consider the patch given by the map $\widetilde{\pi}(u, v, w) = (uw, vw, w)$. In this case, $\breve{E} = \{u = 0\} \cap \{v = 0\}$ pulls up to the exceptional divisor $E = \{u = 0\} \cap \{v = 0\} \cap \{w = 0\}$. We need only show that $\widetilde{\pi}$ is $Fitt_2$-preserving at the double points of E. However, $\widetilde{\pi}^{-1}(e)$ is a copy of $\mathbb{P}^2$, and the only point in $\widetilde{\pi}^{-1}(e)$ that we have not examined through the other patches is the triple point of E. Thus we do not need to examine this patch further, and the proof is complete. ■

3.4 Existence of the Desired Resolution

Let $\breve{\pi}\colon (\breve{U}, \breve{E}) \to (U, v)$ be a resolution of U that factors through the Nash blowup and the blowup of the maximal ideal sheaf, as above. We wish to construct a further resolution $(\widetilde{U}, E) \to (\breve{U}, \breve{E})$ for which the Fitting ideal $Fitt_2(\widetilde{\alpha})$ is locally principal. We begin the construction by finding a blowup $\bar{\pi}\colon (\bar{U}, \bar{E}) \to (\breve{U}, \breve{E})$ for which a certain upper-semicontinuous function ν is locally constant along the k-dimensional strata of $\bar{E}$. We then use a theorem of Hironaka's from [Hir64a] to obtain a transformation $\widetilde{\pi}\colon (\widetilde{U}, E) \to (\bar{U}, \bar{E})$ that is $Fitt_2$-preserving and makes the inverse image of the Fitting ideal on $\bar{U}$ locally principal; thus the Fitting ideal on $\widetilde{U}$ will also be locally principal. To this end we will prove the following two lemmas:

Lemma 3.4.1. *With notation as above, there exists a blowup $\bar{\pi}\colon (\bar{U}, \bar{E}) \to (\breve{U}, \breve{E})$ for which the Hironaka number (see Definition 3.4.3) of the Fitting ideal $Fitt_2(\bar{E})$ is locally constant along the strata of $\bar{E}$.*

Lemma 3.4.2. *With notation as above, there exists a finite sequence of blowups*

$$\widetilde{U} =: \widetilde{U}_r \xrightarrow{\widetilde{\pi}_{r-1}} \widetilde{U}_{r-1} \xrightarrow{\widetilde{\pi}_{r-2}} \cdots \xrightarrow{\widetilde{\pi}_1} \widetilde{U}_1 \xrightarrow{\widetilde{\pi}_0} \widetilde{U}_0 := \bar{U}$$

for which:

(a) *$\widetilde{\pi}_i$ is a $Fitt_2$-preserving blowup in the sense of Lemma 3.3.1, i.e. $\widetilde{\pi}_i$ is the* *blowup of a double line or triple point of the ith level exceptional divisor* $\widetilde{D}_i := \widetilde{\pi}_i^{-1}(\cdots(\widetilde{\pi}_0^{-1}(\bar{E})))$.

(b) $\widetilde{\pi}^{-1}(Fitt_2(\bar{\alpha}))$ *is a locally principal sheaf of ideals over* $\widetilde{U}$.

We will prove Lemma 3.4.1 using the last part of Lemma 3.3.1 from the previous section, and show Lemma 3.4.2 by applying Hironaka's "Main Theorem II" from [Hir64a].

We begin by defining the Hironaka number as in [Hir64a] and discussing its various properties. Suppose X is a non-singular algebraic scheme and J is a coherent sheaf of nonzero ideals on X. Define the "Hironaka number" of J at x to be the highest power p for which the p-th power of the maximal ideal of $\mathcal{O}_{X,x}$ contains J_x, as follows.

Definition 3.4.3. *Given J, X, and x as above, we define the* Hironaka number *of J at x to be the integer*

$$\nu(J_x) := \max\{p \in \mathbb{Z} \mid \mathfrak{m}_x^p \supseteq J_x\},$$

where $\mathfrak{m}_x$ denotes the maximal ideal in $\mathcal{O}_{X,x}$.

The function $x \mapsto \nu(J_x)$ is upper-semicontinuous by Corollary 1 of [Hir64b]. In other words, for every integer r, the set

$$U_r := \{x \in X \mid \nu(J_x) \leq r\}$$

is open in X.

Let $(\breve{U}, \breve{E})$ be a resolution of (U, v) that factors through the Nash blowup and the blowup of the maximal ideal sheaf as discussed in Sections 3.2 and 3.3. For Lemma 3.4.1 we are interested in the sheaf of ideals

$$J := Fitt_2(\breve{\alpha}),$$

the Fitting ideal generated by the 2×2 subdeterminants of a matrix representing the inclusion $\mathcal{N}_{\breve{U}} \hookrightarrow \Omega^1_{\breve{U}}(\log \breve{E})$. Note that J is an ideal sheaf supported on E and locally principal at simple points of E.

For each i, let g_i be the generic point of the component $\breve{E}_i$ of $\breve{E}$, and define

$$\nu_i := \nu(J_{g_i})$$

to be the Hironaka number at this generic point of $\breve{E}_i$. Note that ν_i is also the generic value of $\nu(J_{x_i})$ over all points in $\breve{E}_i$: suppose $\{u, v, w\}$ are coordinates on $\breve{U}$ chosen so that $\breve{E}_i = \{u = 0\}$; then at simple points of $\breve{E}_i$, J is locally principal, say $J = \langle u^a \rangle$. Given a simple point $x \in \breve{E}_i$ we have (by Definition 3.4.3):

$$\nu(J_x) = \max\{m \mid \langle u, v, w\rangle_x^m \supseteq \langle u \rangle_x^a\} = a;$$

thus for a dense open set of points on $\breve{E}_i$ the Hironaka number of J is equal to ν_i.

To prove Lemmas 3.4.1 and 3.4.2 we need to further discuss the notation used in Hironaka's papers [Hir64a] and [Hir64b]. Given a monoidal transform $f\colon X' \to X$ with center B, where B is a nonempty, irreducible, nonsingular subscheme of X, let $B' := f^{-1}(B)$ denote the total preimage of B, and write $I_{B'}$ for the sheaf of ideals on X' which define B'. Suppose x is the generic point of B. Hironaka defines the "weak transform" of J by f as follows (this discussion follows Remark 2 in [Hir64b]).

Definition 3.4.4. *With notation as described above, the* weak transform *of J on X' by the monoidal transformation $f\colon X' \to X$ with center B is*

$$f^{\star}(J) := f^{-1}(J)\, I_{B'}^{-\nu(J_x)},$$

the ideal quotient of $f^{-1}(J)$ by $I_{B'}^{\nu(J_x)}$.

Now suppose we have a finite sequence of monoidal transformations

$$X_r \xrightarrow{f_{r-1}} X_{r-1} \xrightarrow{f_{r-2}} \cdots \xrightarrow{f_1} X_1 \xrightarrow{f_0} X_0 := X$$

with centers $B_i \subset X_i$. We will denote the weak transforms at each level by

$$J_i := f_{i-1}^{\star}(J_{i-1}) \tag{3.4.1}$$

for $0 < i \leq r$, with $J_0 := J$. Given a divisor with normal crossings $D = D_0$ on $X = X_0$, we define

$$D_i := \mathrm{red}(f_{i-1}^{-1}(D_{i-1}) \cup f_{i-1}^{-1}(B_{i-1})) \subset X_i \tag{3.4.2}$$

for $0 < i \leq r$. With this notation, Hironaka's theorem states:

Theorem 3.4.5. *Let X be a nonsingular algebraic scheme, let J be a sheaf of coherent nonzero ideals on X, and let D be a divisor with normal crossings on X.*

Then there exists a finite sequence of monoidal transformations $\{f_i : X_{i+1} \to X_i\}$ *with centers* $B_i \subset X_i$ *with the following properties:*

(a) B_i *is nonsingular and irreducible.*

(b) $\nu((J_i)_{x_i}) \geq d_i$ *for all points* $x_i \in B_i$*, where* d_i *is the maximum Hironaka number for* J_i *over all* $x \in X_i$*.*

(c) D_i *has only normal crossings with* B_i*.*

(d) D_r *has only normal crossings, and* $\nu((J_r)_y) < d$ *for every point* $y \in X_r$*.*

Note that by repeated application of Theorem 3.4.5 we can obtain a space X_R with the property that $\nu((J_R)_y) = 0$ for every point $y \in X_R$ (see part (d) of the theorem). This means that we have "trivialized" the coherent sheaf J in the sense that its (iterated) weak transform J_R is the trivial sheaf: by the definition of the Hironaka number (see Definition 3.4.3), $\nu((J_R)_y) = 0$ means that for all $y \in X_R$ we have $\mathfrak{m}_y^0 = \mathcal{O}_{X,x} \supseteq J_y$ but $\mathfrak{m}_y^1 \not\supseteq J_y$; since $\mathfrak{m}_y$ is maximal we would thus have $J_x = \mathcal{O}_{X,x}$.

It is very important to note that in the sequence of transformations obtained in Theorem 3.4.5, the centers B_i are strata with locally maximum Hironaka number in the stratification of D_i by the Hironaka number ν for J_i. This fact is shown in [BM97], where a constructive proof of Theorem 3.4.5 is given (in particular, see Remark 1.8 and Theorem 1.10 there): at each stage of the resolution, the center of blowing-up is a locally maximum stratum of the local invariant ν. This means in particular that each B_i will have a strictly higher Hironaka number for J_i than any neighboring point in D_i. This fact will be useful in the proof of Lemma 3.4.1 and at the end of the proof of Lemma 3.4.2 below. We are now in a position to prove Lemma 3.4.1.

Proof. Let $(\bar{U}, \bar{E})$ and $J = Fitt(\bar{\alpha})$ as above. Stratify the exceptional divisor $\bar{E}$ by dimension, as follows:

$$\begin{aligned} S_2 &:= \text{simple points of } \bar{E}, \\ S_1 &:= \text{double points of } \bar{E}, \\ S_0 &:= \text{triple points of } \bar{E}. \end{aligned}$$

Note that $\nu(J_x)$ must be locally constant on S_0 (which consists only of isolated points) and on S_2 (since J is locally principal at these points; see the discussion above). Since $\nu(J_x)$ is upper-semicontinuous, the only way that $\nu(J_x)$ can fail to be locally constant on S_1 is if it jumps to a higher value at isolated double points.

We now obtain a blowup $\bar{\pi}\colon (\bar{U}, \bar{E}) \to (\breve{U}, \breve{E})$ for which the Hironaka number is locally constant along double point strata of $\bar{E}$. Suppose b is a double point of $\bar{E}$ with the property that $\nu(J_b) > \nu(J_x)$ for nearby points x on the same double line. Then b has a locally maximal Hironaka number, and thus is an allowable center B_0 for blowing up. Blowing up with center b will result in a new component of the exceptional divisor; above b we will have new simple, double, and triple points.

By the discussion above we are only concerned with making the Hironaka number locally constant along the double point set. If there are more "bad" double points in the new double point set, they are once again allowable centers for blowing up in the Hironaka argument. If we repeat this process of blowing up the bad double points, we will eventually achieve a space $\bar{U}$ for which the Hironaka number of the new Fitting ideal $\bar{J}$ is locally constant along the strata of $\bar{E}$, as follows. By part (d) of 6.2.8, this process of blowing up the "bad" double points will eventually result in a space for which the Hironaka number of the (iterated) weak transform $J^\star$ has been reduced, *i.e.* has been made strictly less than $\nu(J_b)$ at every new double point. By repeating *this* process we can obtain a space $\bar{U}$ for which the weak transform $J^\star$ is locally constant along the strata of $\bar{E}$.

Since the weak transform of J and the inverse image of J by $\bar{\pi}$ differ by a locally principal ideal, we now know that the the inverse image $\bar{\pi}^{-1}(J)$ has locally constant Hironaka number along the strata of $\bar{E}$. Finally, by Lemma 3.3.1 we know that each of the blowups described above is $Fitt_2$-preserving at the new double points; thus we will have $\bar{J}_y = \bar{\pi}^{-1}(J_b)$ for each new double point y above b. Thus the Fitting ideal on $\bar{U}$ will have locally constant Hironaka number along the strata of $\bar{E}$, as desired. ■

In our situation, we wish to take a coherent sheaf ($J = Fitt_2$) and find a sequence of monoidal transformations for which the inverse image of the sheaf is locally principal (while Hironaka's theorem provides a sequence of transformations for which the weak transform of the sheaf is trivial). To make the transition from what we want to what Hironaka's theorem says, we will define an associated sheaf $\widehat{J}$, and apply Hironaka's Theorem to this new sheaf. The following discussion and Claim 3.4.6 serve to define this sheaf $\widehat{J}$.

Let ν_i denote the generic value of the Hironaka number of J_i as defined above. Define the divisor

$$L := \sum \nu_i \bar{U}_i$$

on $\bar{U}$, and let I_L denote the sheaf of ideals defining L. We wish to define the sheaf $\widehat{J}$ discussed above to be

$$\widehat{J} := J\, I_L^{-1},$$

the ideal quotient of J by I_L; in order for this definition to make sense, however, we require the following claim:

Claim 3.4.6. *With notation as above, we have $J \subseteq I_L$.*

Proof. It suffices to prove that $J_x \subseteq (I_L)_x$ for all $x \in \bar{E}$. In the case where x is a simple point in $\bar{E}_i = \{u = 0\}$ the inclusion is in fact an equality: by the discussion above we have $J_x = \langle u^{\nu_i} \rangle_x$ in this case, and $(I_L)_x = (I_{\nu_i E_i})_x = \langle u^{\nu_i} \rangle_x$.

Suppose that x is a double point in $\bar{E}_i \cap \bar{E}_j = \{u = 0\} \cap \{v = 0\}$, and choose an element $j \in J_x$. Since J is supported along $\bar{E}$, j vanishes along E_i and E_j;

thus we have $j = u^a f$ with $u \nmid f$ and $j = v^b g$ with $v \nmid g$. Since $J_y = \langle u^{\nu_i} \rangle_y$ at all nearby simple points $y \in E_i$, we have $a \geq \nu_i$; similarly we have $b \geq \nu_j$. Thus we can write $j = u^{\nu_i} A$ and $j = v^{\nu_i} B$ for some holomorphic A and B; since $(\mathcal{O}_{\bar{U}})_x$ is a unique factorization domain we thus have $j = u^{\nu_i} v^{\nu_j} C$ for some holomorphic C. In other words, we have $j \in \langle u^{\nu_i} v^{\nu_j} \rangle_x = (I_L)_x$. The proof in the triple point case is analogous. ■

Note that $\widehat{J} = \mathcal{O}_X$ if and only if J is locally principal. Moreover, since J is locally principal at the simple points of $\bar{E}$ (see Section 3.2), $\widehat{J}$ is supported on the set of double and triple points of $\bar{E}$.

We are now in a position to prove Lemma 3.4.2.

Proof. By repeated application of Hironaka's theorem 3.4.5 to the coherent sheaf of ideals $\widehat{J}$ and divisor with normal crossings E on $\bar{U}$ we can "trivialize" $\widehat{J}$ to the trivial sheaf (see Corollary 1 in [Hir64a]), as follows. Applying Theorem 3.4.5 to $\widehat{J}$ gives us a sequence of monoidal transformations

$$\bar{U}_r \xrightarrow{\bar{\pi}_{r-1}} \bar{U}_{r-1} \xrightarrow{\bar{\pi}_{r-2}} \cdots \xrightarrow{\bar{\pi}_1} \bar{U}_1 \xrightarrow{\bar{\pi}_0} \bar{U}_0 := \bar{U}$$

with centers $B_i \in \widetilde{U}_i$ satisfying (a)–(c) of Theorem 3.4.5. If $\widehat{J}_r$ is the weak transform $\bar{\pi}^\star_{r-1}(\cdots(\bar{\pi}_0(\widehat{J}))\cdots)$, and d is the maximum Hironaka number for $\widehat{J}$ over all points in $\bar{U}$, then by part (d) of Theorem 3.4.5, $\nu((\widehat{J}_r)_y) < d$ for every point $y \in \bar{U}_r$. Thus the maximum Hironaka number has strictly decreased after this sequence of monoidal transformations. By repeating this process ($\widetilde{U}_1 := \bar{U}_r$ in the sequence below) we can obtain a sequence

$$\widetilde{U} =: \widetilde{U}_R \xrightarrow{\widetilde{\pi}_{R-1}} \widetilde{U}_{R-1} \xrightarrow{\widetilde{\pi}_{R-2}} \cdots \xrightarrow{\widetilde{\pi}_1} \widetilde{U}_1 \xrightarrow{\widetilde{\pi}_0} \widetilde{U}_0 := \bar{U}$$

with the property that the weak transform $\widehat{J}_R = \widetilde{\pi}^{\star}_{R-1}(\cdots(\widetilde{\pi}^{\star}_0(\widehat{J})))$ has a maximum Hironaka number of zero; in other words $\nu((J_R)_x) = 0$ for every $x \in \widetilde{U}$. Thus we have $\widehat{J}_R = \mathcal{O}_{\widetilde{U}_R} = \mathcal{O}_{\widetilde{U}}$.

To prove part (b) of Lemma 3.4.2, we will show that $\widehat{J}_R = \mathcal{O}_{\widetilde{U}}$ implies that $\pi^{-1}(J)$ is locally principal. We will denote the divisor $\bar{E}$ by D_0, and use the notation given in 3.4.1 and 3.4.2 to express the weak transforms of $\widehat{J}$ and the inverse images of D_0, respectively. Let b_i denote the Hironaka number of the weak transform $\widehat{J}_i$ at the generic point y_i of B_i; in other words, define

$$b_i := \nu((\widehat{J}_i)_{y_i}).$$

Finally, denote the inverse image of B_i under $\widetilde{\pi}_i$ as $B'_i := \widetilde{\pi}^{-1}(B_i) \subset \widetilde{U}_{i+1}$. Using this notation and Definition 3.4.4 we have:

$$\begin{aligned}
\mathcal{O}_{\widetilde{U}} &= \widehat{J}_R \\
&= \widetilde{\pi}^{\star}_{R-1}(\widetilde{\pi}^{\star}_{R-2}(\cdots(\widetilde{\pi}^{\star}_0(\widehat{J})))) \\
&= \widetilde{\pi}^{-1}_{R-1}(\widetilde{\pi}^{-1}_{R-2}(\cdots(\widetilde{\pi}^{-1}_1(\widetilde{\pi}^{-1}_0(\widehat{J})\, I^{-b_0}_{B'_0})\, I^{-b_1}_{B'_1})\cdots)\, I^{-b_{R-2}}_{B'_{R-2}})\, I^{-b_{R-1}}_{B'_{R-1}} \\
&= \widetilde{\pi}^{-1}_{R-1}(\widetilde{\pi}^{-1}_{R-2}(\cdots(\widetilde{\pi}^{-1}_1(\widetilde{\pi}^{-1}_0(\widehat{J})))))\,(\widetilde{\pi}^{-1}_{R-1}\circ\cdots\circ\widetilde{\pi}^{-1}_1)I^{-b_0}_{B'_0} \\
&\qquad (\widetilde{\pi}^{-1}_{R-1}\circ\cdots\circ\widetilde{\pi}^{-1}_2)I^{-b_1}_{B'_1}\cdots(\widetilde{\pi}^{-1}_{R-1})I^{-b_{R-2}}_{B'_{R-2}}\, I^{-b_{R-1}}_{B'_{R-1}} \\
&= \widetilde{\pi}^{-1}(\widehat{J})\,((\widetilde{\pi}^{-1}_{R-1}\circ\cdots\circ\widetilde{\pi}^{-1}_1)I_{B'_0})^{-b_0}\cdots(\widetilde{\pi}_{R-1}I_{B'_{R-2}})^{-b_{R-2}}I^{-b_{R-1}}_{B'_{R-1}}.
\end{aligned}$$

By the definition of $\widehat{J}$, we have

$$\widetilde{\pi}^{-1}(\widehat{J}) \;=\; \widetilde{\pi}^{-1}(J\,I_L^{-1}) \;=\; \widetilde{\pi}^{-1}(J)\,(\widetilde{\pi}^{-1}(I_L))^{-1};$$

hence we can write $\widetilde{\pi}^{-1}(J)$ as the locally principal ideal

$$\widetilde{\pi}^{-1}(J) \;=\; (\widetilde{\pi}^{-1}(I_L))\,((\widetilde{\pi}^{-1}_{R-1}\circ\cdots\circ\widetilde{\pi}^{-1}_0)I_{B_0})^{d_0}\cdots(\widetilde{\pi}_{R-1}I_{B'_{R-2}})^{-b_{R-2}}I^{-b_{R-1}}_{B'_{R-1}}.$$

Finally, we prove part (a) of Lemma 3.4.2. We wish to show that the center B_i of each monoidal transform $\widetilde{\pi}_i$ is a double line or triple point of the exceptional divisor $\widetilde{D}_i := \widetilde{\pi}_i^{-1}(\cdots(\widetilde{\pi}_0^{-1}(\bar{E})))$. It suffices to show that B_i is supported away from the simple point set of $\widetilde{D}_i$ and has a higher Hironaka number for $\widehat{J}_i$ than any neighboring point; note that the second condition will exclude the possibility that B_i is the blowup along a single double point of $\widetilde{D}_i$, since by Lemma 3.4.1 the Hironaka number is locally constant along the strata of $\widetilde{D}_i$.

Clearly B_0 satisfies these conditions: part (b) of Theorem 3.4.5 ensures that $\nu((\widehat{J})_x) \geq d$ for all points $x \in B_0$, and by definition d is the maximum Hironaka number for $\widehat{J}$ over all $x \in \bar{U}$. Since J is locally principal at simple points of $\bar{U}$, $\widehat{J}_x \approx \mathcal{O}_x$ at simple points $x \in D_0$; thus $\nu_{J_x} = 0$ at these simple points, and B_0 must be supported away from the simple point set. Since by Lemma 3.4.1 the Hironaka number is locally constant along the strata of D_0, and B_0 must be a maximal component of that stratification (see the remarks following Theorem 3.4.5), B_0 must have a higher Hironaka number for $\widehat{J}$ than any neighboring point, and thus must be either a double line or a triple point of D_0.

Since B_0 is a double line or triple point of D_0, the monoidal transform $\widetilde{\pi}_0$ with center B_0 is a $Fitt_2$-preserving transformation (in the sense of Lemma 3.3.1). This fact will show that B_1 does not intersect the simple points of D_1: since $\widetilde{\pi}_0$ is $Fitt_2$-preserving, at a simple point $x \in D_1 \subset \widetilde{U}_1$ we have:

$$\begin{aligned}
(\widehat{J}_1)_x &= (\widetilde{\pi}_0^{\star} \widehat{J}_0)_x \\
&= (\pi_0^{-1} \widehat{J}_0)_x \, (I^{-b_0}_{\widetilde{\pi}_0^{-1}(B_0)})_x \\
&= (\widetilde{\pi}_0^{-1} J)_x \, (\widetilde{\pi}_0^{-1} I_L)^{-1}_x \, (I_{\widetilde{\pi}_0^{-1}(B_0)})^{-b_0}_x .
\end{aligned}$$

Since the blowup of B_0 is $Fitt_2$-preserving, the pulled-up Fitting ideal $(\widetilde{\pi}_0^{-1} J)_x$ is isomorphic to the Fitting ideal on $\widetilde{U}_1$, which by Lemma 3.2.1 is locally principal. Thus $(\widehat{J}_1)_x$ is the product of the locally principal ideal $\pi_0^{-1} \widehat{J}_0 \approx (\widetilde{\pi}_0^{-1} J)_x \, (\widetilde{\pi}_0^{-1} I_L)^{-1}_x$ with the ideal $(I_{\widetilde{\pi}_0^{-1}(B_0)})^{-b_0}_x$. Since $\nu(\widetilde{\pi}_0^{-1}(\widehat{J})_y) = b_0$ for the generic point $y \in \widetilde{\pi}_0^{-1} B_0$

(see the remarks above and on page 142 of [Hir64a]), b_0 is the largest integer for which the locally principal ideal $I_{\widetilde{\pi}_0^{-1}(B_0)}$ divides $\pi_0^{-1}\widehat{J}_0$. Therefore we must have $(\widehat{J}_1)_x \approx \mathcal{O}_x$ at the simple point $x \in D_1$, and thus $\nu((\widehat{J}_1)_x) = 0$. By part (b) of Theorem 3.4.5 the Hironaka number of $\widehat{J}_1$ cannot be zero at any point of the new center B_1, we know that $x \notin B_1$; thus the intersection of B_1 with the simple point set of D_1 is empty.

The rest of the proof involves repeating the arguments above: since B_1 is supported away from simple points and must be a maximal component of D_i according to the stratification of D_i by ν, B_1 must be a double line or triple point of D_1. B_2 can be shown to be away from simple points by the same argument as above, and so on for the rest of the B_i. ■

Chapter 4

The Flag Proposition

4.1 The Flag Proposition and Nash-minimality

Let $\pi\colon (\widetilde{U}, E) \to (U, v)$ be a complete resolution of singularities as defined in Section 3.1 above. In this Chapter we will be working in the analytic category, on the associated analytic variety $\widetilde{U}^h$ to $\widetilde{U}$ (see Proposition 2.4.1). To avoid excessive notation we will abuse notation and also denote the analytic variety $\widetilde{U}^h$ by $\widetilde{U}$. We also abuse notation by denoting the sheaf of holomorphic functions (see Proposition 2.4.2) $\mathcal{H}_{\widetilde{U}^h}$ by $\mathcal{O}_{\widetilde{U}}$. We will also write the analytic analogues (see Definition 2.4.3) of other sheaves in algebraic notation; *e.g.* the analytic sheaf $\mathcal{N}^h$ corresponding to the algebraic Nash sheaf $\mathcal{N}$ will be denoted simply as $\mathcal{N}$. We will remain in the analytic category until the end of Chapter 5.

With a view to constructing Hsiang-Pati coordinates on $\widetilde{U}$ in an analytic neighborhood W in $\widetilde{U}$ of some $e \in E$, we will prove the following:

Proposition 4.1.1. *Given $e \in E$ and an analytic neighborhood W of e in $\widetilde{U}$, there exist linear functions $h\colon \mathbb{C}^N \to \mathbb{C}$, and $(j,k,l)\colon \mathbb{C}^N \to \mathbb{C}^3$, and $(\alpha,\beta)\colon \mathbb{C}^N \to \mathbb{C}^2$ so that over W:*

- **a.** *$h \circ \pi$ generates $\pi^{-1}(\mathfrak{m}_v)(W)$,*
- **b.** *$\{d(j\circ\pi), d(k\circ\pi), d(l\circ\pi)\}$ generates $\mathcal{N}_{\widetilde{U}}(W)$, and*
- **c.** *$d(\alpha\circ\pi)\wedge d(\beta\circ\pi)$ is a minimal element of $\Lambda^2\mathcal{N}_{\widetilde{U}}(W)$*

  (i.e. vanishes to least order at the point e as in Definition 4.3.2).

Moreover, we can choose these linear functions generically, and so that if we define the hyperplane $H := \ker h$, codimension 2 plane $D_2 := \ker\alpha \cap \ker\beta$, and codimension 3 plane $D_3 := \ker j \cap \ker k \cap \ker l$, we have a flag:

- **d.** *$D_3 \subset D_2 \subset H$.*

The last condition in this proposition gives it its name: the “Flag” Proposition.

This proposition will allow us to choose linear functions $h = j = \alpha$, $k = \beta$, and l that are “Nash-minimal” in the following sense.

Definition 4.1.2. *Given a point $e \in E$, an analytic neighborhood W of e in $\widetilde{U}$, and a triple of linear functions $\{j, k, l\}$ on $\mathbb{C}^N$, define*

$$\phi := j \circ \pi, \quad \psi := k \circ \pi, \quad \text{and} \quad \rho := l \circ \pi.$$

We will say that such a choice $\{j, k, l\}$ is Nash-minimal *(with respect to the triple point $e \in E$) if, near e (i.e. in W), the functions ϕ, ψ, and ρ satisfy the following three conditions:*

(i) *ϕ is a generator for $\pi^{-1}(\mathfrak{m}_v)(W)$;*

(ii) *$\{d\phi, d\psi, d\rho\}$ is a generating set for $\mathcal{N}_{\widetilde{U}}(W)$; and*

(iii) *$d\phi \wedge d\psi$ is minimal in $\Lambda^2\mathcal{N}_{\widetilde{U}}(W)$.*

We will also say that the functions ϕ, ψ, and ρ are Nash-minimal *(when they arise from a Nash-minimal set of linear functions).*

Clearly the Flag Proposition above enables us to generically choose, for any $e \in E$, linear functions $h = j = \alpha$, $k = \beta$, and l that are Nash-minimal. It is this ϕ, ψ, and ρ (arising from a Nash-minimal choice of linear functions j, k, and l) that will be shown in Chapter 5 to be Hsiang-Pati coordinates on $W \subset \widetilde{U}$, and thus to be "monomial" generators for the Nash sheaf (see Definition 5.1.1) over W.

4.2 Trivializations

Let $\mathcal{F}$ be a coherent sheaf of rank r, and let $\widehat{\pi}\colon \widehat{U} \to U$ be the blowup of U relative to $\mathcal{F}$; denote the corresponding exceptional set by $\widehat{E}$. From the construction of $\widehat{U}$ we get a canonical map $\gamma\colon \widehat{U} \to \mathrm{Gr}(N-r, N)$ (see Section 2.1.3).

Given a codimension r subspace $D_r \subseteq \mathbb{C}^N$, define the Schubert variety

$$S(D_r) := \{E^r \in \mathrm{Gr}(r, N) \mid \dim(E^r \cap D_r) \geq 1\}.$$

Note that $S(D_r)$ is the codimension 1 subset of $\mathrm{Gr}(r, N)$ consisting of the r-planes in $\mathbb{C}^N$ that are *not* transverse to D_r.

Given any linear projection $p\colon \mathbb{C}^N \to \mathbb{C}^r$, define $D_r := \ker p$. Let Υ be the universal subsheaf over $\mathrm{Gr}(r, N)$. The map p induces (see the discussion after Proposition A3.11 in [PS97]) a trivialization of Υ over $\mathrm{Gr}(r, N) - S(D_r)$. We can pull this back to a trivialization of $\widehat{\pi}^*\mathcal{F}/\mathrm{Tors}(\widehat{\pi}^*\mathcal{F})$ over $\widehat{U} - \gamma^{-1}S(D_r)$, since this is isomorphic to the pullback of the universal quotient sheaf over $\mathrm{Gr}(r, N)$ (see Section 2.1.3); note we are making use of the isomorphism between $\mathrm{Gr}(r, N)$ and $\mathrm{Gr}(N-r, N)$ under which the universal subsheaf is pulled back to the dual of the universal quotient sheaf. Moreover, if $\widetilde{\pi}\colon \widetilde{U} \to \widehat{U}$ is a further blowup of $\widehat{U}$, we can pull this trivialization up to a trivialization of $(\widehat{\pi} \circ \widetilde{\pi})^*\mathcal{F}/\mathrm{Tors}((\widehat{\pi} \circ \widetilde{\pi})^*\mathcal{F})$ over $\widetilde{U} - (\gamma \circ \widetilde{\pi})^{-1}S(D_r)$.

4.3 Proof of the Flag Proposition

We will now prove Proposition 4.1.1.

Proof. We first prove part (a) of the proposition; as in Section 3.1, let

$$\pi_0 \,:\; (Bl(U), C) \;\;\rightarrow\;\; (U, v)$$

be the blowup along the maximal ideal sheaf $\mathfrak{m}_v$ of the singularity v. As explained in Section 4.2, above, any linear projection $h\colon \mathbb{C}^m \to \mathbb{C}$ induces a trivialization of

$$\pi_0^*\mathfrak{m}_v \big/ \mathrm{Tors}(\pi_0^*\mathfrak{m}_v) = \pi_0^{-1}(\mathfrak{m}_v)$$

over $Bl(U) - \gamma^{-1}S(H)$, where $H := D_1 :=$ ker h. The trivialization in this case is given by the global section $h \circ \pi_0$; in other words, $h \circ \pi_0$ generates $\pi_0^{-1}(\mathfrak{m}_v)$ over $Bl(U) - \gamma^{-1}S(H)$. Now if we blow up the rest of the way to $\widetilde{U}$ as in Section 3.1 (*i.e.* blow up via $\widetilde{\pi}_0 = \check{\pi}_0 \circ \breve{\pi} \circ \widetilde{\pi}$, so that $\pi = \pi_0 \circ \widetilde{\pi}_0$), then $h \circ \pi$ generates

$$\pi^*\mathfrak{m}_v \big/ \mathrm{Tors}(\pi^*\mathfrak{m}_v) = \pi^{-1}(\mathfrak{m}_v)$$

over $Bl(\widetilde{U}) - (\gamma \circ \widetilde{\pi}_0)^{-1}S(H)$. Now given an $e \in E$, we wish to choose h so that $h \circ \pi$ generates $\pi^{-1}(\mathfrak{m}_v)$ near e; therefore we must choose h so that $\widetilde{\pi}_0(e)$ is not in $\gamma^{-1}S(H)$ (and when this condition holds we will say that H is a *good* hyperplane relative to the point $e \in E$ in the resolution $\widetilde{U}$). This choice is possible and generic, by the following transversality lemma, adapted from [Kle74] in Proposition A3.11 of [PS97]:

Lemma 4.3.1. *For generic D_r in* $\mathrm{Gr}(N-r, N)$ *(with notation as in Section 4.2), $\gamma^{-1}S(D_r) \cap \widehat{E}$ is either empty or has codimension 1 in $\widehat{E}$, and can be arranged to miss any finite set of points in $\widehat{E}$.*

A similar process will prove parts (b) and (c) of Proposition 4.1.1; we prove part (b) first. Let

$$\pi_1 : (\widehat{U}, \widehat{E}) \rightarrow (U, v)$$

be the Nash blowup of U, *i.e.* the blowup of the sheaf of 1-forms Ω^1_U on U. Define, in the notation of Section 3.1, $\widetilde{\pi}_1 := \check{\pi}_1 \circ \breve{\pi} \circ \widetilde{\pi}$. By Section 4.2, any linear projection $p = (j, k, l)$ of $\mathbb{C}^n$ onto $\mathbb{C}^3$ with ker p = ker $j \cap$ ker $k \cap$ ker $l =: D_3$ will induce a trivialization of the Nash sheaf

$$\pi_1^* \Omega^1_U \big/ \mathrm{Tors}(\pi_1^* \Omega^1_U) = \mathcal{N}_{\hat{U}}$$

over $\hat{U} - \gamma^{-1}S(D_3)$. Pulling up by $\widetilde{\pi}_1$, this in turn induces a trivialization of the generalized Nash sheaf

$$\pi^* \Omega^1_U \big/ \mathrm{Tors}(\pi^* \Omega^1_U) = \mathcal{N}_{\widetilde{U}}$$

over $\widetilde{U} - (\gamma \circ \widetilde{\pi}_1)^{-1}S(D_3)$. In other words, the projection (j, k, l) gives us a system of generators for $\mathcal{N}_{\widetilde{U}}$ over $\widetilde{U} - (\gamma \circ \widetilde{\pi}_1)^{-1}S(D_3)$, namely $\{d(j \circ \pi), d(k \circ \pi), d(l \circ \pi)\}$. If we wish this to be a basis of $\mathcal{N}_{\widetilde{U}}$ near a given point $e \in E$, we must choose (j, k, l) so that $\widetilde{\pi}_1(e)$ is not in $\gamma^{-1}S(D_3)$; we can do this generically by Lemma 4.3.1 (again, we call a codimension 3 plane D_3 *good* relative to $e \in E$ when this condition is satisfied).

We now prove part (c) of Proposition 4.1.1. Let $e \in E$ have sufficiently small analytic neighborhood W in $\widetilde{U}$. Since $\widetilde{U}$ is a complete resolution, $Fitt_2$ is a locally principal sheaf of ideals in $\widetilde{U}$; thus over W, $Fitt_2$ has a single generator, say, g. Note that g will involve u, v, and w at a triple point $e \in E$, but will only involve u and v (respectively u) at a double point (respectively simple point) $e \in E$. Recall that $Fitt_2$ is the Fitting invariant corresponding to the inclusion

$$\Lambda^2 \mathcal{N}_{\widetilde{U}} \quad \hookrightarrow \quad \Omega^2_{\widetilde{U}}(\log E);$$

thus the coefficients of any 2-form $\omega \in \Lambda^2 \mathcal{N}_{\widetilde{U}}(W)$, expressed in terms of the standard basis for $\Omega^2_{\widetilde{U}}(\log E)(W)$, must all be divisible by g. We now define precisely what it means for such a 2-form ω to vanish to "minimum order". We first deal with the case where e is a triple point.

Definition 4.3.2. *Given a complete resolution $\pi\colon (\widetilde{U}, E) \to (U, v)$, and a point $e \in E$ with analytic neighborhood W in $\widetilde{U}$, choose coordinates $\{u, v, w\}$ about e in W. Let g denote the generator of the locally principal ideal sheaf $Fitt_2$. Suppose $\omega \in \Lambda^2 \mathcal{N}_{\widetilde{U}}(W)$ is written in the form*

$$\omega = \omega_1 \frac{du\, dv}{uv} + \omega_2 \frac{dv\, dw}{vw} + \omega_3 \frac{du\, dw}{uw}.$$

We say that ω vanishes to minimum order *at a point $e \in E$ if one of its coefficients ω_i vanishes to the minimum order at the point e, i.e. if $\omega_i = \mu g$ for some local unit μ. (Note that the other coefficients ω_j, $j \neq i$, must be divisible by such an*

ω_i, since all the ω_j are divisible by g). Thus ω is minimal if it vanishes to the minimum order along each of the components of E.

When e is a double or simple point of e we will similarly say that a 2-form ω is minimal if it vanishes to the minimum order (*i.e.* that of the generator g of $Fitt_2$) along each of the components of E that pass through the point e.

The following lemma proves that minimality is generic.

Lemma 4.3.3. *Choose a point $e \in E$ (with analytic neighborhood $W \subset \widetilde{U}$). A generic 2-form $\omega \in \Lambda^2 \mathcal{N}_{\widetilde{U}}(W)$ vanishes to the minimum order, namely that of g, at the point e.*

Proof. We handle only the triple point case; the simple and double point cases are similar. Let g be the generator of $Fitt_2$ as above, and let

$$\tau = \tau_1 \frac{du\, dv}{uv} + \tau_2 \frac{dv\, dw}{vw} + \tau_3 \frac{du\, dw}{uw}$$

be any 2-form in $\Lambda^2 \mathcal{N}_{\widetilde{U}}(W)$ that vanishes to minimum order at e as in Definition 4.3.2. Such a τ exists because we know $Fitt_2$ to be generated by the single element g, and $Fitt_2$ is generated by the coefficients that appear when we write a basis for $\Lambda^2 \mathcal{N}_{\widetilde{U}}(W)$ in terms of the basis for $\Omega^2(\log E)$; thus, given a basis for $\Lambda^2 \mathcal{N}_{\widetilde{U}}(W)$, one of the 2-forms in the basis will be such a τ. Now choose an arbitrary $\omega \in \Lambda^2 \mathcal{N}_{\widetilde{U}}(W)$, and suppose that it does not vanish to the minimum order at the given e. It suffices to prove that the perturbation $\omega + \epsilon\tau$ does vanish to the minimum order at e. By definition the coefficients of $\omega + \epsilon\tau$ are given by $\omega_i + \epsilon\tau_i$. Since τ vanishes to minimum order at e, one of the τ_i can be written as a unit times g; without loss of generality suppose $\tau_1 = \mu g$. Since the coefficients of any 2-form in $\Lambda^2\mathcal{N}_{\widetilde{U}}(W)$ are divisible by g, we can write $\omega_i = g\omega_i'$ for some holomorphic function ω_i', $i = 1, 2, 3$. Thus we have

$$\omega_1 + \epsilon\tau_1 = g\omega_i' + \epsilon\mu g = g(\omega_i' + \epsilon\mu);$$

since $\omega_i' + \epsilon\mu$ is a local unit in W, we are done. ■

Given j, k, and l as chosen in part (b) of Proposition 4.1.1 above, the set $\{d(j\circ\pi)\wedge d(k\circ\pi),\ d(k\circ\pi)\wedge d(l\circ\pi),\ d(j\circ\pi)\wedge d(l\circ\pi)\}$ will generate $\Lambda^2\mathcal{N}_{\widetilde{U}}$ near any point $e \notin \gamma^{-1}S(D_3)$. One of these generators must vanish to the minimum order; choose α and β from j, k, and l so that $d(\alpha\circ\pi)\wedge d(\beta\circ\pi)$ is this minimum generator, and define $D_2 := \ker\,\alpha \cap \ker\,\beta$. We will say that such a codimension 2 plane D_2 is *good* when it comes from a "good" codimension 3 plane D_3 as defined above. Note that, since minimality is generic, "goodness" is generic for these D_2 planes: given a good plane D_2, any sufficiently nearby codimension 2 plane $D_2' \in \mathrm{Gr}(N-2, N)$ is also good (since such a D_2' will be defined by α', β' that will induce a generator of $\Lambda^2\mathcal{N}_{\widetilde{U}}$; that generator must also be minimal since it will be a perturbation of the minimal generator $d(\alpha\circ\pi)\wedge d(\beta\circ\pi)$).

Since the choices of the good D_k in the proofs to parts (a)-(c) of Proposition 4.1.1 were generic, the set of good D_k's is open and dense in $\mathrm{Gr}(N-k, N)$ for

$k = 1, 2, 3$ (recall that $H = D_1$). To prove part (d) of Proposition 4.1.1 we must show that we can find a flag of *good* planes $D_3 \subset D_2 \subset H$. We will need the following lemma concerning the genericity of the D_k (taken from the proof of Proposition A3.14 in [PS97]):

Lemma 4.3.4. *Given $D_k \in \mathrm{Gr}(N-k, N)$, let $\mathcal{G} \subset \mathrm{Gr}(N-k, N)$ be a neighborhood of D_k, and let $D_{k-1} \in \mathrm{Gr}(N-(k-1), N)$ be any codimension $(k-1)$ plane containing D_k. Then any D'_{k-1} sufficiently close to D_{k-1} contains some $D'_k \in \mathcal{G}$.*

Choose any good D_3 by choosing j, k, and l as above, and choose from j, k, and l the linear functions α and β to get a minimal generator of $\Lambda^2\mathcal{N}_{\tilde{U}}$ (the D_2 thus defined is *a priori* good and contains D_3). It suffices to show that we can find a good H containing D_2.

Begin by choosing any H' (not necessarily good) containing D_2. Since the set of good hyperplanes is open and dense in $\mathrm{Gr}(N-1, N)$, we can find a sequence of good planes $\{H'_i\}$ that converges to H'. Let $\mathcal{G}_2$ be a neighborhood of good codimension 2 planes in $\mathrm{Gr}(N-2, N)$ about D_2. Lemma 4.3.4 above says that we can choose i sufficiently large so that H'_i contains some $D'_2 \in \mathcal{G}_2$. Update our choice of planes to $D_2 := D'_2$ and $H := H'_i$. Finally, rechoose a good D_3 (set j to be the new α, set k to be the new β, and choose a suitable new l) contained in this new D_2; we now have the desired flag of good planes. ■

Chapter 5

The Main Proposition

5.1 Assumptions

Suppose V is a three-dimensional complex algebraic variety with isolated singular point v; let U be a neighborhood of v in V chosen small enough so that we have an embedding of U into $\mathbb{C}^N$ (*i.e.* an affine neighborhood of v). Let $\pi\colon (\widetilde{U}, E) \to (U, v)$ be a complete resolution of U (as described in Section 3.1). As in Chapter 4, in this Chapter we will be working entirely in the analytic category (see Section 2.4) and abusing notation: for example, we denote $\widetilde{U}^h$ by $\widetilde{U}$, $\mathcal{H}_{\widetilde{U}^h}$ by $\mathcal{O}_{\widetilde{U}}$, and $\mathcal{N}^h$ simply by $\mathcal{N}$, *et cetera.*

By assumption E is a divisor with normal crossings $\sum E_i$, and the resolution $\widetilde{U}$ is complete, *i.e.* satisfies the following three properties:

1. the inverse image $\pi^{-1}(\mathfrak{m}_v)$ of the maximal ideal sheaf is locally free,
2. the Nash sheaf $\mathcal{N}_{\widetilde{U}}$ is locally free, and
3. the Fitting ideal $Fitt_2$ locally defined by the 2×2 subdeterminants of a matrix for the inclusion $\mathcal{N} \hookrightarrow \Omega^1(\log E)$ is locally principal.

Given a point $e \in E$, and an analytic neighborhood W of e in $\widetilde{U}$, we assume that we have chosen a Nash-minimal set of linear functions (in the sense of Definition 4.1.2) j, k, and $l \colon \mathbb{C}^n \to \mathbb{C}$ as in Section 4.1. Thus if we define $\phi := j \circ \pi$, $\psi := k \circ \pi$, and $\rho := l \circ \pi$, we have (near the point e):

1. ϕ generates $\pi^{-1}\mathfrak{m}_v(W)$,
2. $\{d\phi, d\psi, d\rho\}$ generates $\mathcal{N}_{\widetilde{U}}(W)$, and
3. $d\phi \wedge d\psi$ is minimal in $\Lambda^2\mathcal{N}_{\widetilde{U}}(W)$,

where minimality in (3) means that among the generators $d\phi \wedge d\psi$, $d\psi \wedge d\rho$, and $d\phi \wedge d\rho$ of $\Lambda^2\mathcal{N}_{\widetilde{U}}$, $d\phi \wedge d\psi$ vanishes to the least order at the point e (see Definition 4.3.2).

We first state (Section 5.2) and prove (Section 5.4) the Main Proposition in the case where e is a triple point of the exceptional divisor E (and along the way prove a couple of key facts; see Section 5.3). The double and simple point cases differ slightly from the triple point case, and we prove them separately (see Sections 5.5 and.5.6) Finally, we will show that the multiplicities obtained from the Main Proposition are "minimal" (Section 5.7).

In each case, the Main Proposition will be the existence of "monomial" generators for the Nash sheaf (as defined in Definition 5.1.1 below) in an analytic neighborhood of any point in the exceptional divisor of a complete resolution. These monomial generators will be obtained by taking a distinguished monomial part of so-called Hsiang-Pati coordinates on the resolution. The construction and existence of Hsiang-Pati coordinates will form the bulk of the proof of the Main Proposition (see, *e.g.* , Proposition 5.2.2). We define monomial coordinates as follows:

Definition 5.1.1. *We will say that a 1-form ω is* monomial *if it is the differential of a monomial function, i.e. if $\omega = df$ for some monomial function f. A triple $\{\omega_1, \omega_2, \omega_3\}$ of monomial 1-forms will be called a set of* monomial generators *for the Nash sheaf $\mathcal{N}$ if they are monomial 1-forms arising from Hsiang-Pati coordinates $\{\phi, \psi, \rho\}$ (i.e. $\omega_1 = d\phi$, $\omega_2 = d\psi$, and $\omega_3 = d\rho$).*

As we will see in the statement of the Main Proposition, the exponents of these monomial coordinates will satisfy various ordering and linear independence conditions.

5.2 Statement (Triple Point)

The Main Proposition stated in this section will show that there exists a resolution over which we can find monomial generators for the Nash sheaf in an analytic neighborhood of any triple point in the exceptional divisor. As above, let V be a three-dimensional complex algebraic variety with isolated singular point v. Choose a neighborhood U of v small enough so that we have an embedding $U \hookrightarrow \mathbb{C}^n$. Given a complete resolution $\pi\colon (\widetilde{U}, E) \to (U, v)$, a triple point $e \in E$, and an analytic neighborhood W of e in U, we choose coordinates $\{u, v, w\}$ on W for which the components of the exceptional divisor passing through e are $E_i = \{u = 0\}$, $E_j = \{v = 0\}$, $E_k = \{w = 0\}$. With this notation we are ready to state the Main Proposition in the triple point case.

Main Proposition 5.2.1. *With notation as above, after possible change of coordinates, we can find "monomial" generators for the Nash sheaf in the analytic neighborhood W of the triple point $e \in E$, i.e. generators that can be locally written in the form $d\phi$, $d\psi$, $d\rho$ for some functions ϕ ψ, ρ that are of the form*

$$\begin{aligned} \phi &= u^{m_i} v^{m_j} w^{m_k} \\ \psi &= u^{n_i} v^{n_j} w^{n_k} \\ \rho &= u^{p_i} v^{p_j} w^{p_k}, \end{aligned}$$

where the exponents m_l, n_l, p_l satisfy the following ordering and linear independence conditions:

(a) $\begin{vmatrix} m_i & n_i & p_i \\ m_j & n_j & p_j \\ m_k & n_k & p_k \end{vmatrix} \neq 0$; *and*

(b) $m_l \leq n_l \leq p_l$ *for* $l = i, j, k$.

The Main Proposition above is in fact a simple corollary of the following more technical proposition, which proves that, given a triple point $e \in E$, any choice of Nash-minimal functions ϕ, ψ, ρ (see Definition 4.1.2) will give rise to Hsiang-Pati coordinates in an analytic neighborhood of e.

Proposition 5.2.2. *Given a complete resolution* $\widetilde{U}$, *triple point* e *with analytic neighborhood* W, *and Nash-minimal* ϕ, ψ, *and* ρ *as described above, there exist coordinates* $\{u, v, w\}$ *on* W *with* $E_i = \{u = 0\}$, $E_j = \{v = 0\}$, *and* $E_k = \{w = 0\}$ *so that there exist triples of positive integers* $\{m_i, m_j, m_k\}$, $\{n_i, n_j, n_k\}$, *and* $\{p_i, p_j, p_k\}$ *satisfying:*

(a) $\phi = u^{m_i} v^{m_j} w^{m_k}$;

(b) $\psi = S + \psi'$, *where*

i. $S = \sum s_l \phi^{\epsilon_l}$ *is a rational series with each* $\epsilon_l \geq 1$, *and*

ii. $\psi' = u^{n_i} v^{n_j} w^{n_k}$;

(c) $\rho = T + \rho'$, *where*

i. $T = \sum t_l \phi^{\delta_l} (\psi')^{\tau_l}$ *is a rational series with* $\delta_l \geq 1$ *when* $\tau_l = 0$, *and when* $\tau_l \neq 0$, $\delta_l m_i + \tau_l n_i \geq n_i$ *(and similarly for* j, k*), and*

ii. $\rho' = u^{p_i} v^{p_j} w^{p_k}$;

(d) $\begin{vmatrix} m_i & n_i & p_i \\ m_j & n_j & p_j \\ m_k & n_k & p_k \end{vmatrix} \neq 0$;

(e) $m_i \leq n_i$, $m_j \leq n_j$, *and* $m_k \leq n_k$; *and*

(f) $n_i \leq p_i$, $n_j \leq p_j$, *and* $n_k \leq p_k$.

Main Proposition 5.2.1 is now clearly a simple corollary to Proposition 5.2.2:

Corollary 5.2.3. *We can rechoose* ϕ, ψ, *and* ρ *to be* ϕ, ψ', *and* ρ', *respectively, and these new choices will be a set of monomial generators for* $\mathcal{N}_{\widetilde{U}}(W)$.

5.3 Key Facts

In this section we will state and prove the two key facts that will allow us to define the n_i and p_i in the proof of proposition 5.2.2.

Fact 5.3.1. *Let* e *be a triple point with analytic neighborhood* W *in* $\widetilde{U}$. *Given Nash-minimal* ϕ, ψ, *and* ρ, *we have (in* W*):*

$$d\phi \wedge d\psi = u^{g_i} v^{g_j} w^{g_k} \left(A \frac{du \wedge dv}{uv} + B \frac{dv \wedge dw}{vw} + C \frac{du \wedge dw}{uw} \right),$$

where g_i, g_j, *and* g_j *are positive integers, and at least one of* A, B, *and* C *is a local unit.*

Proof. We will use the assumption made in section 5.1 that the ideal sheaf $Fitt_2$ is locally principal, where the Fitting invariant $Fitt_2 := Fitt_2(\alpha_1) = Fitt_1(\alpha_2)$ is the sheaf of ideals locally (say, over W) generated by the entries of a matrix representing the inclusion α_2 of $\Lambda^2 \mathcal{N}_{\widetilde{U}}(W)$ into $\Omega^2_{\widetilde{U}}(\log E)(W)$. Using the bases $\{d\phi \wedge d\psi,\, d\psi \wedge d\rho,\, d\phi \wedge d\rho\}$ and $\{\frac{du \wedge dv}{uv}, \frac{dv \wedge dw}{vw}, \frac{du \wedge dw}{uw}\}$, this matrix is:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} := \begin{pmatrix} \begin{vmatrix} u\phi_u & u\psi_u \\ v\phi_v & v\psi_v \end{vmatrix} & \begin{vmatrix} v\psi_v & v\rho_v \\ w\psi_w & w\rho_w \end{vmatrix} & \begin{vmatrix} u\phi_u & u\rho_u \\ w\phi_w & w\rho_w \end{vmatrix} \\ \\ \begin{vmatrix} u\phi_u & u\psi_u \\ v\phi_v & v\psi_v \end{vmatrix} & \begin{vmatrix} v\psi_v & v\rho_v \\ w\psi_w & w\rho_w \end{vmatrix} & \begin{vmatrix} u\phi_u & u\rho_u \\ w\phi_w & w\rho_w \end{vmatrix} \\ \\ \begin{vmatrix} u\phi_u & u\psi_u \\ v\phi_v & v\psi_v \end{vmatrix} & \begin{vmatrix} v\psi_v & v\rho_v \\ w\psi_w & w\rho_w \end{vmatrix} & \begin{vmatrix} u\phi_u & u\rho_u \\ w\phi_w & w\rho_w \end{vmatrix} \end{pmatrix}$$

In other words, $Fitt_2(W)$ is the ideal generated by the 2×2 minors of the matrix

$$\begin{vmatrix} u\phi_u & u\psi_u & u\rho_u \\ v\phi_v & v\psi_v & v\rho_v \\ w\phi_w & w\psi_w & w\rho_w \end{vmatrix}$$

that represents the inclusion of $\mathcal{N}_{\widetilde{U}}(W)$ into $\Omega^1_{\widetilde{U}}(\log E)(W)$ (thus $Fitt_2(W)$ is the second Fitting ideal, or first Fitting invariant, of this inclusion).

Since $Fitt_2$ is locally principal, with local generator, say, g over W, we can write each 2×2 subdeterminant as a multiple of g; denote these multiples as $a_{ij} =: gb_{ij}$, where each b_{ij} is a holomorphic function on $\widetilde{U}$ (note that to prove Fact 5.3.1 we must show that one of b_{11}, b_{21}, and b_{31} is a local unit). Moreover, since $\langle g \rangle = \langle a_{11}, \ldots, a_{33} \rangle$, we also have

$$\begin{aligned} g &= \sum \alpha_{ij}\, a_{ij} \\ &= \sum \alpha_{ij}\, gb_{ij} \end{aligned}$$

for some holomorphic functions α_{ij}, and thus

$$1 = \sum \alpha_{ij}\, b_{ij}.$$

Thus $\sum \alpha_{ij} b_{ij}$ must be a local unit, *i.e.* have a nonzero constant term; in order for this to happen, at least one of the terms $\alpha_{ij} b_{ij}$ must have a nonzero constant term, and thus (for this ij) both α_{ij} and b_{ij} must have nonzero constant term. We have thus shown that at least one of the b_{ij} must be a local unit.

Since by assumption ϕ, ψ, and ρ are Nash-minimal, the 2-form $d\phi \wedge d\psi$ is minimal, *i.e.* vanishes to the least order at the point e (see Definition 4.3.2), and in the notation above

$$d\phi \wedge d\psi = g\left(b_{11}\,\frac{du \wedge dv}{uv} + b_{21}\,\frac{dv \wedge dw}{vw} + b_{31}\,\frac{du \wedge dw}{uw}\right).$$

One of b_{11}, b_{21}, and b_{31} (in the notation used in the statement of the fact, A, B, and C) must be a local unit (else one of the other generators of $\Lambda^2\mathcal{N}_{\widetilde{U}}(W)$ would vanish to a lower order, namely the order of g, at e).

It now suffices to show that $g = \mu u^{g_i} v^{g_j} w^{g_k}$ for some positive integers g_i, g_j, and g_k and local unit μ. The inclusion map $\Lambda^2\mathcal{N}_{\widetilde{U}} \hookrightarrow \Omega^2_{\widetilde{U}}(\log E)$ is an isomorphism if and only if we are away from E (by Claim 2.2.8 and the fact that the inclusion $\Omega^2_{\widetilde{U}} \hookrightarrow \Omega^2_{\widetilde{U}}(\log E)$ is never an isomorphism at points contained in E). Thus g cannot vanish or have poles away from $E|_{\widetilde{U}} = \{u = 0\} \cup \{v = 0\} \cup \{w = 0\}$, and moreover g must vanish or have poles along all of $E|_{\widetilde{U}}$. Hence $g = \mu u^{g_i} v^{g_j} w^{g_k}$ for some integers g_i, g_j, and g_k (where μ is a local unit in W); these integers must be positive because $Fitt_2(W) = \langle g \rangle$, and by construction $Fitt_2$ is a coherent sheaf of ideals in $\mathcal{O}_{\widetilde{U}}$. ■

Fact 5.3.2. *Let e be a triple point with analytic neighborhood W in $\widetilde{U}$. Given*

Nash-minimal ϕ, ψ, and ρ, we have (in W):

$$d\phi \wedge d\psi \wedge d\rho = u^{d_i} v^{d_j} w^{d_k} (\mu\, du \wedge dv \wedge dw),$$

where d_i, d_j, and d_k are positive integers and μ is a local unit.

Proof. We have $\Lambda^3 \mathcal{N}_{\widetilde{U}} \approx \Omega^3_{\widetilde{U}} \otimes \mathcal{O}(-D)$ for some positive divisor D supported on E since $\Lambda^3 \mathcal{N}_{\widetilde{U}} \hookrightarrow \Omega^3_{\widetilde{U}}$ is an inclusion of locally free rank one sheaves that is an isomorphism everywhere away from E. Writing the local defining function for D near e (*i.e.* in W) as $u^{d_i} v^{d_j} w^{d_k}$, we have the desired fact. ■

5.4 Proof of the Proposition (Triple Point)

Armed with the assumptions made in section 5.1 and the two key facts proved in section 5.3, we are ready to prove Proposition 5.2.2.

Proof. **(a)** Note that all computations that follow take place in the neighborhood $W \subset \widetilde{U}$ of the chosen triple point $e \in E$. Since ϕ generates $\pi^{-1}\mathfrak{m}_v(W)$ we can write $\phi = \alpha\, u^{m_i} v^{m_j} w^{m_k}$, where α is a local unit and $Z =: \sum m_i E_i$ is the divisor supported on E determined by the sheaf of locally principal ideals $\pi^{-1}\mathfrak{m}_v$. Note that by definition m_i, m_j and m_k are positive integers. Rechoose coordinates $\{u, v, w\}$ by mapping $u \mapsto u\alpha^{1/m_i}$ (and fixing v, w) so that $\phi = u^{m_i} v^{m_j} w^{m_k}$ as desired. Note that this change of coordinates does not violate the requirement that $E_i = \{u = 0\}$, $E_j = \{v = 0\}$, and $E_k = \{w = 0\}$.

(b) *A priori* we can write ψ as a convergent power series

$$\psi = \sum_{(a,b,c)} r_{a,b,c}\, u^a v^b w^c.$$

Define S and ψ' by separating out the terms in ψ where (a, b, c) is linearly dependent on (*i.e.* a rational multiple of) (m_i, m_j, m_k) as follows:

$$\begin{aligned} \psi &= \sum_{\substack{(a,b,c) \\ \text{dep. on} \\ (m_i,m_j,m_k)}} r_{a,b,c}\, u^a v^b w^c + \sum_{\substack{(a,b,c) \\ \text{ind. of} \\ (m_i,m_j,m_k)}} r_{a,b,c}\, u^a v^b w^c \\ &=: \sum_l s_l (u^{m_i} v^{m_j} w^{m_k})^{\epsilon_l} + \sum_{\star_m} r_{a,b,c}\, u^a v^b w^c \\ &=: S + \psi' \end{aligned} \tag{5.4.1}$$

where, in the first term above, the s_l are the coefficients $r_{a,b,c}$ with (a, b, c) linearly dependent on (m_i, m_j, m_k); and in the second term, a triple (a, b, c) satisfies $(\star_m)$ if it is linearly independent of the triple (m_i, m_j, m_k), *i.e.* if at least one of $am_j - bm_i$, $am_k - cm_i$, and $bm_k - cm_i$ is nonzero.

Part (i) of (b) now follows since ψ must vanish along E to at least the order of ϕ (since ϕ generates $\pi^{-1}\mathfrak{m}_v$ and thus vanishes to minimum order), and thus $a \le m_i$, $b \le m_j$, and $c \le m_k$ for every triple (a, b, c) with nonzero coefficient in ψ. Hence each $\epsilon_l := a/m_i = b/m_j = c/m_k$ must be ≥ 1 as desired.

To prove part (ii) of (b) we calculate $d\phi\, d\psi$ and apply Fact 5.3.1. Using the fact that

$$u\psi_u = \sum r_{a,b,c}\, a\, u^a v^b w^c$$

(we will henceforth denote the $r_{a,b,c}$ by r when convenient) and the multilinearity of the determinant, we have

$$\begin{aligned}
d\phi\, d\psi &= \left|\begin{smallmatrix} u\phi_u & u\psi_u \\ v\phi_v & v\psi_v \end{smallmatrix}\right| \frac{du\, dv}{uv} + \left|\begin{smallmatrix} v\phi_v & v\psi_v \\ w\phi_w & w\psi_w \end{smallmatrix}\right| \frac{dv\, dw}{vw} + \left|\begin{smallmatrix} u\phi_u & u\psi_u \\ w\phi_w & w\psi_w \end{smallmatrix}\right| \frac{du\, dw}{uw} \\
&= u^{m_i} v^{m_j} w^{m_k} \left(\left|\begin{smallmatrix} m_i & u\psi_u \\ m_j & v\psi_v \end{smallmatrix}\right| \frac{du\, dv}{uv} \right. \\
&\qquad \left. + \left|\begin{smallmatrix} m_j & v\psi_v \\ m_k & w\psi_w \end{smallmatrix}\right| \frac{dv\, dw}{vw} + \left|\begin{smallmatrix} m_i & u\psi_u \\ m_k & w\psi_w \end{smallmatrix}\right| \frac{du\, dw}{uw} \right) \\
&= u^{m_i} v^{m_j} w^{m_k} \left(\left|\begin{smallmatrix} m_i & \sum r(a) u^a v^b w^c \\ m_j & \sum r(b) u^a v^b w^c \end{smallmatrix}\right| \frac{du\, dv}{uv} \right. \\
&\qquad \left. + \left|\begin{smallmatrix} m_j & \sum r(b) u^a v^b w^c \\ m_k & \sum r(c) u^a v^b w^c \end{smallmatrix}\right| \frac{dv\, dw}{vw} + \left|\begin{smallmatrix} m_i & \sum r(a) u^a v^b w^c \\ m_k & \sum r(c) u^a v^b w^c \end{smallmatrix}\right| \frac{du\, dw}{uw} \right) \\
&= u^{m_i} v^{m_j} w^{m_k} \\
&\qquad \sum_{(a,b,c)} r u^a v^b w^c \left(\left|\begin{smallmatrix} m_i a \\ m_j b \end{smallmatrix}\right| \frac{du\, dv}{uv} + \left|\begin{smallmatrix} m_j b \\ m_k c \end{smallmatrix}\right| \frac{dv\, dw}{vw} + \left|\begin{smallmatrix} m_i a \\ m_k c \end{smallmatrix}\right| \frac{du\, dw}{uw} \right).
\end{aligned}$$

For (a, b, c) in the sum that do not satisfy $(\star_m)$, all the subdeterminants above are zero; thus we can write :

$$\begin{aligned}
d\phi\, d\psi &= u^{m_i} v^{m_j} w^{m_k} \\
&\qquad \sum_{\star_m} r u^a v^b w^c \left(\left|\begin{smallmatrix} m_i a \\ m_j b \end{smallmatrix}\right| \frac{du\, dv}{uv} + \left|\begin{smallmatrix} m_j b \\ m_k c \end{smallmatrix}\right| \frac{dv\, dw}{vw} + \left|\begin{smallmatrix} m_i a \\ m_k c \end{smallmatrix}\right| \frac{du\, dw}{uw} \right) \\
&= u^{m_i} v^{m_j} w^{m_k} \left(\left(\sum_{\star_m} r u^a v^b w^c \left|\begin{smallmatrix} m_i a \\ m_j b \end{smallmatrix}\right| \right) \frac{du\, dv}{uv} \right. \\
&\qquad \left. + \left(\sum_{\star_m} r u^a v^b w^c \left|\begin{smallmatrix} m_j b \\ m_k c \end{smallmatrix}\right| \right) \frac{dv\, dw}{vw} + \left(\sum_{\star_m} r u^a v^b w^c \left|\begin{smallmatrix} m_i a \\ m_k c \end{smallmatrix}\right| \right) \frac{du\, dw}{uw} \right).
\end{aligned}$$

Note that we abuse notation by using $(\star_m)$ to denote both the property of being linearly independent of $\{m_i, m_j, m_k\}$ and the set of (a, b, c) appearing as exponents in ψ that satisfy this property. We can factor out

$$n_i := \min_{\substack{(a,b,c)\\ \text{with } \star_m}} (a) \qquad n_j := \min_{\substack{(a,b,c)\\ \text{with } \star_m}} (b) \qquad n_k := \min_{\substack{(a,b,c)\\ \text{with } \star_m}} (c)$$

from our expression for $d\phi \wedge d\psi$ (where it is assumed that each (a, b, c) considered has nonzero coefficient in ψ), and by Fact 5.3.1at least one of the remaining coefficients (A, B, or C in the notation below) must be a local unit:

$$d\phi\, d\psi = u^{m_i+n_i} v^{m_j+n_j} w^{m_k+n_k} \left(A\,\frac{du\,dv}{uv} + B\,\frac{dv\,dw}{vw} + C\,\frac{du\,dw}{uw} \right)$$

Note that this is the same as defining $n_i := g_i - m_i$ (and similarly for j and k), where g_i is the exponent defined in Fact 5.3.1. Since at least one of A, B, or C must be a local unit, (n_i, n_j, n_k) must be one of the triples (a, b, c) appearing in ψ' (in other words, the minimum powers of u, v, and w all appear in the same monomial). Looking back at our expression for ψ' in (5.4.1) it is now clear that:

$$\begin{aligned} \psi' &= \sum_{\star_m} r_{a,b,c} u^a v^b w^c \\ &= u^{n_i} v^{n_j} w^{n_k} \sum_{\star_m} r_{a,b,c} u^{a-n_i} v^{b-n_j} w^{c-n_k} \\ &=: u^{n_i} v^{n_j} w^{n_k}\, R \end{aligned}$$

where R is a local unit (so the (n_i, n_j, n_k) is one of the triples (a, b, c)).

To finish the proof of part (b) we must absorb the unit R into the coordinates $\{u, v, w\}$ without disrupting the form of $\phi = u^{m_i} v^{m_j} w^{m_k}$. Since by definition (n_i, n_j, n_k) is a triple (a, b, c) satisfying $(\star_m)$, it is linearly independent of

(m_i, m_j, m_k); thus one of $m_i n_j - m_j n_i$, $m_j n_k - m_k n_j$, $m_i n_k - m_k n_i$ is nonzero. Without loss of generality say $\Delta := m_i n_j - m_j n_i \neq 0$, and rechoose coordinates by mapping

$$\begin{cases} u & \mapsto & u\, R^{m_j/\Delta}, \\ v & \mapsto & v\, R^{-m_i/\Delta}, \\ w & \mapsto & w. \end{cases}$$

Note that this fixes ψ:

$$\begin{aligned} u^{m_i} v^{m_j} w^{m_k} &\mapsto u^{m_i} R^{m_i m_j/\Delta} v^{m_j} R^{-m_j m_i/\Delta} w^{m_k} \\ &= u^{m_i} v^{m_j} w^{m_k}, \end{aligned}$$

and fixes S (since S is a function of ϕ), and absorbs R into ψ', giving us the form we need to finish the proof of (b):

$$\begin{aligned} u^{n_i} v^{n_j} w^{n_k} R &\mapsto u^{n_i} R^{n_i m_j/\Delta} v^{n_j} R^{-n_j m_i/\Delta} w^{n_k} R \\ &= u^{n_i} v^{n_j} w^{n_k} R^{-\Delta/\Delta} R \\ &= u^{n_i} v^{n_j} w^{n_k}. \end{aligned}$$

Note that, since ψ must vanish to at least the order or ϕ, we have $m_i \leq n_i$, $m_j \leq n_j$, and $m_k \leq n_k$, and thus have proved part (e) of the proposition.

(c) *A priori* we have ρ as a series $\rho = \sum_{(\alpha,\beta,\gamma)} \widehat{r}_{\alpha,\beta,\gamma}\, u^\alpha v^\beta w^\gamma$. Define T and ρ' by separating out the terms in ρ where (α, β, γ) is linearly dependent on $\{(m_i, m_j, m_k), (n_i, n_j, n_k)\}$ (*i.e.* a rational linear combination of the elements of the set) as follows:

$$\begin{aligned}
\rho &= \sum_{\substack{(\alpha,\beta,\gamma)\\ \text{dep. on}\\ \{\vec{m},\vec{n}\}}} \widehat{r}_{\alpha,\beta,\gamma}\, u^\alpha v^\beta w^\gamma + \sum_{\substack{(\alpha,\beta,\gamma)\\ \text{ind. of}\\ \{\vec{m},\vec{n}\}}} \widehat{r}_{\alpha,\beta,\gamma}\, u^\alpha v^\beta w^\gamma \\
&=: \sum_l t_l (u^{m_i} v^{m_j} w^{m_k})^{\delta_l} (u^{n_i} v^{n_j} w^{n_k})^{\tau_l} + \sum_{\star\{m,n\}} \widehat{r}_{\alpha,\beta,\gamma}\ u^\alpha v^\beta w^\gamma \\
&=: T + \rho' \qquad (5.4.2)
\end{aligned}$$

where, in the first term above, the t_l are the coefficients $\widehat{r}_{\alpha,\beta,\gamma}$ with (α, β, γ) linearly dependent on the set $\{(m_i, m_j, m_k), (n_i, n_j, n_k)\}$; and in the second term, $(\star_{\{m,n\}})$ means that all the terms (α, β, γ) in the sum are linearly independent of the set $\{(m_i, m_j, m_k), (n_i, n_j, n_k)\}$.

We will first show part (ii) of (c), by comparing Fact 5.3.2 with a calculation of $d\phi\, d\psi\, d\rho$.

$$\begin{aligned}
d\phi\, d\psi\, d\rho &= \begin{vmatrix} u\phi_u & u\psi_u & u\rho_u \\ v\phi_v & v\psi_v & v\rho_v \\ w\phi_w & w\psi_w & w\rho_w \end{vmatrix} \frac{du\, dv\, dw}{uvw} \\
&= u^{m_i} v^{m_j} w^{m_k} \begin{vmatrix} m_i & u\psi_u & u\rho_u \\ m_j & v\psi_v & v\rho_v \\ m_k & w\psi_w & w\rho_w \end{vmatrix} \frac{du\, dv\, dw}{uvw} \\
&= u^{m_i} v^{m_j} w^{m_k} \sum_{(\alpha,\beta,\gamma)} r u^a v^b w^c \begin{vmatrix} m_i & a & u\rho_u \\ m_j & b & v\rho_v \\ m_k & c & w\rho_w \end{vmatrix} \frac{du\, dv\, dw}{uvw},
\end{aligned}$$

where r denotes $r_{\alpha,\beta,\gamma}$ for convenience; the determinant above is zero for any (a, b, c) that is linearly dependent on (m_i, m_j, m_k) (*i.e.* for any terms collected in the S part of ψ), and thus the only nonzero terms come from $\psi' = u^{n_i} v^{n_j} w^{n_k}$;

hence we have:

$$\begin{aligned} d\phi\, d\psi\, d\rho &= u^{m_i} v^{m_j} w^{m_k}\, u^{n_i} v^{n_j} w^{n_k} \begin{vmatrix} m_i & n_i & u\rho_u \\ m_j & n_j & v\rho_v \\ m_k & n_k & w\rho_w \end{vmatrix} \frac{du\, dv\, dw}{uvw} \\ &= u^{m_i+n_i} v^{m_j+n_j} w^{m_k+n_k} \sum_{(\alpha,\beta,\gamma)} \widehat{r} u^\alpha v^\beta w^\gamma \begin{vmatrix} m_i & n_i & \alpha \\ m_j & n_j & \beta \\ m_k & n_k & \gamma \end{vmatrix} \frac{du\, dv\, dw}{uvw}. \end{aligned}$$

For every (α, β, γ) that does not satisfy $(\star_{\{m,n\}})$, the corresponding determinant in the expression above is zero; thus we can write:

$$d\phi\, d\psi\, d\rho = u^{m_i+n_i} v^{m_j+n_j} w^{m_k+n_k} \sum_{\star_{\{m,n\}}} \widehat{r} u^\alpha v^\beta w^\gamma \begin{vmatrix} m_i & n_i & \alpha \\ m_j & n_j & \beta \\ m_k & n_k & \gamma \end{vmatrix} \frac{du\, dv\, dw}{uvw}.$$

We now apply Fact 5.3.2 to factor out

$$p_i := \min_{\substack{(\alpha,\beta,\gamma) \\ \text{with } \star_{\{m,n\}}}} (\alpha) \qquad p_j := \min_{\substack{(\alpha,\beta,\gamma) \\ \text{with } \star_{\{m,n\}}}} (\beta) \qquad p_k := \min_{\substack{(\alpha,\beta,\gamma) \\ \text{with } \star_{\{m,n\}}}} (\gamma)$$

from our expression for $d\phi\, d\psi\, d\rho$, leaving us with

$$d\phi\, d\psi\, d\rho =: u^{m_i+n_i+p_i-1} v^{m_j+n_j+p_j-1} w^{m_k+n_k+p_k-1}\, \mu\, du\, dv\, dw \tag{5.4.3}$$

where μ is a local unit. (Note this is equivalent to defining $p_i := d_i - m_i - n_i + 1$, where d_i is the exponent defined in Fact 5.3.2.) The local unit μ here is

$$\mu = \sum_{\star_{\{m,n\}}} \widehat{r} u^{\alpha-p_i} v^{\beta-p_j} w^{\gamma-p_k} \begin{vmatrix} m_i & n_i & \alpha \\ m_j & n_j & \beta \\ m_k & n_k & \gamma \end{vmatrix}$$

and thus (p_i, p_j, p_k) is actually one of the triples (α, β, γ) satisfying $(\star_{\{m,n\}})$, as well as being componentwise less than or equal to each (α, β, γ). Using 5.4.2 and the above we can now write

$$\begin{aligned} \rho' &= \sum_{\star_{\{m,n\}}} \widehat{r} u^{\alpha} v^{\beta} w^{\gamma} \\ &= u^{p_i} v^{p_j} w^{p_k} \sum_{\star_{\{m,n\}}} \widehat{r} u^{\alpha - p_i} v^{\beta - p_j} w^{\gamma - p_k} \\ &= u^{p_i} v^{p_j} w^{p_k} \widehat{R}, \end{aligned}$$

where $\widehat{R}$ is a local unit.

Note that, since by definition (p_i, p_j, p_k) satisfies $(\star_{\{m,n\}})$ (and, as before, (n_i, n_j, n_k) satisfies $(\star_m)$), we have

$$d := \begin{vmatrix} m_i & n_i & p_i \\ m_j & n_j & p_j \\ m_k & n_k & p_k \end{vmatrix} \neq 0,$$

and thus have proved part (d) of the proposition.

To finish the proof of part (ii) of (c) we must absorb the local unit $\widehat{R}$ into ρ' without disturbing the forms of ϕ and ψ established earlier. To fix ϕ and ψ while absorbing an $\widehat{R}$ into ρ' we rechoose coordinates by mapping

$$\begin{cases} u & \mapsto & u\, R^A, \\ v & \mapsto & v\, R^B, \\ w & \mapsto & w\, R^C. \end{cases}$$

where A, B, and C satisfy

$$\begin{pmatrix} m_i & m_j & m_k \\ n_i & n_j & n_k \\ p_i & p_j & p_k \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}.$$

Since $d \neq 0$ we can do this by choosing

$$\begin{pmatrix} A \\ B \\ C \end{pmatrix} := \begin{pmatrix} m_i & m_j & m_k \\ n_i & n_j & n_k \\ p_i & p_j & p_k \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} (m_j n_k - m_k n_j)/d \\ (m_k n_i - m_i n_k)/d \\ (m_i n_j - m_j n_i)/d \end{pmatrix}.$$

Changing coordinates in this way we have written $\rho' = u^{p_i} v^{p_j} w^{p_k}$ and completed the proof of part (ii) of (c).

We now prove part (i) of (c); we have already written T as

$$T = \sum_{\substack{(\alpha,\beta,\gamma) \\ \text{dep. on} \\ \{\vec{m},\vec{n}\}}} \widehat{r}_{\alpha,\beta,\gamma}\, u^\alpha v^\beta w^\gamma = \sum_l t_l (u^{m_i} v^{m_j} w^{m_k})^{\delta_l} (u^{n_i} v^{n_j} w^{n_k})^{\tau_l}.$$

Since $\rho = T + \rho'$ vanishes to at least order ϕ (*i.e.* $\phi = u^{m_i} v^{m_j} w^{m_k}$ divides ρ, so ρ vanishes to an order greater than or equal to m_i on the component E_i of E), each α, β, and γ above is $\geq m_i$, m_j, and m_k respectively; thus whenever $\tau_l = 0$ in the expression above, we have $\alpha = m_i\delta_l + n_i\tau_l = m_i\delta_l$ (and similarly for j and k), and hence $\delta_l \geq 1$.

To finish the proof of (c) it now suffices to show that, when $\tau_l \neq 0$, we have $m_i\delta_l + n_i\tau_l \geq n_i$ (and similarly for j and k). In other words, we need to show that, when an (α, β, γ) from T (*i.e.* not satisfying $(\star_{m,n})$) is independent of (m_i, m_j, m_k) (although still dependent on the set $\{(m_i, m_j, m_k), (n_i, n_j, n_k)\}$), we have $\alpha \geq n_i$, $\beta \geq n_j$, and $\gamma \geq n_k$. To do this we will compare expressions for $d\phi\, d\psi$ and $d\phi\, d\rho$ and invoke the minimality of $d\phi\, d\psi$.

Since $\psi = S + \psi' = S + u^{n_i} v^{n_j} w^{n_k}$, we have (see the proof of part (ii) of part (b))

$$\begin{aligned} d\phi\, d\psi &= d\phi\, d\psi' \\ &= u^{m_i+n_i} v^{m_j+n_j} w^{m_k+n_k} \\ &\qquad \left(\begin{vmatrix} m_i & n_i \\ m_j & n_j \end{vmatrix} \frac{du\, dv}{uv} + \begin{vmatrix} m_j & n_j \\ m_k & n_k \end{vmatrix} \frac{dv dv}{vw} + \begin{vmatrix} m_i & n_i \\ m_k & n_k \end{vmatrix} \frac{du\, dw}{uw} \right). \end{aligned}$$

On the other hand,

$$\begin{aligned} d\phi\, d\rho &= u^{m_i} v^{m_j} w^{m_k} \\ &\qquad \left(\begin{vmatrix} m_i & u\rho_u \\ m_j & v\rho_v \end{vmatrix} \frac{du\, dv}{uv} + \begin{vmatrix} m_j & v\rho_v \\ m_k & w\rho_w \end{vmatrix} \frac{dv dv}{vw} + \begin{vmatrix} m_i & u\rho_u \\ m_k & w\rho_w \end{vmatrix} \frac{du\, dw}{uw} \right) \\ &= u^{m_i} v^{m_j} w^{m_k} \sum_{(\alpha,\beta,\gamma)} \widehat{r} u^\alpha v^\beta w^\gamma \\ &\qquad \left(\begin{vmatrix} m_i & \alpha \\ m_j & \beta \end{vmatrix} \frac{du\, dv}{uv} + \begin{vmatrix} m_j & \beta \\ m_k & \gamma \end{vmatrix} \frac{dv dv}{vw} + \begin{vmatrix} m_i & \alpha \\ m_k & \gamma \end{vmatrix} \frac{du\, dw}{uw} \right) \\ &= u^{m_i} v^{m_j} w^{m_k} \sum_{\star_m} \widehat{r} u^\alpha v^\beta w^\gamma \\ &\qquad \left(\begin{vmatrix} m_i & \alpha \\ m_j & \beta \end{vmatrix} \frac{du\, dv}{uv} + \begin{vmatrix} m_j & \beta \\ m_k & \gamma \end{vmatrix} \frac{dv dv}{vw} + \begin{vmatrix} m_i & \alpha \\ m_k & \gamma \end{vmatrix} \frac{du\, dw}{uw} \right) \end{aligned}$$

because if (α, β, γ) is dependent on (m_i, m_j, m_k) (*i.e.* does not satisfy $(\star_m)$), then all three 2×2 determinants above are zero. Since $d\phi\, d\psi$ vanishes to the least possible order at the point e, we must have $m_i + n_i \leq m_i + \alpha$, and thus $n_i \leq \alpha$ for each α in a triple satisfying $(\star_m)$ (and similarly, $n_j \leq \beta$ and $n_k \leq \gamma$ for such triples). Thus we have the desired inequality for triples (α, β, γ) appearing in T that are independent of (m_i, m_j, m_k), and so have completed the proof of part (i) of (c).

Moreover, the argument above also shows that $n_i \leq p_i$, $n_j \leq p_j$, and $n_k \leq p_k$ since it holds for any triple (α, β, γ) satisfying $(\star_m)$, and (p_i, p_j, p_k) is such a triple; thus we have shown part (f) of the proposition.

Since part (d) of the proposition was proved during the proof of part (ii) of (c), part (e) was proved at the end of the proof of (b), and part (f) was proved immediately above, the proof of the proposition is complete. ■

We now prove Corollary 5.2.3.

Proof. It is clear that ϕ, ψ', and ρ' as defined above will satisfy parts (i) and (iii) of Nash-minimality (see Definition 4.1.2), as long as we can show part (ii) of Nash-minimality; in other words we must show that $\{d\phi, d\psi', d\rho'\}$ is a basis for $\mathcal{N}$. We will do this by writing each element of the original basis $\{d\phi, d\psi, d\rho\}$ in terms of $d\phi$, $d\psi'$, and $d\rho'$. As in the proof of Proposition 5.2.2 above, we perform all computations in an analytic neighborhood $W \subset \widetilde{U}$ of the triple point $e \in E$.

Obviously $d\phi = d\phi$ is already written in the new basis. We write for later use (using the fact that $\phi = u^{m_i} v^{m_j} w^{m_k}$ by Proposition 5.2.2(a)):

$$d\phi = u^{m_i} v^{m_j} w^{m_k} \left(m_i \frac{du}{u} + m_j \frac{dv}{v} + m_k \frac{dw}{w} \right).$$

Since $d\psi = dS + d\psi'$, we must only show that dS can be written in the new basis; we have

$$\begin{aligned} dS &= d\left(\sum s_l \phi^{\epsilon_l}\right) \\ &= \sum s_l \epsilon_l \phi^{\epsilon_l} \left(m_i \frac{du}{u} + m_j \frac{dv}{v} + m_k \frac{dw}{w}\right) \\ &= \sum s_l \epsilon_l \phi^{\epsilon_l} \left(u^{-m_i} v^{-m_j} w^{-m_k}\, d\phi\right) \\ &= \left(\sum s_l \epsilon_l u^{m_i\epsilon_l - m_i} v^{m_j\epsilon_l - m_j} w^{m_k\epsilon_l - m_k}\right) d\phi, \end{aligned}$$

and since each ϵ_l is ≥ 1, dS is the product of a holomorphic function and $d\phi$. Note that by construction, $m_i\epsilon_l$, $m_j\epsilon_l$, and $m_k\epsilon_l$ are integers. For the computation below, note that (by the definition of ψ' in Proposition 5.2.2(b)):

$$d\psi' = u^{n_i} v^{n_j} w^{n_k} \left(n_i \frac{du}{u} + n_j \frac{dv}{v} + n_k \frac{dw}{w}\right).$$

Finally, since $d\rho = dT + d\rho'$, we will show that dT can be written as a combination of $d\phi$ and $d\psi'$. To do this we will separate T into terms where $\tau_l = 0$ and terms where $\tau_l \neq 0$; coefficients t_l where $\tau_l = 0$ are renamed s_l in the computation below.

$$\begin{aligned} dT &= d\left(\sum t_l \phi^{\delta_l} (\psi')^{\tau_l}\right) \\ &= d\left(\sum s_l \phi^{\delta_l}\right) + d\left(\sum_{\tau_l \neq 0} t_l \phi^{\delta_l} (\psi')^{\tau_l}\right) \\ &= \sum s_l \delta_l \phi^{\delta_l} \left(m_i \frac{du}{u} + m_j \frac{dv}{v} + m_k \frac{dw}{w}\right) + \sum_{\tau_l \neq 0} t_l \phi^{\delta_l} (\psi')^{\tau_l} \\ &\qquad \left((\delta_l m_i + \tau_l n_i)\frac{du}{u} + (\delta_l m_j + \tau_l n_j)\frac{dv}{v} + (\delta_l m_k + \tau_l n_k)\frac{dw}{w}\right); \end{aligned}$$

by the expressions for $d\phi$ and $d\psi'$ above, this becomes:

$$\begin{aligned}
dT &= \sum s_l \delta_l \phi^{\delta_l} \left(u^{-m_i} v^{-m_j} w^{-m_k}\, d\phi\right) \\
&\qquad + \sum_{\tau_l \neq 0} t_l \delta_l \phi^{\delta_l} (\psi')^{\tau_l} \left(u^{-m_i} v^{-m_j} w^{-m_k}\, d\phi\right) \\
&\qquad\qquad + \sum_{\tau_l \neq 0} t_l \tau_l \phi^{\delta_l} (\psi')^{\tau_l} \left(u^{-n_i} v^{-n_j} w^{-n_k}\, d\psi'\right) \\
&= \left(\sum s_l \delta_l u^{m_i\delta_l - m_i} v^{m_j\delta_l - m_j} w^{m_j\delta_l - m_j}\right) d\phi \\
&\quad + \left(\sum_{\tau_l \neq 0} t_l \delta_l u^{m_i\delta_l + n_i\tau_l - m_i} v^{m_j\delta_l + n_j\tau_l - m_i} w^{m_k\delta_l + n_k\tau_l - m_i}\right) d\phi \\
&\quad + \left(\sum_{\tau_l \neq 0} t_l \tau_l u^{m_i\delta_l + n_i\tau_l - n_i} v^{m_j\delta_l + n_j\tau_l - n_i} w^{m_k\delta_l + n_k\tau_l - n_i}\right) d\psi'.
\end{aligned}$$

The first term in the expression above is a holomorphic multiple of $d\phi$ since, by part (i) of Proposition 5.2.2(c), when $\tau_l = 0$ we have $\delta_l \geq 1$. The second term is a holomorphic multiple of $d\phi$, and the third a holomorphic multiple of $d\psi'$, because, again by part (i) of Proposition 5.2.2(c), when $\tau_l \neq 0$, we have $\delta_l m_i + \tau_l n_i \geq n_i \geq m_i$ (and similarly for j and k). This completes the proof of the corollary. ■

5.5 Double Point Case

About a double point $e \in E$, with coordinates $\{u, v, w\}$ in an analytic neighborhood W of e for which the two components of E that pass through e are $E_i = \{u = 0\}$, $E_j = \{v = 0\}$, Main Proposition 5.2.1 has the form:

Main Proposition 5.5.1. *With notation as above, after possible change of coordinates, we can find "monomial" generators for the Nash sheaf in the analytic neighborhood W of the double point $e \in E$, i.e. generators that can be locally written in the form $d\phi$, $d\psi$, $d\rho$ for some functions ϕ ψ, ρ that that are of one of the following two forms (which we call "Case I" and "Case II", respectively):*

$$\begin{array}{lcl} \phi & = & u^{m_i} v^{m_j} \\ \psi & = & u^{n_i} v^{n_j} \\ \rho & = & u^{p_i} v^{p_j} w \end{array} \qquad\qquad \begin{array}{lcl} \phi & = & u^{m_i} v^{m_j} \\ \psi & = & u^{n_i} v^{n_j} w \\ \rho & = & u^{p_i} v^{p_j}, \end{array}$$

where the exponents m_l, n_l, p_l satisfy the following ordering and linear independence conditions:

(a) *In Case I, $\left|\begin{smallmatrix} m_i & n_i \\ m_j & n_j \end{smallmatrix}\right| \neq 0$, and in Case II, $\left|\begin{smallmatrix} m_i & p_i \\ m_j & p_j \end{smallmatrix}\right| \neq 0$; and*

(b) *$m_l \leq n_l \leq p_l$ for $l = i, j, k$.*

As in the triple point case, the Main Proposition above is in fact a simple corollary to the existence of Hsiang-Pati coordinates guaranteed by the following proposition.

Proposition 5.5.2. *Given a complete resolution $\widetilde{U}$, double point e with analytic neighborhood W, and Nash-minimal ϕ, ψ, and ρ as described above, there exist coordinates $\{u, v, w\}$ on W with $E_i = \{u = 0\}$ and $E_j = \{v = 0\}$ so that there exist pairs of positive integers $\{m_i, m_j\}$, $\{n_i, n_j\}$, and $\{p_i, p_j\}$ satisfying:*

(a) *$\phi = u^{m_i} v^{m_j}$;*

(b) *$\psi = S + \psi'$, where*

i. *$S = \sum s_l \phi^{\epsilon_l}$ is a rational series with each $\epsilon_l \geq 1$, and*

ii. *either $\psi' = u^{n_i} v^{n_j}$ or $\psi' = u^{n_i} v^{n_j} w$;*

(c) *$\rho = T + \rho'$, where*

i. *$T = \sum t_l \phi^{\delta_l} (\psi')^{\tau_l}$ is a rational series with $\delta_l \geq 1$ when $\tau_l = 0$, and when $\tau_l \neq 0$, $\delta_l m_i + \tau_l n_i \geq n_i$ (and similarly for j), and*

ii. $\rho' = \begin{cases} u^{p_i} v^{p_j} w & \text{if } \psi' = u^{n_i} v^{n_j}, \\ u^{p_i} v^{p_j} & \text{if } \psi' = u^{n_i} v^{n_j} w; \end{cases}$

(d) *if $\psi' = u^{n_i} v^{n_j}$ and $\rho' = u^{p_i} v^{p_j} w$, then $\left| \begin{smallmatrix} m_i & n_i \\ m_j & n_j \end{smallmatrix} \right| \neq 0$, and if $\psi' = u^{n_i} v^{n_j} w$ and $\rho' = u^{p_i} v^{p_j}$, then $\left| \begin{smallmatrix} m_i & p_i \\ m_j & p_j \end{smallmatrix} \right| \neq 0$;*

(e) *$m_i \leq n_i$ and $m_j \leq n_j$; and*

(f) *$n_i \leq p_i$ and $n_j \leq p_j$.*

Main Proposition 5.5.1 is now clearly a simple corollary to Proposition 5.5.2:

Corollary 5.5.3. *We can rechoose ϕ, ψ, and ρ to be ϕ, ψ', and ρ', respectively (in either case), and these new choices will be a set of monomial generators for $\mathcal{N}_{\widetilde{U}}(W)$.*

It is worth remarking that in Pati's double-point statement of the 3-dimensional case (see Section 2.5.3 and [Pat94]), the functions playing the roles of ϕ, ψ', and ρ are always of the "first" type, *i.e.* Pati always blows up enough to get $\psi' = u^{n_i} v^{n_j}$ and $\rho' = u^{p_i} v^{p_j} w$. Pati blows up as much as is necessary to achieve this result. Here we only want to blow up enough so that our resolution is "complete" (in the sense of Definition 3.1.1); such a resolution will guarantee that we are in one of the two types of coordinates above, but cannot ensure that we are in the "first" type as in Pati. This does not, however, make the Hsiang-Pati coordinates any less useful (as will be evident in Chapters 6–9).

As in the proof of Proposition 5.2.2, we will rely on two key facts:

Fact 5.5.4. *Let e be a double point with analytic neighborhood W in $\widetilde{U}$. Given Nash-minimal ϕ, ψ, and ρ, we have (in W):*

$$d\phi \wedge d\psi = u^{g_i} v^{g_j} \left(A \frac{du \wedge dv}{uv} + B \frac{dv \wedge dw}{v} + C \frac{du \wedge dw}{u} \right),$$

where g_i and g_j are positive integers, and at least one of A, B, and C is a local unit.

Proof. The proof of the fact above is entirely analogous to the proof of Fact 5.3.1, with only a couple of differences. First, in the double point case, $Fitt_2(W)$ is the Fitting ideal for the map $\Lambda^2 \mathcal{N}_{\widetilde{U}}(W) \hookrightarrow \Omega^2_{\widetilde{U}}(\log E)(W)$ with bases

$$\{d\phi\, d\psi,\, d\psi\, d\rho,\, d\phi\, d\rho\} \quad \text{and} \quad \{\frac{du\, dv}{uv}, \frac{dv\, dw}{v}, \frac{du\, dw}{u}\};$$

in other words $Fitt_2(W)$ is generated by the 2×2 subdeterminants of the matrix

$$\begin{pmatrix} u\phi_u & u\psi_u & u\rho_u \\ v\phi_v & v\psi_v & v\rho_v \\ \phi_w & \psi_w & \rho_w \end{pmatrix}.$$

Second, in the double point case, g vanishes to positive order along all of (and only along) $E|_{\widetilde{U}} = \{u = 0\} \cap \{v = 0\}$ (see the end of the proof of Fact 5.3.1). ■

Fact 5.5.5. *Let e be a double point with analytic neighborhood W in $\widetilde{U}$. Given Nash-minimal ϕ, ψ, and ρ, we have (in W):*

$$d\phi \wedge d\psi \wedge d\rho = u^{d_i} v^{d_j} (\mu\, du \wedge dv \wedge dw),$$

where d_i and d_j are positive integers and μ is a local unit.

Proof. The proof is analogous to the proof of Fact 5.3.2 except that the local defining function for the divisor $-D$ near the double point e is $u^{d_i} v^{d_j}$. ■

We now prove Proposition 5.5.2.

Proof. **(a)** As usual, all computations take place in the analytic neighborhood $W \subset \widetilde{U}$ of our double point $e \in E$. Let $Z =: \sum m_i E_i$ be the divisor on E corresponding to $\pi^{-1}\mathfrak{m}_v$; by part (i) of Nash-minimality, we can write $\phi = \alpha\, u^{m_i} v^{m_j}$ for some local unit α. Rechoose coordinates by $u \mapsto \alpha^{1/m_i}$ (and fixing v, w) to get $\phi = u^{m_i} v^{m_j}$.

(b) As in the proof of Proposition 5.2.2, we can write ψ as

$$\begin{aligned} \psi &= \sum_{(a,b,c)} r_{a,b,c}\, u^a v^b w^c \\ &= \sum_{\substack{(a,b,c) \\ \text{dep. on} \\ (m_i,m_j,0)}} r_{a,b,c}\, u^a v^b w^c + \sum_{\substack{(a,b,c) \\ \text{ind. of} \\ (m_i,m_j,0)}} r_{a,b,c}\, u^a v^b w^c \\ &=: \sum_l s_l (u^{m_i} v^{m_j})^{\epsilon_l} + \sum_{\star_m} r_{a,b,c}\, u^a v^b w^c \\ &=: S + \psi', \end{aligned}$$

where $(\star_m)$ means that all the terms (a,b,c) in the sum are linearly dependent on $(m_i, m_j, 0)$. Since ψ vanishes to at least order ϕ, we must have $a \leq m_i$ and $b \leq m_j$ for all triples (a,b,c), and thus S is a rational series in ϕ with $\epsilon_l \geq 1$ for all l; this proves part (i) of (b).

Define

$$n_i := \min_{\substack{(a,b,c)\\ \text{with } \star_m}} (a) \qquad n_j := \min_{\substack{(a,b,c)\\ \text{with } \star_m}} (b);$$

then $\phi' = u^{n_i} v^{n_j} R$ where $R = \sum_{\star_m} r u^{a-n_i} v^{b-n_j} w^c$ is a holomorphic function that is not divisible by u or v.

We must now show that R is either a local unit, or a coordinate independent of u and v (that we can then take to be w). We begin by computing:

$$\begin{aligned} d\phi\, d\psi &= \left|\begin{smallmatrix} u\phi_u & u\psi_u \\ v\phi_v & v\psi_v \end{smallmatrix}\right| \frac{du\, dv}{uv} + w^{-1} \left|\begin{smallmatrix} v\phi_v & v\psi_v \\ 0 & w\psi_w \end{smallmatrix}\right| \frac{dv\, dw}{v} + w^{-1} \left|\begin{smallmatrix} u\phi_u & u\psi_u \\ 0 & w\psi_w \end{smallmatrix}\right| \frac{du\, dw}{u} \\ &= u^{m_i} v^{m_j} \sum_{(a,b,c)} r u^a v^b w^c \\ &\qquad \left(\left|\begin{smallmatrix} m_i & a \\ m_j & b \end{smallmatrix}\right| \frac{du\, dv}{uv} + m_j c w^{-1} \frac{dv\, dw}{v} + m_i c w^{-1} \frac{du\, dw}{u} \right), \end{aligned}$$

where if (a, b, c) does not satisfy $(\star_m)$, then $c = 0$ and $m_i a - m_j b = 0$, and thus:

$$\begin{aligned}
d\phi\, d\psi &= u^{m_i} v^{m_j} \sum_{\star_m} r u^a v^b w^c \\
&\qquad \left(\left| \begin{smallmatrix} m_i & a \\ m_j & b \end{smallmatrix} \right| \frac{du\, dv}{uv} + m_j c w^{-1} \frac{dv\, dw}{v} + m_i c w^{-1} \frac{du\, dw}{u} \right) \\
&= u^{m_i+n_i} v^{m_j+m_j} \sum_{\star_m} r u^{a-n_i} v^{b-n_j} w^c \\
&\qquad \left(\left| \begin{smallmatrix} m_i & a \\ m_j & b \end{smallmatrix} \right| \frac{du\, dv}{uv} + m_j c w^{-1} \frac{dv\, dw}{v} + m_i c w^{-1} \frac{du\, dw}{u} \right) \\
&= u^{m_i+n_i} v^{m_j+n_j} \left(\sum_{\star_m} r u^{a-n_i} v^{b-n_j} w^c \left| \begin{smallmatrix} m_i & a \\ m_j & b \end{smallmatrix} \right| \frac{du\, dv}{uv} \right. \\
&\qquad + \sum_{\star_m} r u^{a-n_i} v^{b-n_j} w^{c-1} m_j c \frac{dv\, dw}{v} \\
&\qquad \left. + \sum_{\star_m} r u^{a-n_i} v^{b-n_j} w^{c-1} m_i c \frac{du\, dw}{u} \right) \\
&=: u^{m_i+n_i} v^{m_j+n_j} \left(A \frac{du\, dv}{uv} + B \frac{dv\, dw}{v} + C \frac{du\, dw}{u} \right).
\end{aligned}$$

By Fact 5.5.4, one of A, B, and C must be a local unit. This implies that R is either a local unit, or the product of a local unit with w: if A is a local unit, then $(a, b, c) = (n_i, n_j, 0)$ is one of the triples satisfying $(\star_m)$, thus in ψ', and $m_i n_j - m_j n_i \neq 0$. By the definitions of ψ' and R we see that R must be a local unit in this case (the minimum a, b, and c in ψ' all appear in the same monomial, namely $u^{n_i} v^{n_j}$). If, on the other hand, A is not a local unit, then neither is R; however by Fact 5.5.4 one of B or C must be a local unit. In such a case the triple $(a, b, c) = (n_i, n_j, 1)$ appears in ψ' and we have $c \neq 0$ (note that there is no condition on the determinant $m_i n_j - m_j n_i$ in this case); thus R can be written as the product of w and a local unit, say μ. Let us now examine these two cases more closely.

In the first case, R is a local unit and $\Delta := m_i n_j - m_j n_i$ is nonzero; thus we can rechoose coordinates by mapping

$$\begin{cases} u & \mapsto & u\,R^{m_j/\Delta} \\ v & \mapsto & v\,R^{-m_i/\Delta} \\ w & \mapsto & w. \end{cases}$$

This fixes ϕ and S while giving us $\psi = u^{n_i}v^{n_j}$ (see the corresponding section of the proof of Proposition 5.2.2). Note that we have also shown the appropriate case of part (d) of the proposition, *i.e.* that $m_i n_j - m_j n_i \neq 0$.

In the second case we have $R = \mu w$, where μ is a local unit. Rechoose coordinates by mapping $w \mapsto \mu^{-1}w$; this fixes ϕ and S (since they do not involve w), and puts ψ' in the form $\psi' = u^{n_i}v^{n_j}w$. This completes the proof of part (ii) of (b).

Since ψ vanishes to at least the order of ϕ we also have (in either case) $m_i \leq n_i$ and $m_j \leq n_j$, which proves part (e).

(c) To better handle the two cases, define

$$\epsilon := \begin{cases} 0, & \text{if we are in ``case I'', } i.e.\ \psi' = u^{n_i}v^{n_j}. \\ 1, & \text{if we are in ``case II'', } i.e.\ \psi' = u^{n_i}v^{n_j}w. \end{cases}$$

We can split ρ as:

$$\begin{aligned}\rho &= \sum_{(\alpha,\beta,\gamma)} \widehat{r}_{\alpha,\beta,\gamma}\, u^\alpha v^\beta w^\gamma \\ &= \sum_{\substack{(\alpha,\beta,\gamma)\\ \text{dep. on}\\ \{\vec{m},\vec{n}\}}} \widehat{r}\, u^\alpha v^\beta w^\gamma + \sum_{\substack{(\alpha,\beta,\gamma)\\ \text{ind. of}\\ \{\vec{m},\vec{n}\}}} \widehat{r}\, u^\alpha v^\beta w^\gamma \\ &=: \sum_l t_l (u^{m_i} v^{m_j})^{\delta_l} (u^{n_i} v^{n_j} w^\epsilon)^{\tau_l} + \sum_{\star_{\{m,n\}}} \widehat{r}\;\, u^\alpha v^\beta w^\gamma \\ &=: T + \rho'\end{aligned}$$

where $(\star_{\{m,n\}})$ means that all the terms (α,β,γ) in the sum are linearly independent on the set $\{\vec{m},\vec{n}\} = \{(m_i,m_j,0),(n_i,n_j,\epsilon)\}$, Note, since ρ vanishes to at least order ϕ, that when $\tau_l = 0$, we have $\delta_l \geq 1$; this proves the first part of part (i) of (c). The rest of part (i) of (c) will be shown later.

We now show part (ii) of (c) by an application of Fact 5.5.5 to the triple wedge of $d\phi$, $d\psi$, and $d\rho$. We have:

$$\begin{aligned} d\phi\, d\psi\, d\rho &= \begin{vmatrix} u\phi_u & u\psi_u & u\rho_u \\ v\phi_v & v\psi_v & v\rho_v \\ \phi_w & \psi_w & \rho_w \end{vmatrix} \frac{du\,dv\,dw}{uv} \\ &= u^{m_i} v^{m_j} \begin{vmatrix} m_i & u\psi_u & u\rho_u \\ m_j & v\psi_v & v\rho_v \\ 0 & \psi_w & \rho_w \end{vmatrix} \frac{du\,dv\,dw}{uv} \\ &= u^{m_i} v^{m_j} \sum_{(a,b,c)} r u^a v^b w^c\, w^{-1} \begin{vmatrix} m_i & a & u\rho_u \\ m_j & b & v\rho_v \\ 0 & c & w\rho_w \end{vmatrix} \frac{du\,dv\,dw}{uv}; \end{aligned}$$

the determinants above are zero for triples (a,b,c) that do not satisfy $(\star_m)$, and thus we only need consider terms coming from ψ', which by the proof of part (b) above and the definition of ϵ is equal to $u^{n_i} v^{n_j} w^\epsilon$. Thus

$$\begin{aligned}
d\phi\, d\psi\, d\rho &= u^{m_i} v^{m_j} w^{-1}\, u^{n_i} v^{n_j} \begin{vmatrix} m_i & n_i w^\epsilon & u\rho_u \\ m_j & n_j w^\epsilon & v\rho_v \\ 0 & \epsilon w^\epsilon & w\rho_w \end{vmatrix} \frac{du\, dv\, dw}{uv} \\
&= u^{m_i+n_i} v^{m_j+n_j} w^{-1} \sum_{(\alpha,\beta,\gamma)} \widehat{r} u^\alpha v^\beta w^\gamma \begin{vmatrix} m_i & n_i & \alpha \\ m_j & n_j & \beta \\ 0 & \epsilon w & \gamma \end{vmatrix} \frac{du\, dv\, dw}{uv} \\
&= u^{m_i+n_i} v^{m_j+n_j} w^{-1} \sum_{\star_{\{m,n\}}} \widehat{r} u^\alpha v^\beta w^\gamma \begin{vmatrix} m_i & n_i w^\epsilon & \alpha \\ m_j & n_j w^\epsilon & \beta \\ 0 & \epsilon w^\epsilon & \gamma \end{vmatrix} \frac{du\, dv\, dw}{uv},
\end{aligned}$$

since any triple (α, β, γ) not satisfying $(\star_{\{m,n\}})$ will have a nonzero determinant in its expression above. Thus we will only need to consider those terms of ρ that are in ρ'.

Define

$$p_i := \min_{\substack{(\alpha,\beta,\gamma) \\ \text{with } \star_{\{m,n\}}}} (\alpha) \qquad p_j := \min_{\substack{(\alpha,\beta,\gamma) \\ \text{with } \star_{\{m,n\}}}} (\beta);$$

then $\rho' = u^{p_i} v^{p_j} \widehat{R}$ where $\widehat{R}$ is a holomorphic function that is not divisible by u or by v. We now have

$$d\phi\, d\psi\, d\rho = u^{m_i+n_i+p_i} v^{m_j+n_j+p_j} \sum_{\star_{\{m,n\}}} \widehat{r} u^{\alpha-p_i} v^{\beta-p_j} w^{\gamma-1} \begin{vmatrix} m_i & n_i w^\epsilon & \alpha \\ m_j & n_j w^\epsilon & \beta \\ 0 & \epsilon w^\epsilon & \gamma \end{vmatrix} \frac{du\, dv\, dw}{uv}.$$

We wish to show that, in case I (where $\epsilon = 0$), R is a coordinate independent of u and v (and thus can be taken as the third coordinate w); in case II (where $\epsilon = 1$) we will show that R is a local unit. Let us begin with case I. Then $\epsilon = 0$ and we have

$$\begin{aligned}
d\phi\, d\psi\, d\rho &= u^{m_i+n_i+p_i} v^{m_j+n_j+p_j} \\
&\qquad \sum_{\star_{\{m,n\}}} \widehat{r} u^{\alpha-p_i} v^{\beta-p_j} w^{\gamma-1} (m_i n_j - m_j n_i)\, \gamma\, \frac{du\, dv\, dw}{uv};
\end{aligned}$$

by Fact 5.5.5 and the definitions of p_i and p_j we know that

$$\sum_{\star\{m,n\}} \widehat{r} u^{\alpha-p_i} v^{\beta-p_j} w^{\gamma-1}(m_i n_j - m_j n_i)\,\gamma$$

is a local unit, and thus that $(p_i, p_j, 1)$ is one of the triples appearing in the expression above. By the definition of ρ' and $\widehat{R}$ we now know that $\widehat{R} = \mu\, w$ for some local unit μ. Change coordinates by mapping $w \mapsto \mu^{-1} w$; this will not affect ϕ or ψ (as they do not involve w in this case), but ρ will become $u^{p_i} v^{p_j} w$ as desired.

On the other hand, in case II where $\epsilon = 1$ we have:

$$\begin{aligned} d\phi\, d\psi\, d\rho &= u^{m_i+n_i+p_i} v^{m_j+n_j+p_j} \sum_{\star\{m,n\}} \widehat{r}\Big(u^{\alpha-p_i} v^{\beta-p_j} w^{\gamma-1} \\ &\qquad \big((m_i n_j - m_j n_i)\,\gamma\, w - (m_i\beta - m_j\alpha)\, w\big)\Big)\ \frac{du\, dv\, dw}{uv} \\ &= u^{m_i+n_i+p_i} v^{m_j+n_j+p_j} \sum_{\star\{m,n\}} \widehat{r}\Big(u^{\alpha-p_i} v^{\beta-p_j} w^{\gamma} \\ &\qquad \big((m_i n_j - m_j n_i)\,\gamma - (m_i\beta - m_j\alpha)\big)\Big)\ \frac{du\, dv\, dw}{uv}; \end{aligned}$$

again by Fact 5.5.5 we know that

$$\sum_{\star\{m,n\}} \widehat{r}\Big(u^{\alpha-p_i} v^{\beta-p_j} w^{\gamma} \big((m_i n_j - m_j n_i)\,\gamma - (m_i\beta - m_j\alpha)\big)\Big)$$

must be a local unit. Thus $(\alpha, \beta, \gamma) = (p_i, p_j, 0)$ must be one of the triples in the expression above, so R is a local unit. Moreover, the subdeterminant $\Delta := (m_i p_j - m_j p_i)$ must be nonzero (since the coefficient of the monomial $(p_i, p_j, 0)$

cannot vanish, and $\gamma = 0$ for this monomial). Now change coordinates by

$$\begin{aligned} u &\mapsto u\,R^{m_j/\Delta} \\ v &\mapsto v\,R^{-m_i/\Delta} \\ w &\mapsto w\,R^{(m_i n_j - m_j n_i)/\Delta}; \end{aligned}$$

note that $m_i n_j - m_j n_i$ may or may not be zero here, but it will not matter. Under this change of coordinates we have

$$\begin{aligned} \phi = u^{m_i} v^{m_j} &\mapsto u^{m_i} R^{m_i m_j/\Delta} v^{m_j} R^{-m_j m_i/\Delta} = u^{m_i} v^{m_j}, \\ \psi' = u^{n_i} v^{n_j} w &\mapsto u^{n_i} R^{n_i m_j/\Delta} v^{n_j} R^{-n_j m_i/\Delta} w R^{(m_i n_j - m_j n_i)/\Delta} = u^{n_i} v^{n_j} w, \\ \rho' = u^{p_i} v^{p_j} R &\mapsto u^{p_i} R^{p_i m_j/\Delta} v^{p_j} R^{-p_j m_i/\Delta} R = u^{p_i} v^{p_j} R^{-1} R = u^{p_i} v^{p_j}. \end{aligned}$$

This completes the proof of part (ii) of (c), as well as the rest of the proof of part (d) of the proposition.

It now suffices to prove (the rest of) part (i) of (c), and part (f). We will do both cases at once, as in the triple point case above, by comparing $d\phi\, d\psi$ (which is minimal by the Nash-minimality of ϕ, ψ, and ρ) with $d\phi\, d\rho$. We need to show that first, for all triples (α, β, γ) satisfying $(\star_{\{m,n\}})$, we have $\alpha \geq n_i$ and $\beta \geq n_j$; and second, for all l with $\tau_l \neq 0$ in T, we have $\delta_l m_i + \tau_l n_i \geq n_i$ (and similarly for n_j). Since $\tau_l \neq 0$ for some triple (α, β, γ) means that (α, β, γ) is independent of $(m_i, m_j, 0)$, it thus suffices to prove that, for all triples (α, β, γ) that are independent of $(m_i, m_j, 0)$ (*i.e.* satisfy $(\star_m)$), we have $\alpha \geq n_i$ and $\beta \geq n_j$.

Let ϵ be as above (0 if we are in the first case, 1 in the second case), and let $\widehat{\epsilon}$ be its "opposite", *i.e.* let $\widehat{\epsilon} = 1$ in the first case, and $\widehat{\epsilon} = 0$ in the second case. Thus we have $\psi' = u^{n_i} v^{n_j} w^{\epsilon}$ and $\rho' = u^{p_i} v^{p_j} w^{\widehat{\epsilon}}$. In this notation we have:

$$\begin{aligned}
d\phi\, d\psi &= d\phi\, d\psi' \\
&= u^{m_i+n_i} v^{m_j+n_j} \\
&\quad \left(\begin{vmatrix} m_i & n_i w^{\epsilon} \\ m_j & n_j w^{\epsilon} \end{vmatrix} \frac{du\, dv}{uv} + \begin{vmatrix} m_j & n_j w^{\epsilon} \\ 0 & \epsilon w^{\epsilon} \end{vmatrix} \frac{dv\, dw}{v} + \begin{vmatrix} m_i & n_i w^{\epsilon} \\ 0 & \epsilon w^{\epsilon} \end{vmatrix} \frac{du\, dw}{u} \right) \\
&= u^{m_i+n_i} v^{m_j+n_j} w^{\epsilon} \left((m_i n_j - m_j n_i) \frac{du\, dv}{uv} + m_j \epsilon \frac{dv\, dw}{v} + m_i \epsilon \frac{du\, dw}{u} \right).
\end{aligned}$$

On the other hand, writing ρ as $\sum_{(\alpha,\beta,\gamma)} \widehat{r} u^{\alpha} v^{\beta} w^{\gamma}$, we have

$$\begin{aligned}
d\phi\, d\rho &= u^{m_i} v^{m_j} \left(\begin{vmatrix} m_i & u\rho_u \\ m_j & v\rho_v \end{vmatrix} \frac{du\, dv}{uv} \right. \\
&\qquad \left. + w^{-1} \begin{vmatrix} m_j & v\rho_v \\ 0 & w\rho_w \end{vmatrix} \frac{dvdv}{v} + w^{-1} \begin{vmatrix} m_i & u\rho_u \\ 0 & w\rho_w \end{vmatrix} \frac{du\, dw}{u} \right) \\
&= u^{m_i} v^{m_j} \sum_{(\alpha,\beta,\gamma)} \widehat{r} u^{\alpha} v^{\beta} w^{\gamma} \\
&\quad \left(\begin{vmatrix} m_i & \alpha \\ m_j & \beta \end{vmatrix} \frac{du\, dv}{uv} + w^{-1} \begin{vmatrix} m_j & \beta \\ 0 & \gamma \end{vmatrix} \frac{dvdv}{v} + w^{-1} \begin{vmatrix} m_i & \alpha \\ 0 & \gamma \end{vmatrix} \frac{du\, dw}{u} \right) \\
&= u^{m_i} v^{m_j} \sum_{\star_m} \widehat{r} u^{\alpha} v^{\beta} w^{\gamma} \\
&\quad \left(\begin{vmatrix} m_i & \alpha \\ m_j & \beta \end{vmatrix} \frac{du\, dv}{uv} + w^{-1} \begin{vmatrix} m_j & \beta \\ 0 & \gamma \end{vmatrix} \frac{dvdv}{v} + w^{-1} \begin{vmatrix} m_i & \alpha \\ 0 & \gamma \end{vmatrix} \frac{du\, dw}{u} \right),
\end{aligned}$$

where the last equality follows since triples (α, β, γ) that do not satisfy $(\star_m)$ cannot contribute nonzero terms in the sum above. Thus, since $d\phi\, d\psi$ is minimal, *i.e.* vanishes to least order along each component of the exceptional divisor $E = \{u = 0\} \cup \{v = 0\}$, we must have $m_i + n_i \leq m_i + \alpha$ and $m_j + n_j \leq m_i + \beta$, and thus $n_i \leq \alpha$ and $n_j \leq \beta$, for all (α, β, γ) satisfying $(\star_m)$; this completes the proof of part (i) of (c) as well as the proof of part (f).

Since parts (e) and (d) were shown at the end of the proof of part (ii) of (b), and part (f) is shown directly above, we are done. ■

5.6 Simple Point Case

About a simple point $e \in E_i$, with coordinates $\{u, v, w\}$ in an analytic neighborhood W of e with $E_i = \{u = 0\}$, Main Proposition 5.2.1 takes the form:

Main Proposition 5.6.1. *With notation as above, after possible change of coordinates, we can find "monomial" generators for the Nash sheaf in the analytic neighborhood W of the simple point $e \in E$, i.e. generators that can be locally written in the form $d\phi$, $d\psi$, $d\rho$ for some functions ϕ ψ, ρ that are of the form*

$$\begin{aligned} \phi &= u^{m_i} \\ \psi &= u^{n_i} v \\ \rho &= u^{p_i} w, \end{aligned}$$

where the exponents m_i, n_i, p_i satisfy the following ordering and linear independence conditions:

(a) $m_i \neq 0$*; and*

(b) $m_i \leq n_i \leq p_i$.

As in the double and triple point cases, the Main Proposition above is in fact a simple corollary to the existence of Hsiang-Pati coordinates. Thus we will prove the following proposition.

Proposition 5.6.2. *Given a complete resolution $\widetilde{U}$, double point e with analytic neighborhood W, and Nash-minimal ϕ, ψ, and ρ as described above, there exist coordinates $\{u, v, w\}$ on W with $E_i = \{u = 0\}$ and $E_j = \{v = 0\}$ so that there exist positive integers m_i, n_i, and p_i satisfying:*

- **(a)** $\phi = u^{m_i}$;
- **(b)** $\psi = S + \psi'$, *where*
  - **i.** $S = \sum s_l \phi^{\epsilon_l}$ *is a rational series with each* $\epsilon_l \geq 1$, *and*
  - **ii.** $\psi' = u^{n_i} v$;
- **(c)** $\rho = T + \rho'$, *where*
  - **i.** $T = \sum t_l \phi^{\delta_l} (\psi')^{\tau_l}$ *is a rational series with* $\delta_l \geq 1$ *when* $\tau_l = 0$, *and when* $\tau_l \neq 0$, $\delta_l m_i + \tau_l n_i \geq n_i$, *and*
  - **ii.** $\rho' = u^{p_i} w$;
- **(d)** $m_i \neq 0$;
- **(e)** $m_i \leq n_i$; *and*
- **(f)** $n_i \leq p_i$.

As in the previous cases, Main Proposition 5.6.1 is now clearly a simple corollary to Proposition 5.6.2:

Corollary 5.6.3. *We can rechoose ϕ, ψ, and ρ to be ϕ, ψ', and ρ', respectively, and these new choices will be a set of monomial generators for $\mathcal{N}_{\widetilde{U}}(W)$.*

Many parts of the proof of Proposition 5.6.2 will be similar to the proofs of Propositions 5.2.2 and 5.5.2; we will focus primarily on the points where this proof is significantly different. We will rely on the following two key facts (analogous to Facts 5.3.1 and 5.3.2).

Fact 5.6.4. *Let e be a simple point with analytic neighborhood W in $\widetilde{U}$. Given Nash-minimal ϕ, ψ, and ρ, we have (in W):*

$$d\phi \wedge d\psi = u^{g_i}\left(A\,\frac{du \wedge dv}{u} + B\,dv \wedge dw + C\,\frac{du \wedge dw}{u}\right),$$

where g_i is a positive integer, and at least one of A, B, and C is a local unit.

Fact 5.6.5. *Let e be a simple point with analytic neighborhood W in $\widetilde{U}$. Given Nash-minimal ϕ, ψ, and ρ, we have (in W):*

$$d\phi \wedge d\psi \wedge d\rho = u^{d_i}(\mu\,du \wedge dv \wedge dw),$$

where d_i is a positive integer and μ is a local unit.

The proofs of these facts are exactly as in the triple point case, with the exception that the basis for $\Omega^2(\log E)(W)$ about a simple point $e \in E_i = \{u = 0\}$ is $\{\frac{du\,dv}{u}, dv\,dw, \frac{du\,dw}{u}\}$.

We now prove Proposition 5.6.2.

Proof. **(a)** Once again all computations take place in the analytic neighborhood W of e in $\widetilde{U}$. Let $Z := \sum m_i E_i$ be the divisor on E corresponding to $\pi^{-1}\mathfrak{m}_v$. By part (i) of Nash-minimality we can write $\phi = \alpha u^{m_i}$ where α is a local unit, and absorb a root of this unit into the coordinate u so that $\phi = u^{m_i}$. Note that by definition m_i must be greater than zero, so we have also shown part (d) of the proposition.

(b) As in the double and triple point cases, split ψ as

$$\begin{aligned}
\psi &= \sum_{(a,b,c)} r_{a,b,c}\, u^a v^b w^c \\
&= \sum_{\substack{(a,b,c) \\ \text{dep. on} \\ (m_i,0,0)}} r_{a,b,c}\, u^a v^b w^c + \sum_{\substack{(a,b,c) \\ \text{ind. of} \\ (m_i,0,0)}} r_{a,b,c}\, u^a v^b w^c \\
&=: \sum_l s_l (u^{m_i})^{\epsilon_l} + \sum_{\star_m} r_{a,b,c}\, u^a v^b w^c \\
&=: S + \psi'
\end{aligned}$$

where $(\star_m)$ means that all the terms (a, b, c) in the sum are linearly dependent on $(m_i, 0, 0)$; in other words, one of b or c must be nonzero. Note that S is a rational series in ϕ, and each ϵ_l is ≥ 1 since ψ vanishes to at least order ϕ along E (thus we have shown part (i) of (b)).

We now calculate

$$\begin{aligned}
d\phi\, d\psi &= \left|\begin{smallmatrix} u\phi_u & u\psi_u \\ \phi_v & \psi_v \end{smallmatrix}\right| \frac{du\, dv}{u} + \left|\begin{smallmatrix} \phi_v & \psi_v \\ \phi_w & \psi_w \end{smallmatrix}\right| dv\, dw + \left|\begin{smallmatrix} u\phi_u & u\psi_u \\ \phi_w & \psi_w \end{smallmatrix}\right| \frac{du\, dw}{u} \\
&= u^{m_i} \left|\begin{smallmatrix} m_i & u\psi_u \\ 0 & \psi_v \end{smallmatrix}\right| \frac{du\, dv}{u} + \left|\begin{smallmatrix} 0 & \psi_v \\ 0 & \psi_w \end{smallmatrix}\right| dv\, dw + \left|\begin{smallmatrix} m_i & u\psi_u \\ 0 & \psi_w \end{smallmatrix}\right| \frac{du\, dw}{u} \\
&= u^{m_i} m_i \left(\psi_v \frac{du\, dv}{u} + \psi_w \frac{du\, dw}{u} \right);
\end{aligned}$$

since every term in ψ with both $b = 0$ and $c = 0$ will have zero v- and w-

derivatives, this is equivalent to:

$$d\phi\, d\psi \;=\; u^{m_i} m_i \left(\psi'_v \frac{du\, dv}{u} + \psi'_w \frac{du\, dw}{u} \right).$$

Define

$$n_i := \min_{\substack{(a,b,c)\\ \text{with } \star_m}} (a);$$

then we have $\psi' = u^{n_i}\, R$ where R is a holomorphic function that is not divisible by u. Moreover, R cannot be a unit, since this would contradict the fact that by definition ϕ' cannot have any pure u terms; thus R must vanish at the point e. Since $\psi'_v = u^{n_i}\, R_v$ and $\psi'_w = u^{n_i}\, R_w$ we now have:

$$d\phi\, d\psi \;=\; u^{m_i + n_i} m_i \left(R_v \frac{du\, dv}{u} + R_w \frac{du\, dw}{u} \right).$$

We wish to show that R is a coordinate on $\widetilde{U}$ that is independent of u (we will then change coordinates so that $R = v$). By Fact 5.6.4 and the calculation above, one of R_v and R_w must be a unit. Thus the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ R_u & R_v & R_w \\ 0 & 0 & 0 \end{pmatrix}$$

is rank 2 on all of $\widetilde{U}$. Since this matrix is the Jacobian matrix of the transformation $(u, v, w) \mapsto (u, R, 0)$ (note this transformation fixes e), the transformation $(u, R, 0)$ has a two-dimensional image in $\widetilde{U}$. Thus as desired R is a coordinate independent of u; rechoose the coordinates v and w so that $v = R$. This rechoosing of coordinates fixes ϕ (which only involves u), fixes S (which is a function of ϕ), and changes ϕ' to $u^{n_i}v$, finishing the proof of part (ii) of (b). Note that since ψ vanishes to at least order ϕ, we have $m_i \leq n_i$, and thus have also shown part (e) of the proposition.

(c) We first split ρ as:

$$
\begin{aligned}
\rho &= \sum_{(\alpha,\beta,\gamma)} \widehat{r}_{\alpha,\beta,\gamma}\, u^\alpha v^\beta w^\gamma \\
&= \sum_{\substack{(\alpha,\beta,\gamma)\\ \text{dep. on}\\ \{\vec{m},\vec{n}\}}} \widehat{r}_{\alpha,\beta,\gamma}\, u^\alpha v^\beta w^\gamma + \sum_{\substack{(\alpha,\beta,\gamma)\\ \text{ind. of}\\ \{\vec{m},\vec{n}\}}} \widehat{r}_{\alpha,\beta,\gamma}\, u^\alpha v^\beta w^\gamma \\
&=: \sum_l t_l (u^{m_i})^{\delta_l} (u^{n_i} v)^{\tau_l} + \sum_{\star\{m,n\}} \widehat{r}_{\alpha,\beta,\gamma}\ u^\alpha v^\beta w^\gamma \\
&=: T + \rho',
\end{aligned}
$$

where $\vec{m} = (m_i, 0, 0)$ and $\vec{n} = (n_i, 1, 0)$, and (α, β, γ) satisfies $(\star_{\{m,n\}})$ if it is independent of (the set consisting of) these two vectors. Clearly T is a rational series in ϕ and ψ'; since ρ must vanish to at least order ϕ, δ_l must be ≥ 1 whenever $\tau_l = 0$; this proves the first part of part of part (i) of (c).

We have

$$\begin{aligned}
d\phi\, d\psi\, d\rho &= \begin{vmatrix} u\phi_u & u\psi_u & u\rho_u \\ \phi_v & \psi_v & \rho_v \\ \phi_w & \psi_w & \rho_w \end{vmatrix} \frac{du\, dv\, dw}{u} \\
&= u^{m_i} \begin{vmatrix} m_i & u\psi_u & u\rho_u \\ 0 & \psi_v & \rho_v \\ 0 & \psi_w & \rho_w \end{vmatrix} \frac{du\, dv\, dw}{u} \\
&= u^{m_i} \begin{vmatrix} m_i & u\psi'_u & u\rho_u \\ 0 & \psi'_v & \rho_v \\ 0 & \psi'_w & \rho_w \end{vmatrix} \frac{du\, dv\, dw}{u} \\
&= u^{m_i+n_i} \begin{vmatrix} m_i & n_i v & u\rho_u \\ 0 & 1 & \rho_v \\ 0 & 0 & \rho_w \end{vmatrix} \frac{du\, dv\, dw}{u};
\end{aligned}$$

the penultimate step follows because pure u terms from ψ will have vanishing v- and w-derivatives (and thus the determinant for those terms will be zero). By multiplying various rows and columns by v, v^{-1}, or w we get

$$\begin{aligned}
d\phi\, d\psi\, d\rho &= u^{m_i+n_i} w^{-1} \begin{vmatrix} m_i & n_i & u\rho_u \\ 0 & 1 & v\rho_v \\ 0 & 0 & w\rho_w \end{vmatrix} \frac{du\, dv\, dw}{u} \\
&= u^{m_i+n_i} w^{-1} \sum_{(\alpha,\beta,\gamma)} \widehat{r} u^\alpha v^\beta w^\gamma \begin{vmatrix} m_i & n_i & \alpha \\ 0 & 1 & \beta \\ 0 & 0 & \gamma \end{vmatrix} \frac{du\, dv\, dw}{u} \\
&= u^{m_i+n_i} w^{-1} \sum_{\star_{\{m,n\}}} \widehat{r} u^\alpha v^\beta w^\gamma \begin{vmatrix} m_i & n_i & \alpha \\ 0 & 1 & \beta \\ 0 & 0 & \gamma \end{vmatrix} \frac{du\, dv\, dw}{u} \\
&= u^{m_i+n_i} w^{-1} \sum_{\star_{\{m,n\}}} \widehat{r} u^\alpha v^\beta w^\gamma (m_i \gamma) \frac{du\, dv\, dw}{u};
\end{aligned}$$

where the second to last step follows since the determinant above will be zero for any (α, β, γ) that do not satisfy $(\star_{\{m,n\}})$.

Define

$$p_i := \min_{\substack{(\alpha,\beta,\gamma)\\ \text{with } \star_{\{m,n\}}}} (\alpha);$$

now $\rho' = u^{p_i}\,\widehat{R}$, where $\widehat{R}$ is a holomorphic function that is not divisible by u. We wish to show that (after possibly changing coordinates) we can take $R = w$. Going back to the calculation above, we now have:

$$\begin{aligned} d\phi\, d\psi\, d\rho &= u^{m_i+n_i+p_i} w^{-1} m_i \sum_{\star_{\{m,n\}}} \widehat{r} u^{\alpha-p_i} v^{\beta} w^{\gamma} (\gamma)\, \frac{du\, dv\, dw}{u} \\ &= u^{m_i+n_i+p_i-1} \left(m_i \sum_{\star_{\{m,n\}}} \widehat{r} u^{\alpha-p_i} v^{\beta} w^{\gamma-1} (\gamma) \right) du\, dv\, dw. \end{aligned}$$

By Fact 5.6.5, $m_i+n_i+p_i-1 = d_i$ and $\sum_{\star_{\{m,n\}}} \widehat{r} u^{\alpha-p_i} v^{\beta} w^{\gamma-1}(\gamma)$ is a local unit; thus $(p_i, 0, 1)$ is a triple (α, β, γ) satisfying $(\star_{\{m,n\}})$, and moreover it is componentwise minimal to every such triple (α, β, γ). Hence

$$\widehat{R} = \sum_{\star_{\{m,n\}}} \widehat{r} u^{\alpha-p_i} v^{\beta} w^{\gamma} = w\, \check{R}$$

where $\check{R}$ is a local unit, and $\rho' = u^{p_i} w\, \check{R}$. Now rechoose coordinates by mapping $w \mapsto w\, R^{-1}$; this clearly fixes ϕ and ψ (as they do not involve w), and gives $\rho' = u^{p_i} w$, completing the proof of part (ii) of (c).

It still remains to show (the second part of) part (i) of (c), and part (f); these will be shown simultaneously as in double and triple point cases, using the minimality of $d\phi\, d\psi$. It suffices to prove that, for all triples (α, β, γ) that are independent of $(m_i, 0, 0)$ (note this includes all triples satisfying $(\star_{\{m,n\}})$), and the triples in T where $\tau_l \neq 0$), we have $\alpha \geq n_i$. We compute:

$$\begin{aligned} d\phi\, d\psi &= d\phi\, d\psi' \\ &= \left(m_i u^{m_i} \frac{du}{u} \right) \wedge \left(n_i u^{n_i} v \frac{du}{u} + u^{n_i}\, dv \right) \\ &= m_i u^{m_i+n_i} \frac{du\, dv}{u} \end{aligned}$$

and

$$\begin{aligned} d\phi\, d\rho &= \left(m_i u^{m_i} \frac{du}{u} \right) \wedge \left(u\rho_u \frac{du}{u} + \rho_v\, dv + \rho_w\, dw \right) \\ &= m_i u^{m_i} \left(\rho_v \frac{du\, dv}{u} + \rho_w \frac{du\, dw}{u} \right); \end{aligned}$$

any pure u term in ρ will not contribute to the quantity above (since ρ_v and ρ_w will be zero); thus we have:

$$d\phi\, d\rho = m_i u^{m_i} \sum_{\star_m} \widehat{r} u^\alpha v^\beta w^\gamma.$$

Since $d\phi\, d\psi$ vanishes to minimum order at e, we must have $m_i + n_i \leq m_i + \alpha$, and thus $n_i \leq \alpha$, for each α in a triple satisfying $(\star_m)$. This completes the proof of part (i) of (c) and the proof of part (f).

Since part (d) was shown in the proof of part (a), part (e) was shown after the proof of part (ii) of (b), and part (f) is shown directly above, the proof of the proposition is complete. ■

5.7 Minimality

In this section we show that the multiplicities defined in Propositions 5.2.2, 5.5.2, and 5.6.2 are minimal. Suppose j, k, and l are Nash-linear functions defining Hsiang-Pati coordinates ϕ, ψ', and ρ' on an analytic neighborhood $W \subset \widetilde{U}$ of a point $e \in E$, and let m_l, n_l, and p_l ($l = i, j, k$ or i, j or simply i depending on whether we are at a triple, double, or simple point, respectively) be the corresponding multiplicities. The integers m_l are minimal by definition (since they come from the divisor associated to $\pi^{-1}\mathfrak{m}_v$, and thus are the minimum order of vanishing on each component of E). We will first show that the n_i and p_i are minimal in the simple point case. As above, all computations take place in the analytic neighborhood W of e in $\widetilde{U}$.

5.7.1 Simple Point Case

Proposition 5.7.1. *If $\widehat{k}$ and $\widehat{l}$ are linear functions on $\mathbb{C}^n$ that satisfy $\widehat{\psi} := \widehat{k} \circ \pi = u^{\widehat{n}_i} v$ and $\widehat{\rho} := \widehat{l} \circ \pi = u^{\widehat{p}_i} w$ for some integers $\widehat{n}_i \leq \widehat{p}_i$, then $\widehat{n}_i \geq n_i$ and $\widehat{p}_i \geq p_i$.*

Proof. We first show that n_i is minimal; a simple calculation shows that:

$$d\phi \wedge d\widehat{\psi} = m_i \, u^{m_i + \widehat{n}_i} \, \frac{du \, dv}{uv}$$

and

$$d\phi \wedge d\psi' = m_i \, u^{m_i + n_i} \, \frac{du \, dv}{uv}.$$

Since $d\phi \wedge d\psi$ by assumption is the minimal generator of $\Lambda^2\mathcal{N}_{\widetilde{U}}(W)$, *i.e.* vanishes to the minimum order at the point e, we must have $n_i \leq \widehat{n}_i$.

To prove that p_i is minimal, we compute:

$$d\phi \wedge d\psi' \wedge d\widehat{\rho} = u^{m_i+n_i+\widehat{p}_i} \begin{vmatrix} m_i & n_i & \widehat{p}_i \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \frac{du\, dv\, dw}{u};$$

the result for $d\phi \wedge d\psi' \wedge d\rho'$ is the same, with the omission of the "hats". Since $d\phi \wedge d\psi' \wedge d\rho'$ generates $\Lambda^3\mathcal{N}_{\widetilde{U}}(W)$, it vanishes to minimum order, and thus we have $p_i \leq \widehat{p}_i$. ■

5.7.2 Double Point Case

The double point case is similar, with the addition of a linear independence condition.

Proposition 5.7.2. *Suppose $\widehat{k}$ and $\widehat{l}$ are linear functions on $\mathbb{C}^n$ that satisfy $\widehat{\psi} := \widehat{k} \circ \pi = u^{\widehat{n}_i} v^{\widehat{n}_j} w^{\epsilon}$ and $\widehat{\rho} := \widehat{l} \circ \pi = u^{\widehat{p}_i} v^{\widehat{p}_j} w^{\widehat{\epsilon}}$ for some integers $\widehat{n}_i$, $\widehat{n}_j$ and $\epsilon = 1$ or 0, $\widehat{\epsilon} = 0$ or 1, respectively, such that $\begin{vmatrix} m_i & \widehat{n}_i \\ m_j & \widehat{n}_j \end{vmatrix} \neq 0$, and some integers $\widehat{p}_i$, $\widehat{p}_j$ with $\widehat{p}_i \geq \widehat{n}_i$ and $\widehat{p}_j \geq \widehat{n}_j$. Then we have $\widehat{n}_i \geq n_i$, $\widehat{n}_j \geq n_j$ and $\widehat{p}_i \geq p_i$, $\widehat{p}_j \geq p_j$.*

Proof. As in the simple point case, we compute:

$$d\phi \wedge d\widehat{\psi} = u^{m_i+\widehat{n}_i} v^{m_j+\widehat{n}_j} w^{\epsilon} \left((m_i\widehat{n}_j - m_j\widehat{n}_i) \frac{du\, dv}{uv} + m_j\epsilon \frac{dv\, dw}{v} + m_i\epsilon \frac{du\, dw}{u} \right).$$

In either case ($\epsilon = 0$ or $\epsilon = 1$; see cases I and II, respectively, of Proposition 5.5.2), since $d\phi \wedge d\psi'$ is minimal (*i.e.* vanishes to the least order along every component of the exceptional divisor $E_i \cup E_j$), we must have $\widehat{n}_i \geq n_i$ and $\widehat{n}_j \geq n_j$.

To show that p_i and p_j are minimal, we look at:

$$d\phi \wedge d\psi' \wedge d\widehat{\rho} = u^{m_i+n_i+\widehat{p}_i} v^{m_j+n_j+\widehat{p}_j} \begin{vmatrix} m_i & n_i & \widehat{p}_i \\ m_j & n_j & \widehat{p}_j \\ 0 & \epsilon & \widehat{\epsilon} \end{vmatrix} \frac{du\, dv\, dw}{uv};$$

again, since $d\phi \wedge d\psi' \wedge d\rho'$ is minimal, we must have $\widehat{p}_i \geq p_i$ and $\widehat{p}_j \geq p_j$. ■

5.7.3 Triple Point Case

As one might expect, the triple point case requires yet another linear independence condition.

Proposition 5.7.3. *Suppose $\widehat{k}$ and $\widehat{l}$ are linear functions on $\mathbb{C}^n$ that satisfy $\widehat{\psi} := \widehat{k} \circ \pi = u^{\widehat{n}_i} v^{\widehat{n}_j} w^{\widehat{n}_k}$ and $\widehat{\rho} := \widehat{l} \circ \pi = u^{\widehat{p}_i} v^{\widehat{p}_j} w^{\widehat{p}_k}$ for some integers $\widehat{n}_i$, $\widehat{n}_j$, $\widehat{n}_k$ with $(\widehat{n}_i, \widehat{n}_j, \widehat{n}_k)$ linearly independent of (m_i, m_j, m_k), and some integers $\widehat{p}_i$, $\widehat{p}_j$, $\widehat{p}_k$ with $\widehat{p}_l \geq \widehat{n}_l$ for $l = i, j, k$ for which $(\widehat{p}_i, \widehat{p}_j, \widehat{p}_k)$ is linearly independent of the set $\{(m_i, m_j, m_k), (n_i, n_j, n_k)\}$. Then we have $\widehat{n}_i \geq n_i$, $\widehat{n}_j \geq n_j$, $\widehat{n}_k \geq n_k$ and $\widehat{p}_i \geq p_i$, $\widehat{p}_j \geq p_j$, $\widehat{p}_k \geq p_k$.*

Proof. Once again we calculate:

$$\begin{aligned} d\phi \wedge d\widehat{\psi} &= u^{m_i+\widehat{n}_i} v^{m_j+\widehat{n}_j} w^{m_k+\widehat{n}_k} \left(\begin{vmatrix} m_i & \widehat{n}_i \\ m_j & \widehat{n}_j \end{vmatrix} \frac{du\,dv}{uv} \right. \\ &\qquad \left. + \begin{vmatrix} m_j & \widehat{n}_j \\ m_k & \widehat{n}_k \end{vmatrix} \frac{dv\,dw}{vw} + \begin{vmatrix} m_i & \widehat{n}_i \\ m_k & \widehat{n}_k \end{vmatrix} \frac{du\,dw}{uw} \right); \end{aligned}$$

at least one of the coefficients is nonzero as long as $(\widehat{n}_i, \widehat{n}_j, \widehat{n}_k)$ is linearly independent of (m_i, m_j, m_k); thus by minimality of $d\phi \wedge d\psi'$ we have the desired $n_l \leq \widehat{n}_l$ for $l = i, j, k$.

Finally, to obtain minimality of the p_l for $l = i, j, k$ we compute:

$$d\phi \wedge d\psi' \wedge d\widehat{\rho} = u^{m_i+n_i+\widehat{p}_i} v^{m_j+n_j+\widehat{p}_j} w^{m_k+n_k+\widehat{p}_k} \begin{vmatrix} m_i & n_i & \widehat{p}_i \\ m_j & n_j & \widehat{p}_j \\ m_k & n_k & \widehat{p}_k \end{vmatrix} \frac{du\,dv\,dw}{uvw};$$

as above, by the linear independence condition on the p_l and the fact that $d\phi \wedge d\psi' \wedge d\rho'$ is minimal, we have $\widehat{p}_l \geq p_l$ for $l = i, j, k$. ■

Chapter 6

The Divisor Proposition

In what follows we take $\pi\colon (\widetilde{U}, E) \to (U, v)$ to be a complete resolution of U (see Section 3). In Sections 6.1 and 6.2 we will show that (in a sufficiently fine resolution) a generic hyperplane will intersect the exceptional divisor transversely at simple points. In these sections we will be working in the algebraic category. In Section 6.3 we will show that such a hyperplane will help us choose j or k (as defined in the Main Proposition; see Chapter 5), and will enable us to construct an exact sequence involving the (generalized) Nash sheaf; in that section we will be building on material from Chapters 4 and 5 and thus will work in the analytic category (although we will abuse notation).

6.1 Choosing a Generic Hyperplane

Given a linear function $h\colon \mathbb{C}^N \to \mathbb{C}$, let $H \subset \mathbb{C}^n$ be the hypersurface defined by h. Define $\widetilde{H}$ to be the proper transform of $H \cap U$ in the resolution $\widetilde{U}$; in other words, $\widetilde{H}$ is the closure of the inverse image under π of the subspace $H \cap (U - v)$ of U. In this section we will show that we can generically choose a "nice" hyperplane H (in the sense of the following definition); the conditions satisfied by such a hyperplane will allow us to apply a theorem of Hironaka in the next section.

Definition 6.1.1. *Let $\pi\colon (\widetilde{U}, E) \to (U, v)$ be a complete resolution as above. We will say that a hyperplane $H \subset \mathbb{C}^N$ is* nice *if it satisfies the following conditions:*

(i) $H \cap (U - v)$ *is smooth;*

(ii) $H \cap U$ *is reduced; and*

(iii) the total transform $\pi^{-1}(H \cap U)$ *of* $H \cap U$ *in* $\widetilde{U}$ *vanishes to minimum order along* E.

Claim 6.1.2. *A sufficiently generic hyperplane $H \subset \mathbb{C}^N$ is "nice".*

Note that, since $\widetilde{H} - E \approx H \cap (U - v)$, the proper transform $\widetilde{H}$ of a "nice" hyperplane H is both smooth away from E and reduced. We now prove Claim 6.1.2.

Proof. Parts (a) and (b) follow from Lemma 1.1 in Teissier's paper [Tei73], which states that (in a small enough neighborhood of v) there exists an open, Zariski dense set $\mathcal{G} \subset \mathrm{Gr}(N-1, N)$ of hyperplanes in $\mathbb{C}^N$ passing through v such that, for each $H \in \mathcal{G}$,

$$(H \cap U)_{\mathrm{sing}} = H \cap U_{\mathrm{sing}}$$

(and thus the singular set of $H \cap (U - v)$ is empty). In fact, the proof of Lemma 1.1 from [Tei73] shows that a generic H will meet $U - v$ transversely.

It now suffices to show (c), *i.e.* that $h \circ \pi$ vanishes to the minimum order along E. Let $Z := \sum m_i E_i$ be the divisor on E corresponding to the pullback $\pi^*(\mathfrak{m}_v)$ of the maximal ideal sheaf of the singularity v; this divisor represents the minimum possible order of vanishing on E (as in Section 5.4). Since $\widetilde{U}$ is a complete resolution it factors through the blowup of this maximal ideal, and thus $\pi^*(\mathfrak{m}_v)$ is a locally principal sheaf of ideals on $\widetilde{U}$; let ϕ be the (local) generator of this ideal. Suppose that $h \circ \pi$ vanishes to an order greater than Z; we must show that there exists a perturbation h' of h so that $h' \circ \pi$ vanishes to the minimum order on E. Since h vanishes to greater than the order of Z on E, we can write $h \circ \pi = \lambda \phi$ for some holomorphic function λ. Since ϕ is by definition an element of $\pi^* \mathfrak{m}_v$, there is an $f \in \mathfrak{m}_v$ with $\phi = \pi^* f = f \circ \pi$. Note that since f is an element of the maximal ideal for v, it defines a hyperplane passing through v. Now let

$h' := h + \epsilon f$; then

$$h' \circ f = (h + \epsilon f) \circ \pi = (h \circ \pi) + \epsilon(f \circ \pi) = \lambda\phi + \epsilon\phi = \phi(\lambda + \epsilon).$$

Since $\lambda + \epsilon$ is a local unit, h' vanishes to the minimum order (*i.e.* that of ϕ) along E, and we are done. ■

6.2 A Lemma from Hironaka's Theorem

Let $\pi \colon (\widetilde{U}, E) \to (U, v)$ be a complete resolution and choose a "nice" hyperplane $H \subset \mathbb{C}^N$ as in Definition 6.1.1. As above let $\widetilde{H}$ denote the proper transform of $H \cap U$ in $\widetilde{U}$. Given a further resolution $\bar{\pi} \colon (\bar{U}, \bar{E}) \to (\widetilde{U}, E)$, we will write $\bar{H}$ for the proper transform of $\widetilde{H}$ in $\bar{U}$. The following lemma will enable us to prove the "divisor proposition", *i.e.* Proposition 6.3.1. Note: we phrase this lemma in terms of the general n-dimensional case because it is no harder; to make it apply to the 3-dimensional case we have been discussing, simply let $n = 3$.

Lemma 6.2.1. *Let H be a "nice" hyperplane in $\mathbb{C}^N$ with notation as above. There exists a further resolution $\bar{U}$ of $\widetilde{U}$ in which $\bar{H}$ is reduced and meets $\bar{E}$ transversely at smooth points of $\bar{H}$.*

Proof. We will prove this lemma by putting our notation in the context of Hironaka's paper [Hir64a] and applying his Theorem $I_2^{N,n}$. This theorem involves "permissible resolutions" of "resolution datum with open restriction"; we present these concepts here only in the cases that we need. We start with the definition of a "resolution datum" (*i.e.* an object that we wish to resolve in some fashion) on $\widetilde{U}$ (following Definition 3(I) from [Hir64a]).

Definition 6.2.2. *A* resolution datum *on a dimension n space X is a triple*

$$\mathfrak{R}_I^{n,m} = (D;\, V;\, W)$$

where

- **a.** *D is reduced and codimension 1 in X with normal crossings;*
- **b.** *V is a subvariety of X with $V \supset W$; and*
- **c.** *W is a reduced subvariety of X of dimension m.*

We will also call a pair $\mathfrak{R}_I^{n,m}(D;\, W)$ a resolution datum *if it satisfies conditions (a) and (c) above.*

Clearly the pair $(E;\, \widetilde{H})$ is a resolution datum of type $\mathfrak{R}_I^{n,n-1}$ on $\widetilde{U}$ because E is reduced and codimension 1 in $\widetilde{U}$ with normal crossings, and $\widetilde{H}$ is reduced and dimension $n-1$. We will denote $\mathfrak{R}_I^{n,m}$ simply by $\mathfrak{R}$ when convenient.

We now state what it means for such a datum to be resolved at a point of W (see Definition 4(I) in [Hir64a]).

Definition 6.2.3. *The datum $\mathfrak{R} = (D;\, V;\, W)$ (and similarly, the datum $\mathfrak{R} = (D;\, W)$) is said to be* resolved *at $x \in W$ if:*

a. *x is a smooth point of W; and*

b. *D has only normal crossings with W at x.*

We define a datum with "open restriction" to be a resolution datum that is resolved on a dense open subset (see Definition 5(I.2) of [Hir64a]) as follows.

Definition 6.2.4. *Given a resolution datum $\mathfrak{R} = (D;\ V;\ W)$ (similarly, a datum $(D;\ W)$), a pair $(\mathfrak{R}, Y)$ is a* resolution datum with open restriction *on X if*

a. *Y is a dense open subset of W; and*

b. *$\mathfrak{R}$ is resolved at every point of Y.*

The pair $((E;\ \widetilde{H}),\ \widetilde{H} - E)$ is a resolution datum with open restriction: the subset $\widetilde{H} - E = \widetilde{H} - (\widetilde{H} \cap E)$ is open and dense in $\widetilde{H}$ since $\widetilde{H}$ is the Zariski closure of $\widetilde{H} - E$. The datum $(E;\ \widetilde{H})$ is resolved along all of $\widetilde{H} - E$ because $\widetilde{H}$ is smooth away from E (by our careful choice of H), and E vacuously has only normal crossings with $\widetilde{H}$ along $\widetilde{H} - E$ since $E \cap (\widetilde{H} - E) = \emptyset$.

Given a smooth, irreducible subset $B \subset X$, we say that a map $f\colon X' \to X$ is the *monoidal transformation with center B* if it is the blowup of X along the sheaf of ideals defining B (as in Section 2.1.3). We now define (as in Definition 6 of [Hir64a]) what is means for such a transformation to be "permissible" with respect to some resolution datum.

Definition 6.2.5. *A monoidal transformation* $f\colon X' \to X$ *with center* B *is* permissible *for the resolution datum* $\mathfrak{R} = (D;\, V;\, W)$ *(respectively* $(D;\, W)$*) if*

a. $(D;\, V \cap W;\, B)$ *(respectively* $(D;\, W;\, B)$*) is a resolution datum on* X*; and*

b. *the datum* $(D;\, V \cap W;\, B)$ *(respectively* $(D;\, W;\, B)$*) is resolved everywhere, i.e. on all of* B*.*

Such a monoidal transformation is permissible *for a resolution datum with open restriction* $(\mathfrak{R}, Y)$ *if it is permissible for* $\mathfrak{R}$ *as defined above, with* $B \subset Y$*.*

In our case where $\mathfrak{R} = (E;\, \widetilde{H})$, a monoidal transformation $f\colon \widetilde{U}' \to \widetilde{U}$ with center B is permissible if the triple $(E;\, \widetilde{H};\, B)$ is a resolution datum (and thus B is reduced and contained in $\widetilde{H}$) and E has only normal crossings with B. If f with center B is permissible for the datum with open restriction $((E;\, \widetilde{H});\, \widetilde{H} - E)$ then in addition we have $B \subset \widetilde{H} - (\widetilde{H} - E)$, *i.e.* $B \subset \widetilde{H} \cap E$.

We now define what it means to "pull back" a resolution datum by a permissible monoidal transformation f (as in Definition 7 of [Hir64a]). Given such an f, define

$$\begin{aligned} D' &= \mathrm{pt}_{X'}(D), \\ V' &= \mathrm{pt}_{X'}(V), \text{ and} \\ W' &= \mathrm{pt}_{X'}(W), \end{aligned}$$

where $\mathrm{pt}_{X'}(D)$ denotes the proper transform of D in X', *et cetera*, and

$$\begin{aligned} B' &= \mathrm{tt}_{X'}(B) \text{ and} \\ Y' &= \mathrm{tt}_{X'}(Y), \end{aligned}$$

where $\mathrm{tt}_{X'}(B)$ denotes the total transform (*i.e.* the inverse image $f^{-1}(B)$) of B in X'. We can now define the pullback of a resolution datum $\mathfrak{R}$ by f as follows.

Definition 6.2.6. *Given a resolution datum $\mathfrak{R}$ and a monoidal transformation f as above (permissible with respect to $\mathfrak{R}$), the* pullback *of $\mathfrak{R}$ by f is defined to be the triple*

$$f^*(\mathfrak{R}) := (D' \cup B';\, V';\, W')$$

(simply omit the V' if $\mathfrak{R}$ is a pair rather than a triple). The pullback of the resolution datum with open restriction $(\mathfrak{R},\, Y)$ by such an f is defined to be the pair

$$f^*(\mathfrak{R},\, Y) := (f^*(\mathfrak{R}),\, Y').$$

By the discussion following Definition 7 in [Hir64a], the pullback $f^*(\mathfrak{R})$ is itself a resolution datum (of the same type, *i.e.* the same dimensions) on X (as long as B does not contain any irreducible components of W; in that case the dimension m may be smaller). Let us investigate what this means in our case, where $(\mathfrak{R},\, Y) = ((E;\, \widetilde{H}),\, \widetilde{H} - E)$. In this case we have

$$f^*((E;\, \widetilde{H}),\, \widetilde{H} - E) = ((E' \cup B';\, \widetilde{H}'),\, \widetilde{H}' - (E' \cup B')),$$

since $Y' = (\widetilde{H} - E)' = f^{-1}(\widetilde{H} - E) = \widetilde{H}' - (E' \cup B')$. The fact that this is a resolution datum (with open restriction) means that $E' \cup B'$ is reduced, codimension 1 in $\widetilde{U}'$, and has normal crossings; and moreover, that $\widetilde{H}'$ is reduced and dimension $n - 1$ (note that B cannot contain any irreducible components of H because $B \subset \widetilde{H} \cap E$).

Our final definition describes what it means for a series of monoidal transformations to be "permissible" (following Definition 8 from [Hir64a]).

Definition 6.2.7. *Given a resolution datum $\mathfrak{R}$ on X, a series of monoidal transformations $f = \{f_i\colon X_{i+1} \to X_i\}_{0 \le i < s}$ with centers B_i on X_i (where $X_0 = X$) is* permissible *if there exists, for $0 \le i < s$, a resolution datum $\mathfrak{R}_i$ (with $\mathfrak{R}_0 = \mathfrak{R}$) for X_i such that:*

a. *f_i is permissible with respect to $\mathfrak{R}_i$; and*

b. $\mathfrak{R}_{i+1} = f_i^*(\mathfrak{R}_i)$.

Given such a permissible series $f\colon X' \to X$ of monoidal transformations (with $X' = X_s$), we will define the pullback $f^*(\mathfrak{R})$ *of $\mathcal{R}$ under f to be the final resolution datum $\mathfrak{R}_s$.*

We can now finally state the theorem of Hironaka that we wish to apply (Theorem $I_2^{N,n}$ in [Hir64a]).

Theorem 6.2.8. *There exists a finite succession of monoidal transformations $f\colon X' \to X$ which is permissible for the resolution datum with open restriction $(\mathfrak{R}, Y)$ such that the resolution datum $f^*(\mathcal{R})$ is resolved everywhere.*

Applying this theorem to the special case where $(\mathfrak{R}, Y) = ((E; \widetilde{H}), \widetilde{H} - E)$ completes the proof of the lemma. In other words, we can find a series of monoidal transformations $f\colon \bar{U} \to \widetilde{U}$ (here $\bar{U} = \widetilde{U}'$ from the above), with centers B_i contained in $E_i \cap \widetilde{H}_i$ at each level, so that in $\bar{U}$, $\bar{H} \cup \bar{E}$ is a divisor with normal crossings and $\bar{H}$ is smooth (where $\bar{H}$ is $\widetilde{H}' = \widetilde{H}_s$ in the notation above, and $\bar{E}$ is the union of (the proper transform of) E with (the total transforms of) the centers B_i). This completes the proof of Lemma 6.2.1. ■

6.3 The Divisor Proposition

From now on we assume that we have made a careful choice of linear function h (as in Section 6.2) and a complete resolution $(\widetilde{U}, E)$ that is sufficiently fine as to satisfy Lemma 6.2.1. In other words we are now simply taking $\widetilde{U}$ to be the resolution $\bar{U}$ obtained in Section 6.2. Likewise we now have $\widetilde{H}$ for $\bar{H}$. As in Chapters 4 and 5 we will now be working in the analytic category.

Proposition 6.3.1. *Given h, $(\widetilde{U}, E)$, and $\widetilde{H}$ satisfying Lemma 6.2.1 as described above, we have:*

(a) $\operatorname{div}(h \circ \pi) = Z + \widetilde{H}$*;*

(b) $\widetilde{H}$ *meets* E *only at double or simple points of* E*;*

(c) *near a point* $e \notin \widetilde{H}$ *we can take* j *to be* h*;*

(d) *near a double point* $e \in \widetilde{H} \cap E_i \cap E_j$ *we have* $m_i = n_i$ *and* $m_j = n_j$*, and we can take* k *to be* h*.*

(e) *near a simple point* $e \in \widetilde{H} \cap E_i$ *we have* $m_i = n_i = p_i$*, and we can take either* k *or* l *to be* h*.*

Proof. **(a)** This follows directly from Lemma 6.2.1, which ensures that $\widetilde{H} \cup E$ is a divisor with normal crossings in $\widetilde{U}$, and the fact that H is a "nice" hyperplane, and thus that $h \circ \pi$ vanishes to minimum order along E.

(b) Hironaka's Theorem (Theorem 6.2.8) used in the proof of Lemma 6.2.1 implies that $\widetilde{H} \cup E$ is a divisor with normal crossings; thus we can choose h so that $\widetilde{H}$ misses the triple points of E.

(c) Suppose e is a point that is not contained in $\widetilde{H}$, and let W be an analytic neighborhood of e in $\widetilde{U}$. By part (a) we have $h \circ \pi = u^{m_i} v^{m_j} w^{m_k}$ near e (at a triple point; at double or simple points simply set $m_j = 0$ or $m_j = m_k = 0$, respectively); clearly $h \circ \pi = \phi$ in W and we can take j to be h.

(d) Suppose $e \in \widetilde{H} \cap E_i \cap E_j$ is a double point contained in H. By Lemma 6.2.1 and part (a) we can choose coordinates $\{u, v, w\}$ on $\widetilde{U}$ so that $E_i = \{u = 0\}$,

$E_j = \{v = 0\}$, and $\widetilde{H} = \{w = 0\}$; then by the definition of m_i and m_j we have (after possibly rechoosing coordinates by multiplying w by a local unit) $h \circ \pi = u^{m_i}v^{m_j}w$. There exists a perturbation g of h so that $g \circ \pi = \delta u^{m_i}v^{m_j}$ near e, where δ is a local unit (this corresponds to a hyperplane $\widetilde{G} \subset \widetilde{U}$ that is shifted away from e, off of $\{w = 0\}$, but still transverse to E). Thus by the minimality of $\{n_i, n_j\}$ and $\{p_i, p_j\}$ as described in Proposition 5.7.2, this implies that we have either $m_i = p_i$ and $m_j = p_j$ ("case I" in Proposition 5.5.2, where $\epsilon = 0$) or $m_i = n_i$ and $m_j = n_j$ ("case II", where $\epsilon = 1$). Suppose first that we have $m_i = p_i$ and $m_j = p_j$. Then since $m_i \leq n_i \leq p_i$ and $m_j \leq n_j \leq p_j$ by 5.5.2(e) and 5.5.2(f) we must have $m_i = n_i$ and $m_j = n_j$. But by the corresponding part of 5.5.2(d) we must have $m_i n_j - m_j n_i \neq 0$, and thus we have a contradiction. Therefore we must have $m_i = n_i$ and $m_j = n_j$. We now have $h \circ \pi = u^{m_i}v^{m_j}w = u^{n_i}v^{n_j}w = \psi'$; in other words, we can choose k to be h in our choice of Nash-minimal linear functions.

(e) Given a simple point $e \in \widetilde{H} \cap E_i$, and an analytic neighborhood W of e in $\widetilde{U}$, we can choose coordinates $\{u, v, w\}$ on W so that $E_i = \{u = 0\}$ and $\widetilde{H} = \{v = 0\}$ (by Lemma 6.2.1); then by part (a) we have $h \circ \pi = u^{m_i}v$ near e. There exists a perturbation g of h so that $g \circ \pi = \delta u^{m_i}$ near e, where δ is a local unit (this corresponds to a hyperplane $\widetilde{G} \subset \widetilde{U}$ that is shifted away from e, off of $\{v = 0\}$, but still transverse to E). There also exists a perturbation f of

h so that $f \circ \pi = \tau u^{m_i}$ near e, where τ is a coordinate independent of u and v (this corresponds to a hyperplane $\widetilde{F} \subset \widetilde{U}$ that is rotated off of $\{v = 0\}$, but still transverse to E). Rechoose coordinates by

$$\begin{cases} u & \mapsto & u\,\delta^{-1/m_i}, \\ v & \mapsto & v\,\delta, \\ w & \mapsto & w; \end{cases}$$

with these coordinates we have $h \circ \pi = u^{m_i}v$, $g \circ \pi = u^{m_i}$, and $f \circ \pi = \tau' u^{m_i}$, where τ' is still a coordinate independent of u and v. Finally, redefine the last coordinate to be $w = \tau'$; then $f \circ \pi = u^{m_i}w$. By minimality from 5.6.2, we now have $m_i = n_i = p_i$ on this component E_i, and we can take $j := g$, $k := h$, and $l := f$. Thus we can indeed choose k to be h. We clearly could have also changed coordinates in order to choose l to be h. ■

Chapter 7

The Logarithmic Nash Frame

7.1 Notation

We first collect and extend our notation. Given a point $e \in E$, and a analytic neighborhood W of e in the (associated analytic space) $\widetilde{U}$, choose coordinates $\{u, v, w\}$ for $\widetilde{U}$ near e and Nash-minimal linear functions j, k, and l as in Chapter 5 so that $\phi = j \circ \pi$, $\psi = k \circ \pi = S + \psi'$, and $\rho = l \circ \pi = T + \rho'$. This gives us multiplicities m_l, n_l, and p_l for $l = i, j, k$ about e if e is a triple point of E (and for $l = i, j$ and $l = i$ if e is a double or simple point, respectively), and thus defines divisors $Z := \sum m_i E_i$, $N := \sum n_i E_i$, and $P := \sum p_i E_i$ on E. Using the results from Propositions 5.2.2, 5.5.2, and 5.6.2 we define ξ_1, ξ_2, and ξ_3 to be the distinguished monomials ϕ, ψ', and ρ' from the Hsiang-Pati coordinates ϕ, ψ and

ρ, respectively, on W. To summarize, at a triple point $e \in E_i \cap E_j \cap E_k$, we have

$$\begin{array}{rcccl} \xi_1 & := & \phi & = & u^{m_i}v^{m_j}w^{m_k}, \\ \xi_2 & := & \psi' & = & u^{n_i}v^{n_j}w^{n_k}, \\ \xi_3 & := & \rho' & = & u^{p_i}v^{p_j}w^{p_k}; \end{array}$$

at a double point $e \in E_i \cap E_j$, we have either (depending on whether we are in "case I" or "case II" as in the proof of Proposition 5.5.2):

$$\begin{array}{lcl} \xi_1 := \phi = u^{m_i}v^{m_j}, & & \xi_1 := \phi = u^{m_i}v^{m_j}, \\ \xi_2 := \psi' = u^{n_i}v^{n_j}, & \text{or} & \xi_2 := \psi' = u^{n_i}v^{n_j}w, \\ \xi_3 := \rho' = u^{p_i}v^{p_j}w & & \xi_3 := \rho' = u^{p_i}v^{p_j}; \end{array}$$

and at a simple point $e \in E_i$, we have

$$\begin{array}{rcccl} \xi_1 & := & \phi & = & u^{m_i}, \\ \xi_2 & := & \psi' & = & u^{n_i}v, \\ \xi_3 & := & \rho' & = & u^{p_i}w. \end{array}$$

By definition (and Main Propositions 5.2.1, 5.5.1, and 5.6.1), $\{d\xi_1, d\xi_2, d\xi_3\}$ is a set of monomial generators for the Nash sheaf $\mathcal{N}_{\widetilde{U}}(W)$ over W.

Now let h be a "nice" linear function chosen to satisfy Claim 6.1.2, and assume that $\widetilde{U}$ is a sufficiently fine resolution to satisfy Lemma 6.2.1 and Proposition 6.3.1 with respect to this h. We will now also denote the composition $h \circ \pi$ by h, and abuse notation by letting $H := \widetilde{H}$ denote the proper transform of $H \cap U$ in $\widetilde{U}$. By 6.3.1 we can choose j, k, and l above so that $h = \xi_1$ near any triple point e (since H cannot pass through such points), as well as near any simple or double point e that is not contained in H. Near a simple point $e \in E_i \cap H$ or a double point $e \in E_i \cap E_j \cap H$ we can choose j, k, and l so that $h = \xi_2$. Recall that we have $m_i = n_i$ and $m_j = n_j$ near a double point $e \in E_i \cap E_j \cap H$, and thus in an analytic neighborhood W of such an e we have $Z = N$ (we also know that we are at a "case II" double point; see the proof of Proposition 5.5.2). Similarly, near a simple point $e \in E_i \cap H$ we have $Z = N = P$. It will be useful to note here that, since $\mathrm{div}(h) = Z + H$ (in our new notation), multiplication by h gives us an isomorphism $\mathcal{O}(H) \approx \mathcal{O}(-Z)$.

7.2 The Logarithmic Nash Frame

By Corollary 5.2.3, $\{d\xi_1, d\xi_2, d\xi_3\}$ is a basis for the Nash sheaf $\mathcal{N}_{\widetilde{U}}(W)$ over the analytic neighborhood W of e. The sheaf $\Omega^1_W(\log E)$ has as its standard basis over W the logarithmic frame:

$$\left\{\frac{du}{u}, \frac{dv}{v}, \frac{dw}{w}\right\} \quad \text{if } e \text{ is a triple point;}$$
$$\left\{\frac{du}{u}, \frac{dv}{v}, dw\right\} \quad \text{if } e \text{ is a double point;}$$
$$\left\{\frac{du}{u}, dv, dw\right\} \quad \text{if } e \text{ is a simple point.}$$

To clarify the relationship between $\mathcal{N}_{\widetilde{U}}(W)$ and $\Omega^1_W(\log E)(W)$ we will define a *logarithmic Nash frame* for $\Omega^1_W(\log E)(W)$; to this end define:

$$\xi_2' \;:=\; \begin{cases} \xi_2, & \text{if } e \text{ is a triple point,} \\ \xi_2, & \text{if } e \text{ is a ``case I'' double point,} \\ \xi_2 w^{-1}, & \text{if } e \text{ is a ``case II'' double point,} \\ \xi_2 v^{-1}, & \text{if } e \text{ is a simple point;} \end{cases}$$

$$\xi_3' \;:=\; \begin{cases} \xi_3, & \text{if } e \text{ is a triple point,} \\ \xi_3 w^{-1}, & \text{if } e \text{ is a ``case I'' double point,} \\ \xi_3, & \text{if } e \text{ is a ``case II'' double point,} \\ \xi_3 w^{-1}, & \text{if } e \text{ is a simple point.} \end{cases} \tag{7.2.1}$$

Note that under these definitions, ξ_1, ξ_2', and ξ_3' are local defining functions for the divisors Z, N, and P, respectively, regardless of whether the chosen point $e \in E$ is a simple, double, or triple point. To help with notation we will also define $\xi_1' = \xi_1$. Now define the logarithmic Nash frame to be

$$\left\{ \frac{d\xi_1}{\xi_1}, \frac{d\xi_2}{\xi_2'}, \frac{d\xi_3}{\xi_3'} \right\}.$$

We wish to show that

Fact 7.2.1. *The logarithmic Nash frame is a (local) basis for* $\Omega^1_W(\log E)(W)$.

Proof. We must show that every element of $\Omega^1_{\widetilde{U}}(\log E)(W)$ (written in the standard logarithmic frame) can be written in the logarithmic Nash frame. In each case (triple point, double point, and simple point) we will do this by calculating the transformation from the logarithmic frame to the logarithmic Nash frame and then showing that this transformation has an inverse. As usual all computations here take place over the analytic neighborhood W of our chosen point e.

Near a triple point e, we have (looking at the table in Section 7.1 and the definitions in 7.2.1):

$$\frac{d\xi_1}{\xi_1} = \frac{d(u^{m_i} v^{m_j} w^{m_k})}{u^{m_i} v^{m_j} w^{m_k}} = m_i \frac{du}{u} + m_j \frac{dv}{v} + m_k \frac{dw}{w},$$

$$\frac{d\xi_2}{\xi_2'} = \frac{d(u^{n_i} v^{n_j} w^{n_k})}{u^{n_i} v^{n_j} w^{n_k}} = n_i \frac{du}{u} + n_j \frac{dv}{v} + n_k \frac{dw}{w},$$

$$\frac{d\xi_3}{\xi_3'} = \frac{d(u^{p_i} v^{p_j} w^{p_k})}{u^{p_i} v^{p_j} w^{p_k}} = p_i \frac{du}{u} + p_j \frac{dv}{v} + p_k \frac{dw}{w}.$$

In other words, the change of basis from the logarithmic to the logarithmic Nash frame of $\Omega^1_W(\log E)(W)$ is given by:

$$\begin{pmatrix} m_i & m_j & m_k \\ n_i & n_j & n_k \\ p_i & p_j & p_k \end{pmatrix} \begin{pmatrix} du/u \\ dv/v \\ dw/w \end{pmatrix} = \begin{pmatrix} d\xi_1/\xi_1 \\ d\xi_2/\xi_2' \\ d\xi_3/\xi_3' \end{pmatrix}.$$

Since by 5.2.2(d) we have

$$\begin{vmatrix} m_i & m_j & m_k \\ n_i & n_j & n_k \\ p_i & p_j & p_k \end{vmatrix} \neq 0,$$

the change of basis matrix is invertible; thus the logarithmic Nash frame is a local basis for $\Omega^1_W(\log E)(W)$.

The double point case is similar. The matrix for the change of basis from the logarithmic frame to the logarithmic frame is in this case either (in case I):

$$\begin{pmatrix} m_i & m_j & 0 \\ n_i & n_j & 0 \\ wp_i & wp_j & 1 \end{pmatrix}$$

or, in case II:

$$\begin{pmatrix} m_i & m_j & 0 \\ wn_i & wn_j & 1 \\ p_i & p_j & 0 \end{pmatrix};$$

in either case, by 5.5.2(d) the matrix has nonzero determinant (since we have either $\left|\begin{smallmatrix} m_i & m_j \\ n_i & n_j \end{smallmatrix}\right| \neq 0$ or $\left|\begin{smallmatrix} m_i & m_j \\ p_i & p_j \end{smallmatrix}\right| \neq 0$, respectively), and thus is invertible.

In the simple point case the change of basis matrix is:

$$\begin{pmatrix} m_i & 0 & 0 \\ vn_i & 1 & 0 \\ wp_i & 0 & 1 \end{pmatrix};$$

since by 5.6.2(d) we have $m_i \neq 0$, this matrix has nonzero determinant and is invertible. ■

Chapter 8

The Exact Sequence

In this chapter we will use without further mention the notation gathered in Chapter 7 above (in particular the definitions of ξ_i, ξ_i', and H). We will be working in the analytic category (and abusing notation) as in Chapters 4 and 5 and Section 6.3. Moreover, we will often abuse notation by writing simply $\mathcal{F}$ when we mean $\mathcal{F}(W)$ (the sheaf $\mathcal{F}$ applied to an analytic neighborhood W); we do this because we wish to avoid writing, for example, $\mathcal{N}(Z-E)(W)$. All computations involving coordinates $\{u, v, w\}$ and/or $\{\xi_1, \xi_2, \xi_3\}$ are clearly meant to be taking place over an analytic neighborhood $W \subset \widetilde{U}$ of a point $e \in E$.

8.1 Statement and Introduction

We now propose and prove an exact sequence generalizing the one given in Proposition 3.20 of [PS97] that relates the Nash sheaf $\mathcal{N}_{\widetilde{U}}$ to the resolution data (*i.e.* the Z, N, and P). Note that in the 2-dimensional case (as in [PS97]), the corresponding sequence was enough to *describe* the Nash sheaf in terms of the resolution data. In the 3-dimensional case the sequence only *relates* the Nash sheaf to the resolution data, since the second exterior power of the Nash sheaf is also involved; see immediately below.

Proposition 8.1.1. *There is an exact sequence of sheaves on $\widetilde{U}$:*

$$\begin{aligned} 0 \;\to\; \mathcal{N}(Z-E) \;&\overset{\alpha}{\hookrightarrow}\; \mathcal{I}_E\Omega^1(\log E) \\ &\overset{\beta}{\longrightarrow}\; \left(\Omega^2(\log E)\big/{\textstyle\bigwedge^2}\mathcal{N}(2Z)\right)\otimes\mathcal{O}(-Z-E) \\ &\overset{\gamma}{\twoheadrightarrow}\; \Omega^3\otimes\mathcal{O}_{P+N-2Z}(-2Z) \;\to\; 0. \end{aligned}$$

We will prove that the sequence above is exact by showing that it is exact in the analytic category, *i.e.* that the associated sequence of analytic sheaves (for which we shall use the same notation) over the associated analytic space $\widetilde{U}^h$ is exact. In Section 8.5 we will discuss how this implies that the above sequence is exact in the algebraic category.

We first show that the sequence in Proposition 8.1.1 is equivalent to an exact sequence that will enable us to use the generic hyperplane H discussed in Chapters

6 and 7. Since $\mathcal{O}(H) \approx \mathcal{O}(-Z)$ (by multiplication by h as in Chapter 7), we have:

$$\begin{aligned}
&\left(\Omega^2(\log E)\big/\textstyle\bigwedge^2 \mathcal{N}(2Z)\right) \otimes \mathcal{O}(-Z-E) \\
&\quad\approx \quad \Omega^2(\log E) \otimes \mathcal{O}(-Z-E)\big/\textstyle\bigwedge^2 \mathcal{N}(2Z) \otimes \mathcal{O}(-Z-E) \\
&\quad\approx \quad \mathcal{I}_E\Omega^2(\log E) \otimes \mathcal{O}(H)\big/\textstyle\bigwedge^2 \mathcal{N}(2Z-E) \otimes \mathcal{O}(H)\cdot
\end{aligned}$$

The last term in the sequence can be rewritten using the following simple claim.

Claim 8.1.2. *There is an isomorphism*

$$\Lambda^3\mathcal{N} \approx \Omega^3 \otimes \mathcal{O}(-Z-N-P+E).$$

Proof. Let $e \in E$ be a point with analytic neighborhood $W \subset \widetilde{U}$. By the definition of the ξ_i and Fact 5.3.2 (near triple points; similarly we can apply this argument to double or simple points using Facts 5.5.5 and 5.6.5, respectively), we can write the generator of $\Lambda^3\mathcal{N}(W)$ as

$$d\xi_1 \wedge d\xi_2 \wedge d\xi_3 = u^{d_i}v^{d_j}w^{d_k}(\mu\, du \wedge dv \wedge dw)$$

for some positive integers d_i, d_j, d_k and local unit μ. Clearly these d_l are equal to $m_l + n_l + p_l - 1$ for $l = i, j, k$ (in the triple point case; see the computation (5.4.3) in the proof of Main Proposition 5.2.2) ■

Using the claim above and the fact that $\Omega^3(\log E) \approx \Omega^3 \otimes \mathcal{O}(E)$ we have:

$$
\begin{aligned}
&\Omega^3 \otimes \mathcal{O}_{N+P-2Z}(-2Z) \\
&\quad\approx\ \Omega^3 \otimes \mathcal{O}(2H) \otimes {}^{\mathcal{O}}\!/_{\mathcal{O}(-N-P-2Z)} \\
&\quad\approx\ {}^{\Omega^3 \otimes \mathcal{O}(2H)}\!/_{\Omega^3 \otimes \mathcal{O}(2H-N-P+2Z)} \\
&\quad\approx\ {}^{\mathcal{I}_E\Omega^3(\log E) \otimes \mathcal{O}(2H)}\!/_{\Lambda^3\mathcal{N}(Z+N+P-E) \otimes \mathcal{O}(2H-N-P+2Z)} \\
&\quad\approx\ {}^{\mathcal{I}_E\Omega^3(\log E) \otimes \mathcal{O}(2H)}\!/_{\Lambda^3\mathcal{N}(3Z-E) \otimes \mathcal{O}(2H)}.
\end{aligned}
$$

Thus the sequence in Proposition 8.1.1 is equivalent to the (more complicated-looking but in fact easier to work with) sequence:

$$
\begin{aligned}
0 \ \rightarrow\ \mathcal{N}(Z-E) \ &\overset{\alpha}{\hookrightarrow}\ \mathcal{I}_E\Omega^1(\log E) \qquad (8.1.1)\\
&\xrightarrow{\beta}\ {}^{\mathcal{I}_E\Omega^2(\log E) \otimes \mathcal{O}(H)}\!/_{\wedge^2\mathcal{N}(2Z-E) \otimes \mathcal{O}(H)} \\
&\overset{\gamma}{\twoheadrightarrow}\ {}^{\mathcal{I}_E\Omega^3(\log E) \otimes \mathcal{O}(2H)}\!/_{\wedge^3\mathcal{N}(3Z-E) \otimes \mathcal{O}(2H)} \ \rightarrow\ 0.
\end{aligned}
$$

Thus to prove the proposition it suffices to prove that we have an exact sequence of the form above.

8.2 Proving Exactness

Proof. We will show that the sequence in (8.1.1) is exact; the first parts of the proof are similar to proof the 2-dimensional version (Proposition 3.20 in [PS97]). We first show that we have an injection

$$\alpha :\ \mathcal{N}(Z-E) \ \hookrightarrow\ \mathcal{I}_E\Omega^1(\log E).$$

Since the (dual) Nash sheaf $\mathcal{N}$ is generated by $\{d\xi_1, d\xi_2, d\xi_3\}$, we have (this computation assumes we are at a triple point e of E; for the double and simple point cases, simply replace uvw with uv or u, respectively):

$$\begin{aligned}
\mathcal{N}(Z-E) &= \Big\{(a\,d\xi_1 + b\,d\xi_2 + c\,d\xi_3)\cdot f \;\Big|\; a,b,c \in \mathcal{O},\ f \in \mathcal{O}(Z-E)\Big\} \\
&= \left\{af\xi_1\frac{d\xi_1}{\xi_1'} + bf\xi_2'\frac{d\xi_2}{\xi_2'} + cf\xi_3'\frac{d\xi_3}{\xi_3'} \;\middle|\; a,b,c, \frac{f}{uvw}\,\xi_1 \in \mathcal{O}\right\} \\
&= \left\{k_1\frac{d\xi_1}{\xi_1'} + k_2\frac{d\xi_2}{\xi_2'} + k_2\frac{d\xi_3}{\xi_3'} \;\middle|\; \frac{k_1}{uvw}, \frac{k_2}{uvw}\frac{\xi_1}{\xi_2'}, \frac{k_3}{uvw}\frac{\xi_1}{\xi_3'} \in \mathcal{O}\right\} \\
&= \left\{k_1\frac{d\xi_1}{\xi_1'} + k_2\frac{d\xi_2}{\xi_2'} + k_2\frac{d\xi_3}{\xi_3'} \;\middle|\; k_1 \in \mathcal{O}(-E),\right. \\
&\qquad \left. k_2 \in \mathcal{O}(Z-N-E),\ k_3 \in \mathcal{O}(Z-P-E)\right\}. \qquad (8.2.1)
\end{aligned}$$

Since $\mathcal{O}(Z-P-E) \subset \mathcal{O}(Z-N-E) \subset \mathcal{O}(-E) \approx \mathcal{I}_E$ (recall that $P > N$ since $p_i \geq n_i$ for all i; see Proposition 5.2.2), we have the desired injection α.

To define β, we first define the map

$$\begin{aligned}
\widetilde{\beta}:\ \mathcal{I}_E\Omega^1(\log E) &\longrightarrow \mathcal{I}_E\Omega^2(\log E)\otimes\mathcal{O}(H) \\
\omega &\longmapsto \omega \wedge \frac{dh}{h}.
\end{aligned}$$

Take $\omega \in \mathcal{I}_E\Omega^1(\log E)$. Then $\omega = \sum k_i \frac{d\xi_i}{\xi_i'}$, with $k_i \in \mathcal{O}(-E)$. We need to show that $\widetilde{\beta}(\omega)$ is actually in $\mathcal{I}_E\Omega^2(\log E)\otimes\mathcal{O}(H)$. We do this locally, examining the three possible cases: $e \in E$ away from H, $e \in E_i \cap H$ is a simple point of E on H, and $e \in E_i \cap E_j \cap H$ is a double point of E On H (and necessarily a "case II" double point; see Proposition 5.5.2). By Proposition 6.3.1 we have $\operatorname{div}(h\circ\pi) = Z + H$ (we will also write $h = h\circ\pi$). Therefore away from H, $h = \xi_1$; in this case we have:

$$\begin{aligned}\widetilde{\beta}(\omega) &= \omega \wedge \frac{dh}{h} \\ &= \left(\sum k_i \frac{d\xi_i}{\xi_i'}\right) \wedge \frac{d\xi_1}{\xi_1'} \\ &= -k_2 \frac{d\xi_1 d\xi_2}{\xi_1'\xi_2'} + k_3 \frac{d\xi_3 d\xi_1}{\xi_3'\xi_1'}; \qquad (8.2.2)\end{aligned}$$

this is clearly in $\mathcal{I}_E\Omega^2(\log E)$ since $\frac{d\xi_1 d\xi_2}{\xi_1'\xi_2'}$ and $\frac{d\xi_3 d\xi_1}{\xi_3'\xi_1'}$ are each nontrivial linear combinations of $\frac{dudv}{uv}$, $\frac{dvdw}{vw}$, $\frac{dwdu}{wu}$ (because the logarithmic Nash frame serves as a basis for $\Omega^1_{\widetilde{U}}(\log E)$; see Section 7.2).

At a simple point of E contained in H, say $e \in E_i \cap H$, we can choose coordinates $\{u, v, w\}$ so that $E_i = \{u = 0\}$ and $H = \{v = 0\}$. Since $m_i = n_i = p_i$ on components E_i that intersect H, up to unit we have

$$h = u^{m_i}v = u^{n_i}v = \xi_2'v = \xi_2.$$

In such a case we have:

$$\begin{aligned}\widetilde{\beta}(\omega) &= \omega \wedge \frac{dh}{h} \\ &= \left(\sum k_i \frac{d\xi_i}{\xi_i'}\right) \wedge \frac{d\xi_2}{\xi_2' v} \\ &= \frac{k_1}{v}\frac{d\xi_1 d\xi_2}{\xi_1'\xi_2'} - \frac{k_3}{v}\frac{d\xi_2 d\xi_3}{\xi_2'\xi_3'}, \qquad (8.2.3)\end{aligned}$$

which is clearly in $\mathcal{I}_E\Omega^2(\log E) \otimes \mathcal{O}(H)$.

Finally, at a double point of E contained in H, $e \in E_i \cap E_j \cap H$, we can choose coordinates $\{u, v, w\}$ centered at e so that $E_i = \{u = 0\}$, $E_j = \{v = 0\}$, and

$H = \{w = 0\}$. Since $m_i = n_i$ and $m_j = n_j$ in such a case (see Proposition 6.3.1), and $div(h) = Z + H$, we have (up to unit)

$$h = u^{m_i} v^{m_j} w = u^{n_i} v^{n_j} w = \xi_2 = \xi_2' w.$$

Thus $\widetilde{\beta}(\omega)$ is given in this case by:

$$\begin{aligned} \widetilde{\beta}(\omega) &= \omega \wedge \frac{dh}{h} \\ &= \left(\sum k_i \frac{d\xi_i}{\xi_i'} \right) \wedge \frac{d\xi_2}{\xi_2' w} \\ &= \frac{k_1}{w} \frac{d\xi_1 d\xi_2}{\xi_1' \xi_2'} - \frac{k_3}{w} \frac{d\xi_2 d\xi_3}{\xi_2' \xi_3'}, \end{aligned} \tag{8.2.4}$$

which as above is clearly an element of $\mathcal{I}_E \Omega^2(\log E) \otimes \mathcal{O}(H)$.

Since $\mathcal{O}(H) \approx \mathcal{O}$ away from H, and $\mathcal{O}(H)$ is generated by v^{-1} (respectively w^{-1}) near a point e in the simple (respectively double) point case near H, computations (8.2.2), (8.2.3) and (8.2.4) show that $\omega \wedge \frac{dh}{h}$ is always in $\mathcal{I}_E \Omega^2(\log E) \otimes \mathcal{O}(H)$.

We will define β to the the composition of the map $\widetilde{\beta}$ with the projection

$$\mathcal{I}_E \Omega^2(\log E) \otimes \mathcal{O}(H) \xrightarrow{p} \mathcal{I}_E \Omega^2(\log E) \otimes \mathcal{O}(H) \Big/ \bigwedge^2 \mathcal{N}(2Z - E) \otimes \mathcal{O}(H);$$

however, first we must show that this projection is well-defined; *i.e.* we must show that $\bigwedge^2 \mathcal{N}(2Z - E) \otimes \mathcal{O}(H)$ is a subset of $\mathcal{I}_E \Omega^2(\log E) \otimes \mathcal{O}(H)$. (The following computation assumes we are at a triple point e of E; for the double and simple point cases, simply replace uvw with uv or u, respectively.)

$$
\begin{aligned}
&\textstyle\bigwedge^2 \mathcal{N}(2Z-E)\otimes\mathcal{O}(H) \\
&\quad= \Big\{(a\,d\xi_1 d\xi_2 + b\,d\xi_2 d\xi_3 + c\,d\xi_3 d\xi_1)\cdot g\cdot r \;\Big|\; a,b,c\in\mathcal{O}, \\
&\qquad\qquad g\in\mathcal{O}(2Z-E),\ r\in\mathcal{O}(H)\Big\} \\
&\quad= \Big\{agr\xi_1\xi_2'\frac{d\xi_1 d\xi_2}{\xi_1'\xi_2'} + bgr\xi_2'\xi_3'\frac{d\xi_2 d\xi_3}{\xi_2'\xi_3'} + cgr\xi_3'\xi_1\frac{d\xi_3 d\xi_1}{\xi_3'\xi_1'} \;\Big| \\
&\qquad\qquad a,b,c,\tfrac{g}{uvw}\,\xi_1^2\in\mathcal{O},\ r\in\mathcal{O}(H)\Big\} \\
&\quad= \Big\{Ar\frac{d\xi_1 d\xi_2}{\xi_1'\xi_2'} + Br\frac{d\xi_2 d\xi_3}{\xi_2'\xi_3'} + Cr\frac{d\xi_3 d\xi_1}{\xi_3'\xi_1'} \;\Big|\; r\in\mathcal{O}(H) \\
&\qquad\qquad \tfrac{A}{uvw}\tfrac{\xi_1}{\xi_2'},\ \tfrac{B}{uvw}\tfrac{\xi_1}{\xi_2'}\tfrac{\xi_1}{\xi_3'},\ \tfrac{C}{uvw}\tfrac{\xi_1}{\xi_3'}\in\mathcal{O}\Big\} \\
&\quad= \Big\{Ar\frac{d\xi_1 d\xi_2}{\xi_1'\xi_2'} + Br\frac{d\xi_2 d\xi_3}{\xi_2'\xi_3'} + Cr\frac{d\xi_3 d\xi_1}{\xi_3'\xi_1'} \;\Big| \\
&\qquad\qquad A\in\mathcal{O}(Z-N-E),\ B\in\mathcal{O}(2Z-N-P-E), \\
&\qquad\qquad C\in\mathcal{O}(Z-P-E),\ r\in\mathcal{O}(H)\Big\}
\end{aligned}
\tag{8.2.5}
$$

Since $\mathcal{O}(2Z-N-P-E)\subset\mathcal{O}(Z-P-E)\subset\mathcal{O}(Z-N-E)\approx\mathcal{I}_E\mathcal{O}(Z-N)$, we see from the above computation that

$$\textstyle\bigwedge^2 \mathcal{N}(2Z-E)\otimes\mathcal{O}(H)\subset\mathcal{I}_E\Omega^2(\log E)\otimes\mathcal{O}(Z-N+H).$$

Moreover, since $\mathcal{O}(Z-N)\subset\mathcal{O}$, we have shown that $\bigwedge^2\mathcal{N}(2Z-E)\otimes\mathcal{O}(H)$ is contained in $\mathcal{I}_E\Omega^2(\log E)\otimes\mathcal{O}(H)$. Thus the projection p is well-defined, and we can define $\beta := p\circ\widetilde{\beta}$.

Now we show that the sequence is exact at $\mathcal{I}_E\Omega^1(\log E)$, in other words, that

$\ker(\beta) = \mathrm{im}(\alpha)$. Let $\omega = \sum k_i \frac{d\xi_i}{\xi_i'}$ be any element of $\mathcal{I}_E\Omega^1(\log E)$. We have

$$\omega \in \ker(\beta) \iff \widetilde{\beta}(\omega) \in \mathcal{I}_E\Omega^2(\log E) \otimes \mathcal{O}(H),$$

$$\omega \in \mathrm{im}(\alpha) \iff \omega \in \mathcal{N}(Z - E).$$

Let us first handle the case where we are away from H. Looking back on our computation of $\bigwedge^2 \mathcal{N}(2Z - E) \otimes \mathcal{O}(H)$ in (8.2.5), where r is now equal to 1, and using (8.2.2) and the above, we see that $\omega \in \ker(\beta)$ if and only if $-k_2 = A \in \mathcal{O}(Z-N-E)$ and $k_3 = C \in \mathcal{O}(Z-P-E)$. Comparing this with our computation of $\mathcal{N}(Z - E)$ in (8.2.1), it is clear that this is precisely the condition we need in order to have $\omega \in \mathcal{N}(Z - E)$, *i.e.* $\omega \in \mathrm{im}(\alpha)$.

Near H, say at a simple point $e \in E_i \cap H$, we have $Z = N = P$ (see Section 7.1). From (8.2.3) we see that $\omega \in \ker(\beta)$ if and only if $k_1 = A \in \mathcal{O}(Z - N - E) \approx \mathcal{O}(-E)$ and $-k_3 = B \in \mathcal{O}(2Z - N - P - E) \approx \mathcal{O}(-E)$. Note that since $Z = N = P$, k_2 and k_3 are *a priori* in $\mathcal{O}(Z - N - E) \approx \mathcal{O}(-E)$; thus we have exactly the conditions we need in order to have $\omega \in \mathrm{im}(\alpha)$.

Likewise, at a double point of E contained in H, we have $Z = N$. Computation (8.2.4) shows that $\omega \in \ker(\beta)$ if and only if $k_1 = A \in \mathcal{O}(Z - N - E) \approx \mathcal{O}(-E)$ and $-k_3 = B \in \mathcal{O}(2Z - N - P - E) \approx \mathcal{O}(Z - P - E)$. Looking back at (8.2.1) we wee that these conditions imply that $\omega \in \mathcal{N}(Z - E)$ We have now shown that, in all cases, the sequence is exact at $\mathcal{I}_E\Omega^1(\log E)$.

As a first step towards defining γ, we define the map

$$\begin{aligned} \widetilde{\gamma}:\ \mathcal{I}_E\Omega^2(\log E)\otimes\mathcal{O}(H) &\longrightarrow \mathcal{I}_E\Omega^3(\log E)\otimes\mathcal{O}(2H) \\ \tau &\longmapsto \tau\wedge\frac{dh}{h}. \end{aligned}$$

Take $\tau\in\mathcal{I}_E\Omega^2(\log E)\otimes\mathcal{O}(H)$. Then $\tau = Ar\frac{d\xi_1 d\xi_2}{\xi_1'\xi_2'} + Br\frac{d\xi_2 d\xi_3}{\xi_2'\xi_3'} + Cr\frac{d\xi_3 d\xi_1}{\xi_3'\xi_1'}$, with $A,B,C\in\mathcal{O}(-E)$ and $r\in\mathcal{O}(H)$. We will first show that the map $\widetilde{\gamma}$ is well-defined, *i.e.* that $\widetilde{\gamma}(\tau)$ is in fact an element of $\mathcal{I}_E\Omega^3(\log E)\otimes\mathcal{O}(2H)$. Away from H (so $r=1$), we have $h=\xi_1$, and thus:

$$\begin{aligned} \widetilde{\gamma}(\tau) &= \tau\wedge\frac{dh}{h} \\ &= \left(Ar\frac{d\xi_1 d\xi_2}{\xi_1'\xi_2'} + Br\frac{d\xi_2 d\xi_3}{\xi_2'\xi_3'} + Cr\frac{d\xi_3 d\xi_1}{\xi_3'\xi_1'}\right)\wedge\frac{d\xi_1}{\xi_1'} \\ &= B\,\frac{d\xi_1 d\xi_2 d\xi_3}{\xi_1\xi_2'\xi_3'}; \end{aligned} \tag{8.2.6}$$

which is in $\mathcal{I}_E\Omega^3(\log E)\otimes\mathcal{O}(2H)$ since $\frac{d\xi_1 d\xi_2 d\xi_3}{\xi_1\xi_2'\xi_3'}$ is a nowhere-vanishing multiple of $\frac{du dv dw}{uvw}$ and $\mathcal{O}(H)\approx\mathcal{O}$ away from H.

Near a simple point $e\in E_i$ contained in H, we have (up to unit) $h=\xi_2=\xi_2'v$ (where as above we have chosen coordinates $\{u,v,w\}$ for $\widetilde{U}$ so that $E_i=\{u=0\}$ and $H=\{v=0\}$), and thus:

$$\begin{aligned} \widetilde{\gamma}(\tau) &= \tau\wedge\frac{dh}{h} \\ &= \left(Ar\frac{d\xi_1 d\xi_2}{\xi_1'\xi_2'} + Br\frac{d\xi_2 d\xi_3}{\xi_2'\xi_3'} + Cr\frac{d\xi_3 d\xi_1}{\xi_3'\xi_1'}\right)\wedge\frac{d\xi_2}{\xi_2'v} \\ &= \frac{Cr}{v}\,\frac{d\xi_1 d\xi_2 d\xi_3}{\xi_1\xi_2'\xi_3'}, \end{aligned} \tag{8.2.7}$$

which is clearly in $\mathcal{I}_E\Omega^3(\log E)\otimes\mathcal{O}(2H)$ since r and $\frac{1}{v}$ are in $\mathcal{O}(H)$.

Near a double point $e \in E_i \cap E_j \cap H$, up to unit we have $h = \xi_2 = \xi_2' w$ (in appropriate coordinates). In this case we have:

$$\begin{aligned}
\widetilde{\gamma}(\tau) &= \tau \wedge \frac{dh}{h} \\
&= \left(Ar\frac{d\xi_1 d\xi_2}{\xi_1'\xi_2'} + Br\frac{d\xi_2 d\xi_3}{\xi_2'\xi_3'} + Cr\frac{d\xi_3 d\xi_1}{\xi_3'\xi_1'} \right) \wedge \frac{d\xi_2}{\xi_2' w} \\
&= \frac{Cr}{w}\,\frac{d\xi_1 d\xi_2 d\xi_3}{\xi_1\xi_2'\xi_3'}, \qquad (8.2.8)
\end{aligned}$$

which is an element of $\mathcal{I}_E\Omega^3(\log E) \otimes \mathcal{O}(2H)$. Thus in all cases we have shown that $\widetilde{\gamma}$ is well-defined.

As a further step towards defining γ, we will show that we have a well-defined projection

$$\mathcal{I}_E\Omega^3(\log E) \otimes \mathcal{O}(2H) \xrightarrow{\widetilde{p}} \mathcal{I}_E\Omega^3(\log E) \otimes \mathcal{O}(2H)\Big/ \textstyle\bigwedge^3 \mathcal{N}(3Z - E) \otimes \mathcal{O}(2H).$$

It suffices to prove that we have an injection of $\bigwedge^3 \mathcal{N}(3Z - E) \otimes \mathcal{O}(2H)$ into $\mathcal{I}_E\Omega^3(\log E) \otimes \mathcal{O}(2H)$. (Once again, we assume we are at a triple point e of E; for the double and simple point cases, simply replace uvw with uv or u, respectively.)

$$\begin{aligned}
&\textstyle\bigwedge^3 \mathcal{N}(3Z - E) \otimes \mathcal{O}(2H) \\
&\quad= \left\{ (a\, d\xi_1 d\xi_2 d\xi_3) \cdot f \cdot r^2 \;\middle|\; a \in \mathcal{O}(3Z - E),\ r \in \mathcal{O}(H) \right\} \\
&\quad= \left\{ afr^2 \xi_1\xi_2'\xi_3' \frac{d\xi_1 d\xi_2 d\xi_3}{\xi_1\xi_2'\xi_3'} \;\middle|\; a, \frac{f}{uvw}\,\xi_1^3 \in \mathcal{O},\ \ r \in \mathcal{O}(H) \right\} \\
&\quad= \left\{ Kr^2 \frac{d\xi_1 d\xi_2 d\xi_3}{\xi_1\xi_2'\xi_3'} \;\middle|\; \frac{K}{uvw}\,\frac{\xi_1^2}{\xi_1\xi_2} \in \mathcal{O},\ r \in \mathcal{O}(H) \right\} \\
&\quad= \left\{ Kr^2 \frac{d\xi_1 d\xi_2 d\xi_3}{\xi_1\xi_2'\xi_3'} \;\middle|\; K \in \mathcal{O}(2Z - P - N - E),\ r \in \mathcal{O}(H) \right\} \\
&\quad= \mathcal{I}_E\Omega^3(\log E) \otimes \mathcal{O}(2Z - P - N + 2H). \qquad (8.2.9)
\end{aligned}$$

Since $\mathcal{O}(2Z-N-P) \approx \mathcal{O}(Z-N)\otimes\mathcal{O}(Z-P) \subset \mathcal{O}$, we have $\bigwedge^3 \mathcal{N}(3Z-E)\otimes\mathcal{O}(2H)$ as a subsheaf of $\mathcal{I}_E\Omega^3(\log E) \otimes \mathcal{O}(2H)$, and the projection $\widetilde{p}$ is well-defined.

We will define the map γ using the maps $\widetilde{\gamma}$, p, and $\widetilde{p}$ via the diagram:

$$\begin{array}{ccc} \mathcal{I}_E\Omega^2(\log E) \otimes \mathcal{O}(H) & \xrightarrow{p} & \mathcal{I}_E\Omega^2(\log E) \otimes \mathcal{O}(H)\big/ \bigwedge^2 \mathcal{N}(2Z-E) \otimes \mathcal{O}(H) \\ \widetilde{\gamma}\big\downarrow & & \gamma\big\downarrow \\ \mathcal{I}_E\Omega^3(\log E) \otimes \mathcal{O}(2H) & \xrightarrow{\widetilde{p}} & \mathcal{I}_E\Omega^3(\log E) \otimes \mathcal{O}(2H)\big/ \bigwedge^3 \mathcal{N}(3Z-E) \otimes \mathcal{O}(2H) \end{array}$$

In other words, given

$$\bar{\tau} \in \mathcal{I}_E\Omega^2(\log E) \otimes \mathcal{O}(H)\big/ \bigwedge^2 \mathcal{N}(2Z-E) \otimes \mathcal{O}(H),$$

with representative $\tau \in \mathcal{I}_E\Omega^2(\log E) \otimes \mathcal{O}(H)$, $p(\tau) = \bar{\tau}$, we define $\gamma(\bar{\tau}) = \widetilde{p}(\widetilde{\gamma}(\tau))$. This is well-defined because the restriction of $\widetilde{\gamma}$ to $\bigwedge^2 \mathcal{N}(2Z-E) \otimes \mathcal{O}(H)$ maps into $\bigwedge^3 \mathcal{N}(3Z-E) \otimes \mathcal{O}(2H)$; if

$$\tau = Ar\frac{d\xi_1 d\xi_2}{\xi_1'\xi_2'} + Br\frac{d\xi_2 d\xi_3}{\xi_2'\xi_3'} + Cr\frac{d\xi_3 d\xi_1}{\xi_3'\xi_1'}$$

is an element of $\bigwedge^2 \mathcal{N}(2Z-E) \otimes \mathcal{O}(H)$, then we have $A \in \mathcal{O}(Z-N-E)$, $B \in \mathcal{O}(2Z-N-P-E)$, $C \in \mathcal{O}(Z-P-E)$, and $r \in \mathcal{O}(H)$. Looking at computations (8.2.6), (8.2.7), and (8.2.9) it is clear that in this case we have $\widetilde{\gamma}(\tau) \in \bigwedge^3 \mathcal{N}(3Z-E) \otimes \mathcal{O}(2H)$.

The map γ is surjective because the map $\widetilde{\gamma}$ is: given $\tau \in \mathcal{I}_E\Omega^2(\log E) \otimes \mathcal{O}(H)$ as above, examine (8.2.6) and (8.2.7); clearly if we can choose B (if away from H) or C (if near H) in the coefficients of τ so that $\widetilde{\gamma}(\tau)$ hits any specified element of $\mathcal{I}_E\Omega^3(\log E) \otimes \mathcal{O}(2H)$.

It now remains only to prove that $\ker(\gamma) = \operatorname{im}(\beta)$. It is easy to show that $\operatorname{im}(\beta) \subseteq \ker(\gamma)$; given ω in $\mathcal{I}_E\Omega^1(\log E)$ we must show that $\gamma(\beta(\omega)) = 0$, *i.e.* that $\widetilde{p}(\widetilde{\gamma}(\widetilde{\beta}(\omega))) = 0$:

$$\widetilde{p}(\widetilde{\gamma}(\widetilde{\beta}(\omega))) \;=\; \widetilde{p}\left(\omega \wedge \frac{dh}{h} \wedge \frac{dh}{h}\right) \;=\; \widetilde{p}(0) \;=\; 0.$$

To show that $\ker(\gamma) \subseteq \operatorname{im}(\beta)$, take $\bar{\tau} = [\tau]$ with τ in $\mathcal{I}_E\Omega^2(\log E) \otimes \mathcal{O}(H)$. If $\bar{\tau} \in \ker(\gamma)$, then τ must be in $\ker(\widetilde{p} \circ \widetilde{\gamma})$; that is to say, $\widetilde{\gamma}(\tau)$ is contained in $\bigwedge^3 \mathcal{N}(3Z - E) \otimes \mathcal{O}(2H)$. Suppose

$$\tau = Ar\frac{d\xi_1 d\xi_2}{\xi_1'\xi_2'} + Br\frac{d\xi_2 d\xi_3}{\xi_2'\xi_3'} + Cr\frac{d\xi_3 d\xi_1}{\xi_3'\xi_1'}$$

(*a priori* A, B, and C are in $\mathcal{O}(-E)$, and $r \in \mathcal{O}(H)$). Away from H we have $h = \xi_1$ (and $r = 1$ in τ), and looking at (8.2.6) and (8.2.9) we see that if $\tau \in \ker(\widetilde{p} \circ \widetilde{\gamma})$, then $B \in \mathcal{O}(2Z - N - P - E)$. To show that $\bar{\tau} \in \operatorname{im}(\beta)$, we must show that there exists an $\omega \in \mathcal{I}_E\Omega^1(\log E)$ so that $\bar{\tau} = \beta(\omega) = p(\widetilde{\beta}(\omega))$, *i.e.* $p(\tau) = p(\widetilde{\beta}(\omega))$. Choose $\omega = \sum k_i \frac{d\xi_i}{\xi_i'}$ with $k_2 = -A$ and $k_3 = C$; then by (8.2.2),

$$p(\widetilde{\beta}(\omega)) = p\left(A\,\frac{d\xi_1 d\xi_2}{\xi_1'\xi_2'} + C\,\frac{d\xi_3 d\xi_1}{\xi_3'\xi_1'}\right).$$

On the other hand, since $B \in \mathcal{O}(2Z - N - P - E)$, we have:

$$\begin{aligned} p(\tau) &= p\left(A\,\frac{d\xi_1 d\xi_2}{\xi_1'\xi_2'} + B\,\frac{d\xi_2 d\xi_3}{\xi_2'\xi_3'} + C\,\frac{d\xi_3 d\xi_1}{\xi_3'\xi_1'}\right) \\ &= p\left(A\,\frac{d\xi_1 d\xi_2}{\xi_1'\xi_2'} + C\,\frac{d\xi_3 d\xi_1}{\xi_3'\xi_1'}\right). \end{aligned}$$

Thus we have shown that, away from H, $\ker(\gamma) \subseteq \operatorname{im}(\beta)$.

At a simple point $e \in E_i \cap H$ near H we have coordinates $\{u, v, w\}$ in an analytic neighborhood of e so that $E_i = \{u = 0\}$, $H = \{v = 0\}$, and $h = \xi_2 = \xi_2' v$ (recall that $Z = N = P$ on components E_i that intersect H). By (8.2.7) and (8.2.9) we see that if $\tau \in \ker(\widetilde{p} \circ \widetilde{\gamma})$, then $C \in \mathcal{O}(Z - P - E) \approx \mathcal{O}(-E)$. Again we must find an $\omega \in \mathcal{I}_E \Omega^1(\log E)$ so that $p(\tau) = p(\widetilde{\beta}(\omega))$; choose $\omega = \sum k_i \frac{d\xi_i}{\xi_i'}$ with $k_1 = A$ and $k_3 = -B$; then by (8.2.3),

$$p(\widetilde{\beta}(\omega)) = p\left(\frac{A}{v}\,\frac{d\xi_1 d\xi_2}{\xi_1'\xi_2'} + \frac{B}{v}\,\frac{d\xi_2 d\xi_3}{\xi_2'\xi_3'}\right).$$

On the other hand, since $C \in \mathcal{O}(Z - P - E) \approx \mathcal{O}(-E)$ we have:

$$\begin{aligned} p(\tau) &= p\left(Ar\,\frac{d\xi_1 d\xi_2}{\xi_1'\xi_2'} + Br\,\frac{d\xi_2 d\xi_3}{\xi_2'\xi_3'} + Cr\,\frac{d\xi_3 d\xi_1}{\xi_3'\xi_1'}\right) \\ &= p\left(Ar\,\frac{d\xi_1 d\xi_2}{\xi_1'\xi_2'} + Br\,\frac{d\xi_2 d\xi_3}{\xi_2'\xi_3'}\right). \end{aligned}$$

Since $r = \frac{1}{v}$ this shows that $p(\tau) = p(\widetilde{\beta}(\omega))$, and thus we have shown that, near a simple point of E contained in H, $\ker(\gamma) \subseteq \operatorname{im}(\beta)$.

Finally, let $e \in E_i \cap E_j \cap H$ be a double point of E that is contained in H. With coordinates $\{u, v, w\}$ about e wo that $E_i = \{u = 0\}$, $E_j = \{v = 0\}$, and $H = \{w = 0\}$, we have $h = \xi_2 = \xi_2' w$. Moreover, $Z = N$ on this analytic neighborhood of e. By (8.2.8) and (8.2.9) it is evident that if $\tau \in \ker(\widetilde{p} \circ \widetilde{\gamma})$, then $C \in \mathcal{O}(Z - P - E)$. Once more we wish to find an element ω of $\mathcal{I}_E \Omega^1(\log E)$ with the property that $p(\tau) = p(\widetilde{\beta}(\omega))$. As above, choose $\omega = \sum k_i \frac{d\xi_i}{\xi_i'}$ with $k_1 = A$ and

$k_3 = -B$. Then computation (8.2.4) implies that

$$p(\widetilde{\beta}(\omega)) = p\left(\frac{A}{v}\,\frac{d\xi_1 d\xi_2}{\xi_1'\xi_2'} + \frac{B}{v}\,\frac{d\xi_2 d\xi_3}{\xi_2'\xi_3'}\right).$$

Moreover, since p mods out by $\Lambda^2\mathcal{N}(2Z-E)\otimes\mathcal{O}(H)$, expression (8.2.5) shows that again we have:

$$\begin{aligned} p(\tau) &= p\left(Ar\,\frac{d\xi_1 d\xi_2}{\xi_1'\xi_2'} + Br\,\frac{d\xi_2 d\xi_3}{\xi_2'\xi_3'} + Cr\,\frac{d\xi_3 d\xi_1}{\xi_3'\xi_1'}\right) \\ &= p\left(Ar\,\frac{d\xi_1 d\xi_2}{\xi_1'\xi_2'} + Br\,\frac{d\xi_2 d\xi_3}{\xi_2'\xi_3'}\right). \end{aligned}$$

Thus, in each of the three possible cases, we have $\ker(\gamma) \subseteq \operatorname{im}(\beta)$. This completes the proof. ■

8.3 Weighted Complexes

Looking at the formulation of the exact sequence from Proposition 8.1.1 given in (8.1.1), we see an obvious pattern. To describe this pattern (and thus pave the way for the generalization of this sequence to the general n-dimensional case, which will be presented in a later paper), we define the following two complexes.

Definition 8.3.1. *The* weighted Nash complex *is defined to be the complex whose* k^{th} *level is given by:*

$$\widetilde{\mathcal{N}}^k := \Lambda^k\mathcal{N}\otimes\mathcal{O}(Z-E)$$

with maps $\widetilde{\mathcal{N}}^k \to \widetilde{\mathcal{N}}^{k+1}$ *given by* $\wedge\frac{dh}{h}$. *The* weighted log forms complex *is defined* *to be the complex with k^{th} level:*

$$\widetilde{\Omega}^k := \Omega^k(\log E) \otimes \mathcal{O}(-(k-1)Z - E),$$

with maps $\widetilde{\Omega}^k \to \widetilde{\Omega}^{k+1}$ also given by $\wedge \frac{dh}{h}$.

The utility of these complexes becomes clearer when we utilize the isomorphism $\mathcal{O}(-Z) \approx \mathcal{O}(H)$ to rewrite them as:

$$\widetilde{\mathcal{N}}^k = \Lambda^k \mathcal{N} \otimes \mathcal{O}(kZ + (k-1)H - E)$$

and

$$\widetilde{\Omega}^k = \Omega^k(\log E) \otimes \mathcal{O}((k-1)H - E).$$

In this form it is also more apparent that the maps are well-defined for these complexes. Note that, as in the proof of Proposition 8.1.1, when we are near a point $e \notin H$ we have $h = \xi_1$ and thus $\frac{dh}{h} \in \mathcal{N}(Z)$ (or, from another point of view, $\frac{dh}{h} \in \Omega^1(\log E)$); when we are near a point $e \in H$ we have $h = \xi_2$ and thus $\frac{dh}{h} \in \mathcal{N}(N + H) \approx \mathcal{N}(Z + H)$ (alternately, $\frac{dh}{h} \in \Omega^1(\log E) \otimes \mathcal{O}(H)$).

Clearly the sequence in (8.1.1) can be written in terms of these complexes as

$$0 \to \widetilde{\mathcal{N}}^1 \overset{\alpha}{\hookrightarrow} \widetilde{\Omega}^1 \overset{\beta}{\longrightarrow} \widetilde{\Omega}^2/\widetilde{\mathcal{N}}^2 \overset{\gamma}{\twoheadrightarrow} \widetilde{\Omega}^3/\widetilde{\mathcal{N}}^3 \to 0.$$

8.4 Sequences for the Nash sheaf

As an easy corollary to Proposition 8.1.1 we can obtain an exact sequence relating the Nash sheaf to the resolution data.

Corollary 8.4.1. *We have the following exact sequence of sheaves on* $\widetilde{U}$*:*

$$\begin{aligned} 0 \to \mathcal{N} &\overset{\alpha}{\hookrightarrow} \Omega^1(\log E) \otimes \mathcal{O}(-Z) \\ &\xrightarrow{\beta} \left(\Omega^2(\log E)/\Lambda^2\mathcal{N}(2Z) \otimes \mathcal{O}(-2Z)\right) \\ &\overset{\gamma}{\twoheadrightarrow} \Omega^3 \otimes \mathcal{O}_{P+N-2Z}(E-3Z) \to 0. \end{aligned}$$

Proof. Simply tensor the exact sequence in Proposition 8.1.1 with $\mathcal{O}(E-Z)$. ■

We could also tensor the equivalent sequence in (8.1.1) with $\mathcal{O}(E-Z)$ and apply a little algebra to get the following sequence for $\mathcal{N}$:

$$\begin{aligned} 0 \to \mathcal{N} &\overset{\alpha}{\hookrightarrow} \Omega^1(\log E) \otimes \mathcal{O}(-Z) \\ &\xrightarrow{\beta} \Omega^2(\log E) \otimes \mathcal{O}(H-Z)/\Lambda^2\mathcal{N}(H+Z) \\ &\overset{\gamma}{\twoheadrightarrow} \Omega^3(\log E) \otimes \mathcal{O}(2H-Z)/\Lambda^3\mathcal{N}(2H+2Z) \to 0. \end{aligned}$$

Since $\mathcal{O}(H) \approx \mathcal{O}(-Z)$ this sequence is clearly equivalent to the following sequence (which we also take to be a corollary of Proposition 8.1.1):

Corollary 8.4.2. *We have the following exact sequence of sheaves on* $\widetilde{U}$*:*

$$\begin{aligned} 0 \to \mathcal{N} &\overset{\alpha}{\hookrightarrow} \Omega^1(\log E) \otimes \mathcal{O}(-Z) \\ &\xrightarrow{\beta} \Omega^2(\log E) \otimes \mathcal{O}(-2Z)/\Lambda^2\mathcal{N} \\ &\overset{\gamma}{\twoheadrightarrow} \Omega^3(\log E) \otimes \mathcal{O}(-3Z)/\Lambda^3\mathcal{N} \to 0. \end{aligned}$$

The sequence in Corollary 8.4.1 is the one that we will use in Chapter 9 to get information about the Chern classes of the Nash sheaf over $\widetilde{U}$.

8.5 Changing back to the algebraic category

Throughout this Chapter we have been working in the analytic category (although abusing notation). The sequence in Proposition 8.1.1 that was proved exact was in fact the associated analytic sequence over the associated analytic space $\widetilde{U}^h$. By Theorems 2.4.4 and 2.4.5 and Proposition 2.4.6 the original sequence of algebraic sheaves over $\widetilde{U}$ is thus also exact. We can now revert to the algebraic category for Chapter 9.

Chapter 9

Chern Class Results

In this Chapter we use the exact sequence in Corollary 8.4.1 to find certain Chern classes of the Nash sheaf $\mathcal{N}_{\widetilde{U}}$. We begin by stating some simple formulas involving the Chern classes of sheaves (Section 9.1). The Chern class of a locally free sheaf over $\widetilde{U}$ is defined to be the Chern class of its corresponding vector bundle, and the Chern class of a torsion sheaf is defined from a resolution of the torsion sheaf by locally free sheaves.

In Sections 9.2.1 and 9.2.2 we will use these formulas (and the exact sequence from Corollary 8.4.1) to describe the first and last Chern classes of $\mathcal{N}_{\widetilde{U}}$ in the 2 and 3 dimensional cases. Then in Section 9.2.3 we use two formulas involving the zeroth MacPherson-Chern class (developed in Sections 2.6.4 and 2.6.7) and a formula for the local Euler obstruction (due to Gonzalez-Sprinberg, as described in Section 2.6.5) to relate the middle Chern class of the 3-dimensional case to the Euler characteristic of the exceptional divisor E. In the 2-dimensional case these formulas can be used to obtain a formula for the Euler characteristic of E. In Section 9.2.4 we conjecture a general formula for the Chern classes of the Nash sheaf and show some evidence that it may hold; as a consequence of this conjecture we obtain formulas for the local Euler obstruction $Eu_v(U)$ and Euler characteristic $\chi(E)$ in terms of the resolution data in the 3-dimensional case.

9.1 Chern Class Formulas

9.1.1 Chern classes from an exact sequence

Given a three-term exact sequence of sheaves

$$0 \rightarrow \mathcal{A} \hookrightarrow \mathcal{B} \twoheadrightarrow \mathcal{C} \rightarrow 0,$$

we have the following relation of Chern classes (see C3 in Appendix A.3 of [Har77]):

$$c(\mathcal{B}) = c(\mathcal{A})\, c(\mathcal{C}). \tag{9.1.1}$$

We can apply this relation to the exact sequence obtained by Pardon and Stern from the Hsiang-Pati coordinates in the 2-dimensional case (see Proposition 2.5.3), and thus obtain an expression of the Chern classes of the Nash sheaf in terms of the resolution data. In the 3-dimensional case the corresponding sequence (see

Proposition 8.1.1) has four terms; to examine the Chern classes of the Nash sheaf in this case we first prove the following simple claim.

Claim 9.1.1. *Given a four-term exact sequence of sheaves*

$$0 \rightarrow \mathcal{A} \overset{\alpha}{\hookrightarrow} \mathcal{B} \overset{\beta}{\rightarrow} \mathcal{C} \overset{\gamma}{\twoheadrightarrow} \mathcal{D} \rightarrow 0,$$

the Chern classes of $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$ are related by the formula:

$$c(\mathcal{A})\, c(\mathcal{C}) = c(\mathcal{B})\, c(\mathcal{D}).$$

Proof. The four-term sequence above gives rise to the two short exact sequences

$$0 \rightarrow \mathcal{A} \overset{\alpha}{\hookrightarrow} \mathcal{B} \overset{\beta}{\twoheadrightarrow} \operatorname{im}(\beta) \rightarrow 0$$

and

$$0 \rightarrow \ker(\gamma) \hookrightarrow \mathcal{C} \overset{\gamma}{\twoheadrightarrow} \mathcal{D} \rightarrow 0,\cdot$$

Thus we have

$$c(\mathcal{B}) = c(\mathcal{A})\, c(\operatorname{im}(\beta)) \quad \text{and} \quad c(\mathcal{C}) = c(\ker(\gamma))\, c(\mathcal{D}).$$

Putting these together (and using the fact that, by exactness of the original four-term sequence, we have $\operatorname{im}(\beta) = \ker(\gamma)$), we clearly have

$$\begin{aligned} c(\mathcal{A})\, c(\mathcal{C}) &= \left(c(\mathcal{B}) / c(\operatorname{im}(\beta)) \right) c(\ker(\gamma))\, c(\mathcal{D}) \\ &= c(\mathcal{B})\, c(\mathcal{D}). \end{aligned}$$

■

9.1.2 Chern classes of tensor products

In order to do the computations in Section 9.2 we will need to know how the Chern class of a tensor product of sheaves $\mathcal{F} \otimes \mathcal{G}$ relates to the Chern classes of the sheaves $\mathcal{F}$ and $\mathcal{G}$. Of course with regards to the direct sum of sheaves we have the Whitney sum formula

$$c(\mathcal{F} \oplus \mathcal{G}) = c(\mathcal{F})\, c(\mathcal{G})$$

for all sheaves $\mathcal{F}$ and $\mathcal{G}$. In contrast, with tensor products the desired formula will depend on the ranks of $\mathcal{F}$ and $\mathcal{G}$. In the computations below we will only need to know such a formula in two cases: where $\mathrm{rank}(\mathcal{F}) = 2$ and $\mathrm{rank}(\mathcal{G}) = 1$; and where $\mathrm{rank}(\mathcal{F}) = 3$ and $\mathrm{rank}(\mathcal{G}) = 1$ (and where these are sheaves on a 3-dimensional smooth space).

We will obtain the tensor product formulas on the vector bundle side of the correspondence between locally free sheaves and vector bundles, since in the category of vector bundles we have the following useful fact, called the *Splitting Principle* (from Section IV.21 in [BT82]; a corresponding principle for sheaves can be found in C4 of Appendix A.3 of [Har77]).

Claim 9.1.2. *To prove a polynomial identity in the Chern classes of complex vector bundles, it suffices to prove it under the assumption that the vector bundles are direct sums of line bundles.*

Thus in our calculations we can assume that, for example, a rank k vector bundle ξ over a space X is written in the form:

$$\xi = L_1 \oplus \cdots \oplus L_k,$$

where the L_i are line bundles over X. Note that in this situation, the Whitney sum formula implies that

$$c(\xi) \;=\; \prod_{i=1}^{k} c(L_i) \;=\; \prod_{i=1}^{k}(1 + l_i),$$

where l_i is by definition the first Chern class of the line bundle L_i. In particular, the j^{th} Chern class of ξ is thus given by:

$$c_j(\xi) = \sum_{1\le i_1<\ldots<i_j\le k} l_{i_1}\ldots l_{i_j};$$

in other words, the j^{th} Chern class of ξ is given by the j^{th} elementary symmetric polynomial in the Chern classes $l_1, \ldots, l_k$.

Using the Splitting Principle our computations will all come down to the case where the tensor product involves only line bundles; if L and J are line bundles we have (20.1 in [BT82]):

$$c(L \otimes J) = 1 + c_1(L) + c_1(J). \tag{9.1.2}$$

We first consider the case where $\xi = L_1 \oplus L_2$ is a rank 2 vector bundle and $\eta = J$ is a line bundle (over a 3-dimensional space). Using the Whitney sum formula and 9.1.2 we have

$$\begin{aligned} c(\xi \otimes \eta) &= c((L_1 \oplus L_2) \otimes J) \\ &= c((L_1 \otimes J) \oplus (L_2 \otimes J)) \\ &= c(L_1 \otimes J)\, c(L_2 \otimes J) \\ &= (1 + l_1 + j)(1 + l_2 + j), \end{aligned}$$

where as above, $l_i := c_1(L_i)$ and $j := c_1(J)$. It is well-known that any symmetric polynomial (as is the above) can be written in terms of the elementary symmetric functions (which in this case are the Chern classes of the bundles ξ and η) In this case it is easy enough to do by hand; we simply write the expression above in terms of the Chern classes

$$c_1(\xi) = l_1 + l_2, \quad c_2(\xi) = l_1 l_2, \quad \text{and} \quad c_1(\eta) = j.$$

Then (applying the Splitting Principle), we get the formula we need, namely:

Claim 9.1.3. *Given a rank 2 vector bundle ξ and a line bundle η over a 3-dimensional space X, the Chern class of the tensor product $\xi \otimes \eta$ is*

$$c(\xi \otimes \eta) \;=\; 1 \;+\; (c_1(\xi) + 2c_1(\eta)) \;+\; (c_2(\xi) + c_1(\xi)c_1(\eta) + c_1(\eta)^2).$$

The case where $\xi = L_1 \oplus L_2 \oplus L_3$ is of rank 3 and $\nu = J$ is a line bundle (over a dimension three space X) is quite similar. In that case we have

$$\begin{aligned} c(\xi \otimes \eta) &= c((L_1 \oplus L_2 \oplus L_3) \otimes J) \\ &= c((L_1 \otimes J) \oplus (L_2 \otimes J) \oplus (L_3 \otimes J)) \\ &= c(L_1 \otimes J)\, c(L_2 \otimes J)\, c(L_3 \otimes J) \\ &= (1 + l_1 + j)(1 + l_2 + j)(1 + l_3 + j). \end{aligned}$$

We wish to put this in terms of the elementary symmetric polynomials in $l_1, \ldots, l_3$ (and of j), *i.e.* in terms of the Chern classes

$$\begin{aligned} c_1(\xi) &= l_1 + l_2 + l_3, \\ c_2(\xi) &= l_1 l_2 + l_1 l_3 + l_2 l_3, \\ c_3(\xi) &= l_1 l_2 l_3, \end{aligned}$$

(and $c_1(\eta)j$) of ξ and ν. After a lengthy calculation we arrive at the following result (note we only keep the terms of (complex) degree 3 or less, since the cohomology of the base space X is zero in dimensions higher than six).

Claim 9.1.4. *Given a rank 3 vector bundle ξ and a line bundle η over a 3-dimensional space X, the Chern class of the tensor product $\xi \otimes \eta$ is*

$$\begin{aligned} c(\xi \otimes \eta) &= 1 + (c_1(\xi) + 3c_1(\eta)) + (c_2(\xi) + 2c_1(\xi)c_1(\eta) + 3c_1(\eta)^2) \\ &\quad + (c_3(\xi) + c_2(\xi)c_1(\eta) + c_1(\xi)c_1(\eta)^2 + c_1(\eta)^3). \end{aligned}$$

9.1.3 Chern classes of exterior powers

In this section we find formulas for the second exterior power of a rank 2 vector bundle and the second and third exterior powers of a rank 3 vector bundle. The cases involving the top exterior power are easy:

Claim 9.1.5. *The Chern class of the top exterior power of a vector bundle ξ over X (of rank, say, n) is given by:*

$$c(\Lambda^n \xi) = 1 + c_1(\xi).$$

Proof. We use the Splitting Principle; suppose ξ is written as the direct sum of line bundles $\xi = L_1 \oplus \cdots \oplus L_n$. By definition we have

$$\Lambda^n(\xi) \approx L_1 \otimes \cdots \otimes L_n.$$

Thus by (9.1.2) we have:

$$\begin{aligned} c(\Lambda^n(\xi)) &= c(L_1 \otimes \cdots \otimes L_n) \\ &= 1 + c(L_1) + \ldots + c(L_n) \\ &= 1 + (l_1 + \ldots + l_n) \\ &= 1 + c_1(\xi), \end{aligned}$$

where $l_i := c_1(L_i)$, since the first Chern class of ξ is given by the first elementary symmetric function in the Chern classes l_i. ■

We now examine the Chern class of the second exterior power of a rank three vector bundle, again utilizing the Splitting Principle. Let $\xi = L_1 \oplus L_2 \oplus L_3$ be a rank three vector bundle over X. By definition the second exterior product of ξ is the bundle

$$\Lambda^2(\xi) \;=\; \bigoplus_{1 \leq i_1 < i_2 \leq 3} (L_{i_1} \otimes L_{i_2}) \;=\; (L_1 \otimes L_2) \oplus (L_1 \otimes L_3) \oplus (L_2 \otimes L_3),$$

and thus

$$\begin{aligned} c(\Lambda^2(\xi)) &= c(L_1 \otimes L_2)\, c(L_1 \otimes L_3)\, c(L_2 \otimes L_3) \\ &= (1 + l_1 + l_2)(1 + l_1 + l_3)(1 + l_2 + l_3). \end{aligned}$$

Since this is a symmetric polynomial in the $l_i := c_1(L_i)$, we can rewrite it in terms of the elementary symmetric polynomials in the l_i, *i.e.* in terms of the Chern classes of ξ. Keeping only terms of degree less than or equal to three (as in the previous section) we get the following formula.

Claim 9.1.6. *Given a rank 3 vector bundle ξ over a 3-dimensional space X, the Chern class of the second exterior power of ξ is*

$$c(\Lambda^2(\xi)) \;=\; 1 \;+\; (2c_1(\xi)) \;+\; (c_2(\xi) + c_1(\xi)^2) \;+\; (-c_3(\xi) + c_1(\xi)c_2(\xi)).$$

9.2 Chern Classes of the Nash Sheaf

9.2.1 The 2-dimensional sequence

Recall the sequence of sheaves over $\widetilde{U}$ obtained by Pardon and Stern in the 2-dimensional case (see Proposition 2.5.3):

$$0 \;\to\; \mathcal{N}_{\widetilde{U}} \;\to\; \mathcal{I}_E\Omega^1_{\widetilde{U}}(\log E)\otimes\mathcal{O}_{\widetilde{U}}(E-Z) \;\to\; \Omega^2_{\widetilde{U}}\otimes\mathcal{O}_{N-Z}(E-2Z) \;\to\; 0.$$

In this section we apply the formulas obtained in Section 9.1 to the sequence above, and thereby obtain a formula for the Chern class of the Nash sheaf $\mathcal{N}_{\widetilde{U}}$ in terms of the resolution data. By (9.1.1) we have:

$$c(\mathcal{N}) \;=\; {c(\Omega^1(\log E)\otimes\mathcal{O}(-Z))}\big/{c(\Omega^2_{\widetilde{U}}\otimes\mathcal{O}_{N-Z}(E-2Z))} \tag{9.2.1}$$

(after using the fact that $\mathcal{I}_E \approx \mathcal{O}(-E)$); note that we drop the $\widetilde{U}$ subscripts in the notation above, and will often do so throughout this section. Note also that in this case $\widetilde{U}$ is a 2-dimensional smooth space.

We first examine the Chern classes of the sheaf $\Omega^1(\log E)$, the sheaf of 1-forms on $\widetilde{U}$ with logarithmic poles along E. By definition this sheaf fits into the exact sequence over $\widetilde{U}$:

$$0 \;\to\; \Omega^1 \;\hookrightarrow\; \Omega^1(\log E) \;\twoheadrightarrow\; \oplus_i\mathcal{O}_{E_i} \;\to\; 0,$$

where the surjection above is the residue map. Thus we have:

$$c(\Omega^1(\log E)) \;=\; c(\Omega^1)\,c(\oplus_i\mathcal{O}_{E_i}) \;=\; c(\Omega^1)\,c(\mathcal{O}_{E_1})\cdots c(\mathcal{O}_{E_s}), \tag{9.2.2}$$

where s is the number of components E_i of the exceptional divisor $E \subset \widetilde{U}$. Since Ω^1 is the sheaf of sections of the cotangent bundle $T^*\widetilde{U}$ over $\widetilde{U}$, we have

$$c(\Omega^1) = c(T^*\widetilde{U}) = (-1)^k c_k(T\widetilde{U}) = 1 - c_1(\widetilde{U}) + c_2(\widetilde{U}) =: 1 + K + C.$$

Now to find the Chern class of $\Omega^1(\log E)$ it now suffices to find the Chern classes of the $\mathcal{O}_{E_i}$ and apply 9.2.2. The sheaf $\mathcal{O}_{E_i}$ is by definition the quotient sheaf $\mathcal{O}/\mathcal{O}(-E_i)$ and thus fits into the exact sequence:

$$0 \rightarrow \mathcal{O}(-E_i) \hookrightarrow \mathcal{O} \twoheadrightarrow \mathcal{O}_{E_i} \rightarrow 0,$$

Moreover, we have the following simple fact (see C1 in Appendix A.3 of [Har77]):

Fact 9.2.1. *The Chern class of the line bundle defined by a divisor D is*

$$c(\mathcal{O}(D)) = 1 + D,$$

where D denotes the cohomology class corresponding to the divisor D.

Thus we have

$$\begin{aligned} c(\mathcal{O}_{E_i}) &= {}^{c(\mathcal{O})}\!/_{c(\mathcal{O}(-E_i))} \\ &= {}^{1}\!/_{(1 - E_i)} \\ &= (1 - E_i)^{-1} \\ &= 1 + E_i + E_i^2, \end{aligned}$$

since the inverse of an element $1+x$ in the cohomology ring $\mathrm{H}^*(\widetilde{U})$ is

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + ...,$$

and all cohomology is zero after (real) dimension four (since we are over the complex 2-dimensional space $\widetilde{U}$). Putting all this into (9.2.2) we get:

$$\begin{aligned} c(\Omega^1(\log E)) &= (1+K+C)\,(1+E_1+E_1^2)\,(1+E_2+E_2^2)\cdots(1+E_s+E_s^2) \\ &= (1+K+C)\left(1+(E_1+\ldots E_s)+(E_1^2+\ldots E_s^2)+\sum_{i<j} E_iE_j\right) \\ &= (1+K+C)\left(1+E+E^2-\sum_{i<j} E_iE_j\right) \\ &=: (1+K+C)\,(1+E+E^2-y_2), \end{aligned} \tag{9.2.3}$$

since $E = E_1 + \ldots E_s$ and E^2 can be written as

$$E^2 \;=\; \sum_i E_i^2 + \sum_{i\neq j} E_iE_j \;=\; \sum_i E_i^2 + 2\sum_{i<j} E_iE_j.$$

Multiplying out our computation in 9.2.3, and considering only classes of real rank less than or equal to 4, we have a formula for $c(\Omega^1(\log E))$, namely:

$$c(\Omega^1(\log E)) \;=\; 1 \;+\; (E+K) \;+\; (E^2+C+EK-y_2). \tag{9.2.4}$$

Using the formula above, the fact that (by Fact 9.2.1) $c(\mathcal{O}(-Z)) = 1 - Z$, and the tensor product formula from Claim 9.1.3, we can write the Chern class of the middle term of Pardon and Stern's exact sequence as:

$$\begin{aligned} c(\Omega^1(\log E)\otimes\mathcal{O}(-Z)) &= 1 + c_1(\Omega^1(\log E)) + 2c_1(\mathcal{O}(-Z)) + c_2(\Omega^1(\log E)) \\ &\quad + c_1(\Omega^1(\log E))c_1(\mathcal{O}(-Z)) + c_1(\mathcal{O}(-Z))^2 \\ &= 1 + (E+K) - 2Z + (E^2 + C + EK - y_2) \\ &\quad + (E+K)(-Z) + Z^2 \\ &= 1 + (E + K - 2Z) \qquad (9.2.5) \\ &\quad + (E^2 + C + EK - y_2 - EZ - KZ + Z^2). \end{aligned}$$

We now move on to computing the Chern class of the last term in the sequence; first we use Claim 9.1.5 to write the Chern class of Ω^2 in terms of the Chern class of Ω^1:

$$c(\Omega^2) = c(\Lambda^2\Omega^1) = 1 + c_1(\Omega^1) = 1 + K.$$

Now consider the following free resolution of $\mathcal{O}_{N-Z} \approx \mathcal{O}/\mathcal{O}(Z-N)$:

$$0 \to \mathcal{O}(Z-N) \hookrightarrow \mathcal{O} \twoheadrightarrow \mathcal{O}_{N-Z} \to 0.$$

Tensoring the sequence above by $\Omega^2 \otimes \mathcal{O}(E-2Z)$ gives us the exact sequence:

$$\begin{aligned} 0 \to \Omega^2\otimes\mathcal{O}(E-Z-N) &\hookrightarrow \Omega^2\otimes\mathcal{O}(E-2Z) \\ &\twoheadrightarrow \Omega^2\otimes\mathcal{O}_{N-Z}(E-2Z) \to 0. \end{aligned}$$

Using this sequence and applying (9.1.1), (9.1.2), and Fact 9.2.1 we can now compute the Chern class of the last term of the Pardon-Stern sequence:

$$\begin{aligned}
c(\Omega^2 \otimes \mathcal{O}_{N-Z}(E-2Z)) &= \Omega^2 \otimes \mathcal{O}(E-2Z) \big/ \Omega^2 \otimes \mathcal{O}(E-Z-N) \\
&= (1+K+E-2Z) \big/ (1+K+E-Z-N) \\
&= (1+K+E-2Z) \\
&\qquad (1-(K+E-Z-N)+(K+E-Z-N)^2) \\
&= 1+(N-Z) \\
&\qquad +(-KN+KZ+N^2-Z^2-EN+EZ). \quad (9.2.6)
\end{aligned}$$

We are now in a position to prove the following claim.

Claim 9.2.2. *Given a sufficiently fine resolution $(\widetilde{U}, E)$ of a two-dimensional variety with isolated singular point (U, v), let Z and N be the divisors defined from the Hsiang-Pati coordinates on $\widetilde{U}$. Then the Chern class of the Nash sheaf over $\widetilde{U}$ is given in terms of the resolution data as:*

$$c(\mathcal{N}) = 1 + (E+K-Z-N) + (-KZ+Z^2-EZ+E^2+KE+C-y_2),$$

where $c(\Omega^1_{\widetilde{U}}) =: 1+K+C$ and $y_2 := \sum_{i<j} E_iE_j$.

Proof. We put (9.2.5) and (9.2.6) into expression (9.2.1) to compute:

$$\begin{aligned}
c(\mathcal{N}) &= c(\Omega^1(\log E) \otimes \mathcal{O}(-Z)) \big/ c(\Omega^2_{\widetilde{U}} \otimes \mathcal{O}_{N-Z}(E-2Z)) \\
&= (1+(E+K-2Z)+(E^2+C+EK-y_2-EZ-KZ+Z^2)) \\
&\qquad (1+(N-Z)+(-KN+KZ+N^2-Z^2-EN+EZ))^{-1} \\
&= 1+(E+K-Z-N)+(-KZ+Z^2-EZ+E^2+KE+C-y_2).
\end{aligned}$$

■

We will come back to this formula in Section 9.2.3 and use it to get expressions for the local Euler obstruction of the singularity v and the Euler characteristic of the exceptional divisor E.

It is important to point out that we can actually get the first Chern class of the Nash sheaf by very simple means: we have an isomorphism (see 3.12(d) in [PS97]; the argument is similar to the one above in Claim 8.1.2):

$$\Lambda^2 \mathcal{N} \approx \Omega^2 \otimes \mathcal{O}(E - Z - N).$$

Thus (by Claim 9.1.5, Equation 9.1.2, and Fact 9.2.1) we can write:

$$\begin{aligned} c_1(\mathcal{N}) &= c_1(\Lambda^2 \mathcal{N}) \\ &= c_1(\Omega^2 \otimes \mathcal{O}(E - Z - N)) \\ &= c_1(\Omega^2) + c_1(E - Z - N) \\ &= c_1(\Omega^1) + c_1(E - Z - N) \\ &= K + E - Z - N. \end{aligned}$$

9.2.2 The 3-dimensional sequence

We now use similar means to obtain the first and last Chern classes of the Nash sheaf over $\widetilde{U}$ in the three-dimensional case. Recall from Corollary 8.4.1 of

Proposition 8.1.1 that we have an exact sequence of sheaves over $\widetilde{U}$:

$$\begin{aligned} 0 \to \mathcal{N} &\overset{\alpha}{\hookrightarrow} \Omega^1(\log E) \otimes \mathcal{O}(-Z) \\ &\xrightarrow{\beta} \Omega^2(\log E) \otimes \mathcal{O}(-2Z)/_{\Lambda^2\mathcal{N}} \\ &\overset{\gamma}{\twoheadrightarrow} \Omega^3 \otimes \mathcal{O}_{P+N-2Z}(E-3Z) \to 0. \end{aligned} \tag{9.2.7}$$

By Claim 9.1.1 we have

$$c(\mathcal{N}) = \frac{c(\Omega^1(\log E) \otimes \mathcal{O}(-Z))\, c(\Omega^3 \otimes \mathcal{O}_{P+N-2Z}(E-3Z))}{c\left(\Omega^2(\log E) \otimes \mathcal{O}(-2Z)/_{\Lambda^2\mathcal{N}}\right)}$$

and thus

$$\frac{c(\mathcal{N})}{c(\Lambda^2\mathcal{N})} = \frac{c(\Omega^1(\log E) \otimes \mathcal{O}(-Z))\, c(\Omega^3 \otimes \mathcal{O}_{P+N-2Z}(E-3Z))}{c(\Omega^2(\log E) \otimes \mathcal{O}(-2Z))}. \tag{9.2.8}$$

We first examine the left-hand side of the expression above. Since $\widetilde{U}$ is three-dimensional in this case, Claim 9.1.6 implies that (after a bit of computation):

$$\begin{aligned} \frac{c(\mathcal{N})}{c(\Lambda^2\mathcal{N})} &= \frac{1 + c_1(\mathcal{N}) + c_2(\mathcal{N}) + c_3(\mathcal{N})}{1 + 2c_1(\mathcal{N}) + c_2(\mathcal{N}) + c_1(\mathcal{N})^2 - c_3(\mathcal{N}) + c_1(\mathcal{N})c_2(\mathcal{N})} \\ &= 1 - c_1(\mathcal{N}) + c_1(\mathcal{N})^2 - c_1(\mathcal{N})^3 + 2c_3(\mathcal{N}). \end{aligned} \tag{9.2.9}$$

Note that the expression above does not involve the Chern class $c_2(\mathcal{N})$; thus the sequence (9.2.7) will not give us any information about this Chern class, *i.e.* will only describe the first and third Chern classes of $\mathcal{N}$.

The first Chern class of $\mathcal{N}$ can easily be obtained without the sequence (9.2.7), as we found in the 2-dimensional case at the end of the previous section, using the isomorphism

$$\Lambda^3\mathcal{N} \approx \Omega^3 \otimes \mathcal{O}(E - Z - N - P)$$

from Claim 8.1.2. Applying various facts obtained in Section 9.1, we have:

$$\begin{aligned} c_1(\mathcal{N}) &= c_1(\Lambda^3\mathcal{N}) \\ &= c_1(\Omega^3 \otimes \mathcal{O}(E - Z - N - P)) \\ &= c_1(\Omega^3) + c_1(E - Z - N - P) \\ &= c_1(\Omega^1) + c_1(E - Z - N - P) \\ &= K + E - Z - N - P \end{aligned} \tag{9.2.10}$$

(where $K := c_1(\Omega^1_{\tilde{U}})$). Since we cannot describe $c_2(\mathcal{N})$ from sequence (9.2.7), and we already know $c_1(\mathcal{N})$ as above, we will now concentrate on using the sequence to find the top Chern class of $\mathcal{N}$. Many of the computations below are similar to the ones in the last section and we do them here at a faster pace.

We begin by examining the Chern class of $\Omega^1(\log E)$, which will involve the Chern classes $c(\mathcal{O}_{E_i})$. As above we have

$$\begin{aligned} c(\mathcal{O}_{E_i}) &= {c(\mathcal{O})}/{c(\mathcal{O}(-E_i))} \\ &= (1 - E_i)^{-1} \\ &= 1 + E_i + E_i^2 + E_i^3. \end{aligned}$$

Thus the Chern class of the direct sum $\oplus_i \mathcal{O}_{E_i}$ is

$$\begin{aligned} c(\oplus_i \mathcal{O}_{E_i}) &= \prod_i c(\mathcal{O}_{E_i}) \\ &= (1 + E_1 + E_1^2 + E_1^3) \cdots (1 + E_s + E_s^2 + E_s^3) \\ &= 1 + \sum_i E_i + \sum_i E_i^2 + \sum_i E_i^3 + \sum_{i<j} E_i E_j \\ &\quad + \sum_{i \neq j} E_i^2 E_j + \sum_{i<j<k} E_i E_j E_k \\ &=: 1 + E + E^2 + E^3 - y_2 - y_3, \end{aligned}$$

where

$$y_2 := \sum_{i<j} E_i E_j \quad \text{and} \quad y_3 := 2 \sum_{i \neq j} E_i^2 E_j + 5 \sum_{i<j<k} E_i E_j E_k; \tag{9.2.11}$$

these computations make use of the fact that $E = E_1 + \ldots E_s$ and E^2 can be written as

$$E^2 = \sum_i E_i^2 + 2 \sum_{i<j} E_i E_j,$$

while E^3 can be written

$$E^3 = \sum_i E_i^3 + 3 \sum_{i \neq j} E_i^2 E_j + 6 \sum_{i<j<k} E_i E_j E_k.$$

Denote the Chern class of the sheaf of 1-forms on $\widetilde{U}$ by

$$c(\Omega^1_{\widetilde{U}}) =: 1 + K + C_2 + C_3.$$

Now using the residue sequence for $\Omega^1(\log E)$ we have:

$$\begin{aligned} c(\Omega^1(\log E)) &= c(\Omega^1)\, c(\oplus \mathcal{O}_{E_i}) \\ &= (1 + K + C_2 + C_3)\,(1 + E + E^2 + E^3 - y_2 - y_3); \end{aligned}$$

multiplying this out and considering only terms that sit in cohomology groups of (real) dimension less than or equal to 6, we obtain the following.

$$\begin{aligned} c(\Omega^1(\log E)) &= 1 + (E+K) + (KE + E^2 - y_2 + C_2) \\ &\quad + (KE^2 + E^3 - Ky_2 + C_2E - y_3 + C_3). \end{aligned} \tag{9.2.12}$$

Applying Claim 9.1.4 we can write (after much calculation) the Chern class of the second term of the sequence (9.2.7) as

$$\begin{aligned} c(\Omega^1(\log E) \otimes \mathcal{O}(-Z)) &= 1 + (E+K) + 3(-Z) + (KE + E^2 - y_2 + C_2) \\ &\quad + 2(E+K)(-Z) + 3(-Z)^2 \\ &\quad + (KE^2 + E^3 - Ky_2 + C_2E - y_3 + C_3) \\ &\quad + (KE + E^2 - y_2 + C_2)(-Z) \\ &\quad + (E+K)(-Z)^2 + (-Z)^3 \\ &= 1 + (-3Z + E + K) \qquad (9.2.13) \\ &\quad + (KE + E^2 + 3Z^2 - 2ZE - 2ZK - y_2 + C_2) \\ &\quad + (E^3 + KE^2 - ZE^2 + C_2E + Z^2E - KZE - Z^3 \\ &\qquad + y_2Z + KZ^2 - Ky_2 - C_2Z - y_3 + C_3). \end{aligned}$$

Now we compute the Chern classes of the last term in sequence (9.2.7), using the free resolution for $\mathcal{O}_{P+N-2Z}$ tensored with the sheaf $\Omega^3 \otimes \mathcal{O}(E-3Z)$ (note that, although tensoring is in general only right exact, it is exact when the sheaves in question are locally free):

$$0 \to \Omega^3 \otimes \mathcal{O}(E - Z - P - N) \hookrightarrow \Omega^3 \otimes \mathcal{O}(E - 3Z) \twoheadrightarrow \Omega^3 \otimes \mathcal{O}_{P+N-2Z}(E - 3Z) \to 0.$$

Using this sequence we find

$$\begin{aligned} &\Omega^3 \otimes \mathcal{O}_{P+N-2Z}(E - 3Z) \\ &\quad = \Omega^3 \otimes \mathcal{O}(E - 3Z) / \Omega^3 \otimes \mathcal{O}(E - Z - P - N) \\ &\quad = (1 + K + E - 3Z)(1 + K + E - Z - P - N)^{-1} \\ &\quad = (1 + K + E - 3Z)(1 - (K + E - Z - P - N) + \\ &\qquad (K + E - Z - P - N)^2 - (K + E - Z - P - N)^3) \end{aligned} \tag{9.2.14}$$

Multiplying this out, that the Chern class of the last term of the sequence is:

$$\begin{aligned} &\Omega^3 \otimes \mathcal{O}_{P+N-2Z}(E - 3Z) \\ &\quad = 1 + (-2Z + P + N) \\ &\qquad + (-EN - EP - ZN - ZP - KN - KP + 2PN + P^2 + N^2 \\ &\qquad\quad - 2Z^2 + 2ZE + 2ZK) \\ &\qquad + (-2EN^2 + N^3 - 3NZ^2 - 4KZE + K^2N - 4EPN + 3PN^2 + 2ENZ \\ &\qquad\quad - 3PZ^2 + 2KEN - 2KP^2 + K^2P - 4KPN + 2KEP + 2KNZ \\ &\qquad\quad + E^2P + 2KPZ + 2EPZ - 2KN^2 - 2EP^2 + E^2N + 3P^2N + P^3 \\ &\qquad\quad - 2ZK^2 - 2ZE^2 - 2Z^3 + 4KZ^2 + 4Z^2E). \end{aligned} \tag{9.2.15}$$

It now remains to find the Chern class of $\Omega^2(\log E) \otimes \mathcal{O}(-2Z)$ and apply Equation 9.2.8 to find the Chern classes of the Nash sheaf. By Claim 9.1.6 we have

$$\begin{aligned} c(\Omega^2(\log E)) &= c(\Lambda^2(\Omega^1(\log E))) \\ &= 1 + 2c_1(\Omega^1(\log E)) + c_2(\Omega^1(\log E)) + c_1(\Omega^1(\log E))^2) \\ &\quad - c_3(\Omega^1(\log E)) + c_1(\Omega^1(\log E))c_2(\Omega^1(\log E)). \end{aligned}$$

Applying (9.2.12) and doing a lot of calculations results in

$$\begin{aligned} c(\Omega^2(\log E)) &= 1 + (2E + 2K) \\ &\quad + (2E^2 + 3KE + K^2 - y_2 + C_2) \\ &\quad + (KE^2 + K^2E - Ey_2 + KC_2 + y_3 - C_3). \end{aligned}$$

Finally we apply Claim 9.1.4 to this and $\mathcal{O}(-2Z)$ and find that the chern class of the third term of the sequence is:

$$\begin{aligned} &\Omega^2(\log E) \otimes \mathcal{O}(-2Z) \\ &\quad = 1 + (-6Z + 2E + 2K) \\ &\qquad + (K^2 + 3KE + 2E^2 + 12Z^2 - 8ZE - 8ZK - y_2 + C_2) \\ &\qquad + (-6KZE + K^2E - 2ZK^2 - 4ZE^2 + KE^2 - 8Z^3 + 8KZ^2 \\ &\qquad + 8Z^2E - Ey_2 + 2y_2Z + KC_2 - 2C_2Z + y_3 - C_3) \quad (9.2.16) \end{aligned}$$

To obtain an expression for the right-hand side of (9.2.8), it now remains only to multiply the results found in (9.2.13) and (9.2.14), and then divide that by the result in (9.2.16). Comparing this result with the left-hand side of (9.2.8), and substituting $E + K - Z - N - P$ for $c_1(\mathcal{N})$ as in (9.2.10), we can solve for the third chern class of $\mathcal{N}$. Doing this (and throwing out terms landing in cohomology groups of real dimension greater than six) we obtain the following result.

Claim 9.2.3. *Given a sufficiently fine resolution $(\widetilde{U}, E)$ of a three-dimensional variety with isolated singular point (U, v), let Z, N, and P be the divisors defined from the Hsiang-Pati coordinates on $\widetilde{U}$. Then the third Chern class of the Nash sheaf over $\widetilde{U}$ is:*

$$\begin{aligned} c_3(\mathcal{N}) &= E^3 + KE^2 - ZE^2 + C_2E + Z^2E - KZE - Z^3 \\ &\quad + y_2Z + KZ^2 - Ky_2 - C_2Z - y_3 + C_3, \end{aligned}$$

where $c(\Omega^1_{\widetilde{U}}) =: 1 + K + C_2 + C_3$ and y_2 and y_3 are as defined in Equation 9.2.11.

Note that, as in the 2-dimensional case (see Claim 9.2.2), the top Chern class of $\mathcal{N}$ does not involve the "higher" multiplicities defined by the divisors N and P (in the 2-dimensional case the top Chern class does not involve N). It will, however, be useful in determining the second Chern class of $\mathcal{N}$ that we were not able to obtain directly from the sequence. In the next section this is precisely what we shall do.

9.2.3 Using the MacPherson class

By Corollaries 2.6.9 and 2.6.15 we have two ways of expressing the zeroth MacPherson-Chern class of U in terms of the resolution $\pi\colon (\widetilde{U}, E) \to (U, v)$, namely:

$$\begin{aligned} c_0^{\mathrm{MP}}(U) &= \pi_* \mathrm{Dual}\, c_n(\widetilde{U}) + (1 - \chi(E)), \text{ and} \\ c_0^{\mathrm{MP}}(U) &= \pi_* \mathrm{Dual}\, c_n(\mathfrak{N}) + (1 - Eu_v(U)), \end{aligned}$$

where $\chi(E)$ is the Euler characteristic of the exceptional divisor E, $\mathfrak{N}$ is the Nash bundle over $\widetilde{U}$, and $Eu_v(U)$ is the local Euler obstruction of U at the point v (as defined in Section 2.6.5). Moreover, by 2.6.12 we can write this local Euler obstruction in terms of the Nash bundle as:

$$Eu_v(U) = \pi_*(c_{n-1}(\mathfrak{N} - \xi) \cap [E]),$$

where ξ is the the line bundle over E defined by the divisor Z corresponding to the inverse image of the maximal ideal, and $[E] \in \mathrm{H}_{2n-2}$ is the homology class corresponding to E.

Under the correspondence between vector bundles and locally free sheaves, the tangent bundle $T\widetilde{U}$ corresponds to the dual of the sheaf of 1-forms $\Omega^1_{\widetilde{U}}$; the Nash bundle $\mathfrak{N}$ corresponds to the dual of $\mathcal{N}$; and the bundle ξ corresponds to the sheaf

$\mathcal{O}(Z)$. Thus the three formulas above combine to give us the following:

$$\begin{aligned} 1-\chi(E) &= \pi_*\mathrm{Dual}\, c_n(\mathcal{N}^*) - \pi_*\mathrm{Dual}\, c_n((\Omega^1)^*_{\widetilde{U}}) \\ &\qquad + 1 - \pi_*(c_{n-1}(\mathcal{N}^* - \mathcal{O}(Z)) \cap [E]). \end{aligned}$$

and thus

$$\chi(E) = \pi_*\Big(\mathrm{Dual}\left(c_n((\Omega^1)^*_{\widetilde{U}}) - c_n(\mathcal{N}^*)\right) + \left(c_{n-1}(\mathcal{N}^* - \mathcal{O}(Z)) \cap [E]\right)\Big). \quad (9.2.17)$$

9.2.3.1 The 2-dimensional case

Now assume that U is 2-dimensional, with resolution $\widetilde{U}$ as above, and thus that we have the formula given in Claim 9.2.2 for the Chern classes of the Nash sheaf. We can use these Chern classes and Equation 9.2.17 to express the Euler characteristic of the exceptional divisor in terms of the resolution data. First note that we have (using 9.2.2):

$$\begin{aligned} c_1(\mathcal{N}^* - \mathcal{O}(Z)) \cap [E] &= \Big((-1)^k c(\mathcal{N}) / c(\mathcal{O}(Z))\Big)_1 \cap [E] \\ &= \Big((1 - c_1(\mathcal{N}) + c_2(\mathcal{N}))\,(1 - Z + Z^2)\Big)_1 \cap [E] \\ &= (-c_1(\mathcal{N}) - Z) \cap [E] \\ &= (-(E + K - Z - N) - Z) \cap [E] \\ &= \mathrm{Dual}(-E^2 - EK + EN), \end{aligned}$$

where multiplication in the last line is cup product: for example, since $[E]$ is the dual of E (in $\widetilde{U}$), we have

$$E \cap [E] = E \cap (E \cap [\widetilde{U}]) = (E \cup E) \cap [\widetilde{U}] = \mathrm{Dual}(E^2).$$

We thus have the following result relating the local Euler obstruction to the resolution data.

Claim 9.2.4. *Given a sufficiently fine resolution $(\widetilde{U}, E)$ of a two-dimensional variety with isolated singular point (U, v), let Z and N be the divisors defined from the Hsiang-Pati coordinates on $\widetilde{U}$. The local Euler obstruction of U at the point v can be written in terms of the resolution data as*

$$Eu_v(U) = \pi_* \mathrm{Dual}(-E^2 - EK + EN),$$

where $K := c_1(\Omega^1_{\widetilde{U}})$.

On the other hand, we have

$$\begin{aligned} c_2((\Omega^1)^*_{\widetilde{U}}) - c_2(\mathcal{N}^*) &= c_2(\Omega^1_{\widetilde{U}}) - c_2(\mathcal{N}) \\ &= C - (-KZ + Z^2 - EZ + E^2 + EK + C - y_2) \\ &= KZ - Z^2 + EZ - E^2 - KE + y_2, \end{aligned}$$

where, as in Section 9.2.1, $y_2 := \sum_{i<j} E_i E_j$ and $c(\Omega^1_{\widetilde{U}}) =: 1 + K + C$.

Putting these computations together in Equation 9.2.17, we obtain:

$$\begin{aligned} \chi(E) &= \pi_* \mathrm{Dual}\big((KZ - Z^2 + EZ - E^2 - KE + y_2) \\ &\qquad + (-E^2 - EK + EN)\big) \\ &= \pi_* \mathrm{Dual}(KZ - Z^2 + y_2 - 2E^2 - 2EK + EN + EZ). \end{aligned}$$

In other words, we have the following claim.

Claim 9.2.5. *Given a sufficiently fine resolution* $(\widetilde{U}, E)$ *of a two-dimensional variety with isolated singular point* (U, v)*, let* Z *and* N *be the divisors defined from the Hsiang-Pati coordinates on* $\widetilde{U}$*. The Euler characteristic of the exceptional divisor can be written in terms of the resolution data as*

$$\chi(E) = \pi_* \mathrm{Dual}\left(KZ - Z^2 - 2E^2 - 2EK + EN + EZ + \sum_{i<j} E_i E_j \right),$$

where $K := c_1(\Omega^1_{\widetilde{U}})$.

Claims 9.2.4 and 9.2.5 illustrate that (at least in the 2-dimensional case) the divisor N obtained from the Hsiang-Pati coordinates on $\widetilde{U}$ can be used to obtain numerical invariants of the singularity v.

Note that in the trivial case where $U = \mathbb{C}^2$ and $\widetilde{U}$ is the blowup of the complex plane at the origin, we have $E = Z = N \approx \mathbb{P}^1$ (see Section 2.1.1), $E^2 = -1$ (see Section 4.1.2 of [GH78]), and $EK = -E^2 - 2$ (see Section 3.1 of [GS82]; we use the fact that cup product is dual to intersection product). Thus Claim 9.2.5 above states that we have:

$$\begin{aligned}
\chi(E) &= \pi_* \mathrm{Dual}(EK - E^2 - 2E^2 - 2EK + E^2 + E^2 + 0) \\
&= \pi_* \mathrm{Dual}(-EK - E^2) \\
&= \pi_* \mathrm{Dual}(E^2 + 2 - E^2) \\
&= 2,
\end{aligned}$$

which is what we would expect, since $\chi(E) = \chi(\mathbb{P}^1) = 2$.

9.2.3.2 The 3-dimensional case

We now wish to use Equation 9.2.17 to obtain information in the 3-dimensional case. Recall that in Section 9.2.2 we could only obtain the first and last Chern classes of $\mathcal{N}$; thus instead of using (9.2.17) to get a formula for the Euler characteristic of the exceptional divisor, we will use it to obtain information about the middle Chern class of $\mathcal{N}$.

We start by computing the local Euler obstruction (as given by the Gonzalez-Sprinberg formula 2.6.12). We have (using (9.2.10)):

$$\begin{aligned}
c_2(\mathcal{N}^* - \mathcal{O}(Z)) \cap [E] &= \Big((-1)^k c(\mathcal{N})/c(\mathcal{O}(Z))\Big)_2 \cap [E] \\
&= \Big((1 - c_1(\mathcal{N}) + c_2(\mathcal{N}))\,(1 - Z + Z^2)\Big)_2 \cap [E] \\
&= (c_2(\mathcal{N}) + c_1(\mathcal{N})\,Z + Z^2) \cap [E] \\
&= (c_2(\mathcal{N}) + (K + E - Z - N - P)\,Z + Z^2) \cap [E] \\
&= c_2(\mathcal{N}) \cap [E] \\
&\qquad +(KZ + EZ - NZ - PZ) \cap [E].
\end{aligned}$$

Thus we can write the local Euler obstruction of U at v as

$$Eu_v(U) = \pi_*\big(c_2(\mathcal{N}) \cap [E] + (KZ + EZ - NZ - PZ) \cap [E]\big). \tag{9.2.18}$$

On the other hand, using 9.2.3 we have:

$$\begin{aligned} c_3((\Omega^1)^*_{\widetilde{U}}) - c_3(\mathcal{N}^*) &= -c_3(\Omega^1_{\widetilde{U}}) + c_3(\mathcal{N}) \\ &= -C_3 + (E^3 + KE^2 - ZE^2 + C_2E + Z^2E - KZE \\ &\quad - Z^3 + y_2Z + KZ^2 - Ky_2 - C_2Z - y_3 + C_3) \\ &= E^3 + KE^2 - ZE^2 + C_2E + Z^2E - KZE - Z^3 \\ &\quad + y_2Z + KZ^2 - Ky_2 - C_2Z - y_3, \end{aligned}$$

where as in Section 9.2.2 we define $c(\Omega^1_{\widetilde{U}}) =: 1+K+C_2+C_3$ and $y_2 := \sum_{i<j} E_iE_j$, $y_3 := 2\sum_{i\neq j} E_i^2E_j + 5\sum_{i<j<k} E_iE_jE_k$.

Putting this, and the expression for the local Euler obstruction above, into Equation 9.2.17, we obtain:

$$\begin{aligned} \chi(E) &= \pi_*\Big(\mathrm{Dual}\,\big(E^3 + KE^2 - ZE^2 + C_2E + Z^2E - KZE \\ &\qquad - Z^3 + y_2Z + KZ^2 - Ky_2 - C_2Z - y_3)\big) \\ &\quad + c_2(\mathcal{N}) \cap [E] \\ &\quad + (KZ + EZ - NZ - PZ) \cap [E]\Big) \qquad (9.2.19) \end{aligned}$$

Thus we can write the cup product of $c_2(\mathcal{N})$ with E as follows (where we omit the π_* since it is an isomorphism on the zero level of homology here, and note that Dual is an isomorphism on $\widetilde{U}$ since $\widetilde{U}$ is smooth):

$$\begin{aligned} c_2(\mathcal{N})\,E &= \chi(E) - (E^3 + KE^2 - ZE^2 + C_2E + Z^2E - KZE - Z^3 \\ &\qquad + y_2Z + KZ^2 - Ky_2 - C_2Z - y_3) \\ &\qquad - (KZ + EZ - NZ - PZ)\,E \\ &= \chi(E) - E^2K - C_2E - Z^2E + Z^3 - y_2Z \\ &\qquad - KZ^2 + Ky_2 + C_2Z + y_3 + ENZ + EPZ. \end{aligned}$$

Thus we have the following claim.

Claim 9.2.6. *Given a sufficiently fine resolution $(\widetilde{U}, E)$ of a three-dimensional variety with isolated singular point (U, v), let Z, N, and P be the divisors defined from the Hsiang-Pati coordinates on $\widetilde{U}$. The second Chern class of the Nash sheaf over $\widetilde{U}$ is related to the resolution data by the formula:*

$$\begin{aligned} c_2(\mathcal{N})\,E &= \chi(E) - E^2K - C_2E - Z^2E + Z^3 - y_2Z \\ &\qquad - KZ^2 + Ky_2 + C_2Z + y_3 + ENZ + EPZ, \end{aligned}$$

where we define

$$\begin{aligned} c(\Omega^1_{\widetilde{U}}) &=: 1 + K + C_2 + C_3, \\ y_2 &:= \sum_{i<j} E_iE_j, \quad \text{and} \\ y_3 &:= 2\sum_{i\neq j} E_i^2E_j + 5\sum_{i<j<k} E_iE_jE_k. \end{aligned}$$

The Chern number $c_2(\mathcal{N})\,E$ (pushed down to U by π_*) is a numerical invariant of the singularity v. In the next section we will conjecture a general formula for the Chern classes of the Nash sheaf of $\widetilde{U}$. If this formula holds then the formula above in Claim 9.2.6 will in fact describe the Euler characteristic of E (as well as the local Euler obstruction $Eu_v(U)$) in terms of the resolution data.

9.2.4 A conjectured formula

We first consider the three-dimensional case. As above, let V be a 3-dimensional variety with isolated singular point v and neighborhood U of v in V, and let $\pi\colon (\widetilde{U}, E) \to (U, v)$ be a complete resolution. Suppose Z, N, and P are the divisors obtained from the Hsiang-Pati coordinates on $\widetilde{U}$. We now conjecture a formula for the Chern classes of the Nash sheaf over $\widetilde{U}$ in terms of the resolution data Z, N, and P.

Conjecture 9.2.7. *With notation as above, the Chern classes of the Nash sheaf over the resolution $\widetilde{U}$ are given by:*

$$\begin{aligned} c_1(\mathcal{N}) &= c_1(\Omega^1(\log E) - \mathcal{O}(Z + N + P)), \\ c_2(\mathcal{N}) &= c_2(\Omega^1(\log E) - \mathcal{O}(Z + N)), \quad \text{and} \\ c_3(\mathcal{N}) &= c_3(\Omega^1(\log E) - \mathcal{O}(Z)). \end{aligned}$$

In fact, it is not hard to see that the formulas for $c_1(\mathcal{N})$ and $c_3(\mathcal{N})$ in the conjecture above agree with the formulas found in 9.2.10 and Claim 9.2.3 above; we do this in Section 9.2.4.2 below. Thus Conjecture 9.2.7 is really a Claim for the first and third Chern classes of $\mathcal{N}$; only the statement involving the second Chern class is in question. At the end of Section 9.2.4.2 we will assume the conjecture and use it to obtain (conjectured) formulas for the local Euler obstruction of U at v and the Euler characteristic of the exceptional divisor E.

In the 2-dimensional case, Conjecture 9.2.7 is in fact the following claim:

Claim 9.2.8. *With notation as above (but with V a 2-dimensional variety), the Chern classes of the Nash sheaf over the resolution $\widetilde{U}$ are given by:*

$$\begin{aligned} c_1(\mathcal{N}) &= c_1(\Omega^1(\log E) - \mathcal{O}(Z+N)) \quad \text{and} \\ c_2(\mathcal{N}) &= c_2(\Omega^1(\log E) - \mathcal{O}(Z)). \end{aligned}$$

We prove this claim in the following section, using the formulas for the Chern classes of $\mathcal{N}$ obtained in Section 9.2.1 from the exact sequence in Proposition 3.20 of [PS97].

9.2.4.1 The conjecture in the 2-dimensional case

In this section we prove that the formula conjectured above is true in the 2-dimensional case. To simplify notation let us write

$$c(\Omega^1(\log E)) =: 1 + g_1 + g_2;$$

looking back at (9.2.4) we see that this means

$$g_1 = E + K \quad \text{and} \quad g_2 = E^2 + C + EK - y_2.$$

Now we compute

$$\begin{aligned} c_1(\Omega^1(\log E) - \mathcal{O}(Z+N)) &= \left(c(\Omega^1(\log E)) / c(\mathcal{O}(Z+N)) \right)_1 \\ &= \left((1 + g_1 + g_2)\,(1 - (Z+N) + (Z+N)^2) \right)_1 \\ &= g_1 - Z - N \\ &= E + K - Z - N; \end{aligned}$$

comparing this to Claim 9.2.2 we see that this is precisely $c_1(\mathcal{N})$.

It is also easy to see that

$$\begin{aligned} c_2(\Omega^1(\log E) - \mathcal{O}(Z)) &= \left(c(\Omega^1(\log E)) / c(\mathcal{O}(Z)) \right)_2 \\ &= \left((1 + g_1 + g_2)\,(1 - Z + Z^2) \right)_2 \\ &= -g_1 Z + g_2 + Z^2 \\ &= -(E+K)Z + (E^2 + C + EK - y_2) + Z^2 \\ &= -EZ - KZ + E^2 + C + EK - y_2 + Z^2, \end{aligned}$$

which by Claim 9.2.2 is the second Chern class $c_2(\mathcal{N})$ of the Nash sheaf. We have thus shown Claim 9.2.8.

9.2.4.2 The conjecture in the 3-dimensional case

We now show that Conjecture 9.2.7 gives correct formulas for the first and last Chern classes of the Nash sheaf (in the three-dimensional case). To simplify notation, define

$$c(\Omega^1(\log E)) =: 1 + g_1 + g_2 + g_3;$$

in other words (see (9.2.12)), we have

$$g_1 = E + K, \quad g_2 = KE + E^2 - y_2 + C_2,$$

$$\text{and} \quad g_3 = KE^2 + E^3 - Ky_2 + C_2E - y_3 + C_3$$

(in the notation of Section 9.2.2). Thus we can compute:

$$\begin{aligned} c_1(\Omega^1(\log E) - \mathcal{O}(Z+N+P)) &= \Big(c(\Omega^1(\log E)) / c(\mathcal{O}(Z+N+P)) \Big)_1 \\ &= \Big((1 + g_1 + g_2 + g_3)\ (1 - (Z+N+P) \\ &\qquad + (Z+N+P)^2 - (Z+N+P)^3) \Big)_1 \\ &= g_1 - Z - N - P \\ &= E + K - Z - N - P, \end{aligned}$$

which is the first Chern class $c_1(\mathcal{N})$ of the Nash sheaf by Equation 9.2.10.

Similarly we can compute the third Chern class:

$$\begin{aligned} c_3(\Omega^1(\log E) - \mathcal{O}(Z)) &= \Big(c(\Omega^1(\log E)) / c(\mathcal{O}(Z)) \Big)_3 \\ &= \Big((1 + g_1 + g_2 + g_3)\,(1 - Z + Z^2 - Z^3) \Big)_3 \\ &= g_3 - Z^3 - g_2 Z + g_1 Z^2 \\ &= (KE^2 + E^3 - Ky_2 + C_2 E - y_3 + C_3) - Z^3 \\ &\qquad -(KE + E^2 - y_2 + C_2)Z + (E+K)Z^2 \\ &= E^2K + E^3 - Ky_2 + C_2E - y_3 + C_3 - Z^3 \\ &\qquad -EKZ - E^2Z + y_2Z - C_2Z + EZ^2 + KZ^2; \end{aligned}$$

comparing this to Claim 9.2.3 it is clear that this is indeed the third Chern class $c_3(\mathcal{N})$ of the Nash sheaf over $\widetilde{U}$.

We finish this section with a discussion of the results that would follow if the second part of Conjecture 9.2.7 was indeed true; for the remainder of this section we assume that that is the case. We would then have, in the notation above,

$$\begin{aligned} c_2(\mathcal{N}) &= c_2(\Omega^1(\log E) - \mathcal{O}(Z+N)) \qquad (9.2.20) \\ &= \Big(c(\Omega^1(\log E)) / c(\mathcal{O}(Z+N)) \Big)_2 \\ &= \Big((1 + g_1 + g_2 + g_3)\,(1 - (Z+N) + (Z+N)^2 - (Z+N)^3) \Big)_2 \\ &= g_2 + (Z+N)^2 - g_1(Z+N) \\ &= (KE + E^2 - y_2 + C_2) + (Z+N)^2 - (E+K)(Z+N) \\ &= EK + E^2 - y_2 + C_2 + Z^2 + 2NZ + N^2 - EZ - EN - KZ - KN. \end{aligned}$$

Assuming this, we can use Equation 9.2.18 to write a formula for the local Euler obstruction of U at v in terms of the resolution data, namely:

$$\begin{aligned} Eu_v(U) &= \pi_*(c_2(\mathcal{N}) \cap [E] + (KZ + EZ - NZ - PZ) \cap [E]) \\ &= \pi_*\mathrm{Dual}(E^2K + E^3 - Ey_2 + C_2E + EZ^2 + 2ENZ \\ &\qquad + EN^2 - E^2Z - E^2N - EKZ - EKN \\ &\qquad + EKZ + E^2Z - ENZ - EPZ) \\ &= \pi_*\mathrm{Dual}(E^2K + E^3 - Ey_2 + C_2E + EZ^2 + ENZ \\ &\qquad + EN^2 - E^2N - EKN - EPZ) \end{aligned}$$

Moreover, assuming Conjecture 9.2.7 allows us to compute the following formula for the Euler characteristic of the exceptional divisor E (using Equation

9.2.19 but omitting the π_* and Dual maps since they are isomorphisms here):

$$\begin{aligned}
\chi(E) &= E^3 + KE^2 - ZE^2 + C_2E + Z^2E - KZE \\
&\quad - Z^3 + y_2Z + KZ^2 - Ky_2 - C_2Z - y_3 \\
&\quad + c_2(\mathcal{N})\,E + (KZ + EZ - NZ - PZ)\,E \\
&= E^3 + KE^2 - ZE^2 + C_2E + Z^2E - KZE \\
&\quad - Z^3 + y_2Z + KZ^2 - Ky_2 - C_2Z - y_3 \\
&\quad + (EK + E^2 - y_2 + C_2 + Z^2 + 2NZ \\
&\quad\quad + N^2 - EZ - EN - KZ - KN)\,E \\
&\quad + EKZ + E^2Z - ENZ - EPZ \\
&= 2E^3 + 2E^2K - E^2Z + 2C_2E + 2EZ^2 - EKZ \\
&\quad - Z^3 + y_2Z + KZ^2 - Ky_2 - C_2Z - y_3 \\
&\quad - Ey_2 + ENZ + EN^2 - E^2N - EKN - EPZ.
\end{aligned}$$

Thus, assuming Conjecture 9.2.7, we can obtain formulas for the local Euler obstruction and the Euler characteristic in terms of the resolution data. In Section 10.1.5 we will verify Conjecture 9.2.7 in a particular example.

Chapter 10

Examples

In this chapter we examine two examples: first, the trivial example of a cone; second, a nontrivial example (in a patch of a complete resolution). In each example we obtain a complete resolution of the given variety, show that there exist Nash-minimal linear functions $\{j, k, l\}$ that pull up to be Hsiang-Pati coordinates on the resolution $\widetilde{U}$ (which in turn induce monomial generators of the Nash sheaf), and examine the multiplicities obtained from such coordinates. In the cone case we will examine all charts of the resolution and will thus be able to obtain an exact sequence of sheaves on $\widetilde{U}$ and formulas for the first and last Chern classes of the Nash sheaf. We then verify, in this example, the formula conjectured in (9.2.7) for the second such Chern class.

10.1 Cone Example

In this section we consider the cone given locally by

$$f(z_1, z_2, z_3, z_4) = z_1 z_2 + z_3 z_4.$$

in $U \subset \mathbb{C}^4$ (with coordinates $\{z_1, z_2, z_3, z_4\}$). Clearly this has an isolated singular point at the origin (where all of its partial derivatives simultaneously vanish). We begin by obtaining a complete resolution of $(U, 0)$.

10.1.1 Blowing up

The maximal ideal and the Jacobian ideal for this variety coincide:

$$\mathfrak{m}_0 = Jac_f = (z_1, z_2, z_3, z_4);$$

thus the blowup of the maximal ideal is in fact the Nash blowup (see Section 2.2.1.2). This blowup $\pi\colon \widetilde{U} \to U$ is the subvariety of $\mathbb{C}^4 \times \mathbb{P}^3$ given by (as in Section 2.1.1):

$$Bl(U) = \widetilde{U} = \big\{ \big((z_1, z_2, z_3, z_4), [y_1, y_2, y_3, y_4]\big) \mid z_i y_j = z_j y_i,\ z_1 z_2 + z_3 z_4 = 0 \big\}.$$

We will only examine one patch of this blowup (the other patches are similar). In the first patch $\widetilde{U}_1$ of $\widetilde{U}$ we have

$$\widetilde{U}_1 = \big\{ \big((z_1, z_2, z_3, z_4), [1, b, c, d]\big) \mid z_2 = bz_1,\ z_3 = cz_1,\ z_4 = dz_1,\ z_1 z_2 + z_3 z_4 = 0 \big\};$$

in other words, we have

$$\widetilde{U}_1 = \big\{ \big((a, ab, ac, ad), [1, b, c, d]\big) \mid b - cd = 0 \big\}.$$

We can think of this as the subvariety of $\mathbb{C}^4\{a, b, c, d\}$ given by the equation $b - cd = 0$. The exceptional divisor E has one component in this patch, namely $E = \{a = 0\}$. Since the partials of $b - cd = 0$ never simultaneously vanish, this is a smooth variety in $\mathbb{C}^4$ (thus is a resolution of U). In this notation, the map π is clearly

$$\begin{array}{ccc} \widetilde{U} & \xrightarrow{\pi} & U \\ (a, b, c, d) & \longmapsto & (a, ab, ac, ad). \end{array}$$

We wish to have coordinates $\{u, v, w\}$ on $\widetilde{U}_1$ about some point $e \in E$. Let $u := a$, $v := c$, and $w := d$ be coordinates about the point given by $b = c = d = 0$ (so $v = w = 0$) in $E = \{a = 0\}$ (*i.e.* $u = 0$). Then (considering $b = cd$ in this patch) a point (u, v, w) represents the point

$$\big((u, uvw, uv, uw), [1, vw, v, w]\big) \in \mathbb{C}^4 \times \mathbb{P}^3.$$

The coordinate functions z_1, z_2, z_3, and z_4 on U pull up to u, uvw, uv, and uw, respectively.

Similarly, the other patches $\widetilde{U}_2$, $\widetilde{U}_3$, and $\widetilde{U}_4$ of $\widetilde{U}$ are smooth and given by coordinates

$$\begin{aligned} (a_2, b_2, c_2, d_2) &= \big((a_2b_2, b_2, b_2c_2, b_2d_2), [a_2, 1, c_2, d_2]\big), \\ (a_3, b_3, c_3, d_3) &= \big((a_3c_3, b_3c_3, c_3, c_3d_3), [a_3, b_3, 1, d_3]\big), \\ (a_4, b_4, c_4, d_4) &= \big((a_4d_4, b_4d_4, c_4d_4, d_4), [a_4, b_4, c_4, 1]\big), \end{aligned}$$

respectively, with $b_2^2(a_2 - c_2 d_2) = 0$, $c_3^2(a_3 b_3 - d_3) = 0$, and $d_4^2(a_4 b_4 - c_4) = 0$. Redefining these coordinates as in the first patch above we obtain coordinates

$$\begin{aligned}(u_2, v_2, w_2) &= \big((u_2 v_2 w_2, u_2, u_2 v_2, u_2 w_2), [v_2 w_2, 1, v_2, w_2]\big),\\ (u_3, v_3, w_3) &= \big((u_3 w_3, v_3 w_3, w_3, u_3 v_3 w_3), [u_3, v_3, 1, u_3 v_3]\big),\\ (u_4, v_4, w_4) &= \big((u_4 w_4, v_4 w_4, u_4 v_4 w_4, w_4), [u_4, v_4, u_4 v_4, 1]\big),\end{aligned}$$

respectively. These coordinates patch together via

$$\begin{array}{lll} u_2 = uvw & u_3 = v^{-1} & u_4 = w^{-1} \\ v_2 = w^{-1} & v_3 = w & v_4 = v \\ w_2 = v^{-1} & w_3 = uv & w_4 = uw \end{array}$$

and the exceptional divisor E is given by $u_2 = 0$ in the second patch, $w_3 = 0$ in the third, and $w_4 = 0$ in the fourth.

10.1.2 Choosing Nash-minimal functions

Consider again the $\widetilde{U}_1$ patch of $\widetilde{U}$, and let $e \in E = \{u = 0\}$ be the point $u = v = w = 0$ in the exceptional divisor. We now show that the linear functions $\{z_1, z_3, z_4\}$ are Nash-minimal (as in Definition 4.1.2) with respect to e. Clearly $z_1 \circ \pi = u$ is the generator of $\pi^{-1}\mathfrak{m}_0(\widetilde{U}_1)$ and thus the triple $\{z_1, z_3, z_4\}$ satisfies condition (i) of Definition 4.1.2. Define ϕ, ψ, and ρ as

$$\phi := z_1 \circ \pi = u, \quad \psi := z_3 \circ \pi = uv, \quad \text{and} \quad \rho := z_4 \circ \pi = uw.$$

To show that $\{\phi, \psi, \rho\}$ satisfies condition (ii) of Definition 4.1.2 we must show that $\{d\phi, d\psi, d\rho\}$ is a generating set for $\mathcal{N}_{\widetilde{U}}(\widetilde{U}_1)$. By definition the Nash sheaf over $\widetilde{U}_1$ is (see Section 2.2.3)

$$\mathcal{N}_{\widetilde{U}_1} = \pi^*\Omega^1_U \big/ \text{Torsion}(\pi^*\Omega^1_U),$$

and thus is generated by the 1-forms

$$\begin{aligned}
d\phi \;=\; \pi^* dz_1 \;&=\; d(z_1 \circ \pi) \;=\; u\,\frac{du}{u},\\
\pi^* dz_2 \;&=\; d(z_2 \circ \pi) \;=\; uvw\,\frac{du}{u} + uw\,dv + uv\,dw,\\
d\psi \;=\; \pi^* dz_3 \;&=\; d(z_3 \circ \pi) \;=\; uv\,\frac{du}{u} + u\,dv,\\
d\rho \;=\; \pi^* dz_4 \;&=\; d(z_4 \circ \pi) \;=\; uw\,\frac{du}{u} + u\,dw
\end{aligned}$$

(written in terms of the basis $\{du/u, dv, dw\}$ of the sheaf of logarithmic 1-forms $\Omega^1(\log E)(\widetilde{U}_1)$ over $\widetilde{U}_1$). The second 1-form in the list above can be written as a combination of the other three:

$$\pi^* dz_2 = v d\rho + w d\psi - vw d\phi.$$

Thus $\{d\phi, d\psi, d\rho\}$ do indeed generate $\mathcal{N}_{\widetilde{U}}(\widetilde{U}_1)$. It is now easy to see that we also have condition (iii) of Nash-minimality; namely, that $d\phi \wedge d\psi$ is minimal in

$\Lambda^2\mathcal{N}_{\widetilde{U}}(\widetilde{U}_1)$: the second exterior power of the Nash sheaf over $\widetilde{U}_1$ is generated by

$$\begin{aligned} d\phi \wedge d\psi &= u^2\,\frac{dudv}{u} \\ d\psi \wedge d\rho &= u^2 w\,\frac{dudv}{u} + u^2\,dvdw + u^2 v\,\frac{dudw}{u} \\ d\psi \wedge d\rho &= u^2\,\frac{dudw}{u} \end{aligned}$$

(again, written in the logarithmic Nash frame). Note that the Fitting invariant $Fitt_2(\alpha_1)$ corresponding to the inclusion

$$\alpha_2 \colon \Lambda^2\mathcal{N}_{\widetilde{U}}(\widetilde{U}_1) \hookrightarrow \Omega^2_{\widetilde{U}}(\log E)(\widetilde{U}_1)$$

is locally principal: in fact it is

$$Fitt_2(\alpha_1) \;=\; \langle u^2, u^2 w, u^2, u^2 v, u^2\rangle \;=\; \langle u^2\rangle.$$

10.1.3 Multiplicities

The functions

$$\phi = u, \quad \psi = uv, \quad \rho = uw$$

as defined above (in the $\widetilde{U}_1$ patch) clearly satisfy all the conditions of the Main Proposition (in the simple point case, *i.e.* Proposition 5.2.2) and are thus Hsiang-Pati coordinates on $\widetilde{U}_1$. From these coordinates we obtain the multiplicities

$$m = 1, \quad n = 1, \quad p = 1$$

which in turn define divisors $P = N = Z$ on E. Moreover, the 1-forms $d\phi$, $d\psi$, and $d\rho$ are monomial generators for the Nash sheaf (see Definition 5.1.1).

10.1.4 The hyperplane H

To explore some of the applications of these monomial generators, we choose a hyperplane H (as in Chapter 6) that will enable us to construct an exact sequence of sheaves over $\widetilde{U}_1$ (as in Chapter 8). The hyperplane $H \subset \mathbb{C}^4$ defined by $h := z_3 = 0$ is "nice" in the sense of Definition 6.1.1: $H \cap U$ is singular only at the origin and reduced (it is defined by the equation $z_1 z_2 = 0$), and the total transform of H in $\widetilde{U}$ vanishes to minimum order along E (since it is defined by the equation $h \circ \pi = uv$). Note that, as proved in the Divisor Proposition (6.3.1), we have $\mathrm{div}(h \circ \pi) = Z + \widetilde{H}$, and (trivially) $\widetilde{H}$ meets E only in simple points.

Near $\widetilde{H}$, *i.e.* at the point e given by $v = w = 0$ in $E = \{u = 0\}$, we indeed have $m = n = p$ and can take (as above) k to be $h = z_3$. The remaining linear functions can then be chosen (after rechoosing the coordinates $\{u, v, w\}$) of $\widetilde{U}_1$ as perturbations of this $k = h$: define $j := h + \epsilon z_1$ and $l := h + \epsilon z_4$. Pulled up to $\widetilde{U}_1$ these functions have the form

$$j \circ \pi \;=\; h \circ \pi + \epsilon z_1 \circ \pi \;=\; uv + \epsilon u \;=\; u(v + \epsilon)$$

and

$$l \circ \pi \;=\; h \circ \pi + \epsilon z_4 \circ \pi \;=\; uv + \epsilon uw \;=\; u(v + \epsilon w).$$

Since $v + \epsilon$ is a local unit and $v + \epsilon w$ is a coordinate independent of u and v, these linear functions are Nash-minimal and give rise to Hsiang-Pati coordinates ϕ, ψ, and ρ as described above.

On the other hand, away from $\widetilde{H}$, say at the point e given by $v = w = 1$ in $E = \{u = 0\}$, we can take j to be h: clearly v is a local unit in a neighborhood of $v = w = 1$, and thus $h = uv$ is a (local) generator of the sheaf $\pi^{-1}\mathfrak{m}_0$.

10.1.5 Chern Classes and Numbers

We are now in a position to discuss the exact sequence relating the Nash sheaf $\mathcal{N}$ over $\widetilde{U}$ to the resolution data $Z = N = P$ (which are all E in this example; the $\widetilde{U}_2, \widetilde{U}_3, and \widetilde{U}_4$ patches are similar to the $\widetilde{U}_1$ patch described above). The exact sequence from Corollary 8.4.1 is, in this example:

$$\begin{aligned} 0 \;\to\; \mathcal{N} \;&\overset{\alpha}{\hookrightarrow}\; \Omega^1(\log E) \otimes \mathcal{O}(-E) \\ &\overset{\beta}{\longrightarrow}\; \Omega^2(\log E) \otimes \mathcal{O}(-2E) \big/ \Lambda^2\mathcal{N} \;\overset{\gamma}{\twoheadrightarrow}\; \Omega^3 \otimes \mathcal{O}(-2E) \;\to\; 0. \end{aligned}$$

We can compute from this sequence the first and last Chern classes of the Nash sheaf $\mathcal{N}_{\widetilde{U}}$, as we did in Sections 9.2.2 and 9.2.3.2. By Equation 9.2.10 the first such Chern class is, in this example,

$$c_1(\mathcal{N}) \;=\; E + K - Z - N - P \;=\; K - 2E,$$

where $K := c_1(\Omega^1_{\widetilde{U}})$. We can compute the third Chern class of the Nash sheaf using Claim 9.2.3:

$$\begin{aligned} c_3(\mathcal{N}) &= E^3 + KE^2 - ZE^2 + C_2E + Z^2E - KZE - Z^3 \\ &\quad + y_2Z + KZ^2 - Ky_2 - C_2Z - y_3 + C_3 \\ &= E^3 + E^2K - E^3 + C_2E + E^3 - KE^2 - E^3 \\ &\quad + 0 + KE^2 + 0 - C_2E - 0 + C_3 \\ &= E^2K + C_3, \end{aligned}$$

where y_2 and y_3 are expressions involving the products of distinct components of E (see (9.2.11)) and are thus zero in this example (since E consists of only one component), and where the C_i denote the Chern classes $c_i(\widetilde{U})$.

In Section 9.2.3.2 we obtained a formula for $c_2(\mathcal{N})\,E$, the cup product of the second Chern class of $\mathcal{N}_{\widetilde{U}}$ with the cohomology class corresponding to the exceptional divisor, by using the classes above, a formula for the local Euler obstruction, and two formulas for the zeroth MacPherson-Chern class of U. As in Claim 9.2.6, we have:

$$\begin{aligned} c_2(\mathcal{N})\,E &= \chi(E) - E^2K - C_2E - Z^2E + Z^3 - y_2Z \\ &\quad - KZ^2 + Ky_2 + C_2Z + y_3 + ENZ + EPZ \\ &= \chi(E) - E^2K - C_2E - E^3 + E^3 - 0 \\ &\quad - E^2K + 0 + C_2E + 0 + E^3 + E^3 \\ &= \chi(E) - 2E^2K + 2E^3 \qquad (10.1.1) \end{aligned}$$

where K, C_i, and y_i are as above.

If Conjecture 9.2.7 does indeed hold for $c_2(\mathcal{N})$, then we can write the second Chern class of the Nash sheaf in terms of the resolution data Z and N and the Chern classes of $\Omega^1(\log E)$. As in Equation 9.2.20 we would then have:

$$\begin{aligned} c_2(\mathcal{N}) &= EK + E^2 - y_2 + C_2 + Z^2 + 2NZ + N^2 - EZ - EN - KZ - KN \\ &= EK + E^2 - 0 + C_2 + E^2 + 2E^2 + E^2 - E^2 - E^2 - EK - EK \\ &= 3E^2 + C_2 - EK. \end{aligned} \tag{10.1.2}$$

We can use Equation 10.1.1 and the equation above (after cupping with E) to check the validity of Conjecture 9.2.7: if the conjecture holds then we must have

$$\chi(E) - 2E^2K + 2E^3 = (3E^2 + C_2 - EK)\,E.$$

In other words, we would have the following formula for the Euler characteristic of E (which we could also obtain by specializing the corresponding formula for the Euler characteristic given at the end of Section 9.2.4.2, which also relies on the conjectured Equation 9.2.20):

$$\chi(E) = \mathrm{Dual}_{\widetilde{U}}(E^3 + C_2E + E^2K). \tag{10.1.3}$$

(Recall that in Claim 9.2.6 we dropped the dual map to simplify notation; we reinsert it here as it will play a part in the following calculations.) By definition, the Euler characteristic of E is the dual (in $E \subset \widetilde{U}$) of the top Chern class of the tangent bundle T_E to E. We thus have

$$\begin{aligned}\chi(E) &= \mathrm{Dual}_E(c_2(T_E)) \\ &= c_2(T_E) \cap [E] \\ &= c_2(T_E) \cap (E \cap [\widetilde{U}]) \\ &= c_2(T_E)\, E \cap [\widetilde{U}] \\ &= \mathrm{Dual}_{\widetilde{\mathrm{U}}}(c_2(T_E)\, E).\end{aligned}$$

On the other hand, we can write the tangent bundle to $\widetilde{U}$ in terms of the tangent and normal bundles of E:

$$T_{\widetilde{U}} \approx p^* T_E \oplus p^* \nu_E,$$

where $p\colon \widetilde{U} \to E$ is projection to E. Since p is a homotopy equivalence we will omit it in all future calculations, *i.e.* we will write simply $T_{\widetilde{U}} \approx T_E \oplus \nu_E$. Thus we can write the Chern class of $\widetilde{U}$ as follows:

$$\begin{aligned}c(\widetilde{U}) &= c(T_E)\, c(\nu_E) \\ &= (1 + c_1(T_E) + c_2(T_E))\,(1+E) \\ &= 1 + (E + c_1(T_E)) + (c_2(T_E) + c_1(T_E)\, E) + (c_2(T_E)\, E).\end{aligned}$$

Recall that in our notation above we have

$$c(\widetilde{U}) = \sum_k (-1)^k c(\Omega^1_{\widetilde{U}}) = 1 - K + C_2 - C_3;$$

Putting this together with the expression for $c(\widetilde{U}$ above, we have

$$\begin{aligned} K &= -E - c_1(T_E) \text{ and} \\ C_2 &= c_2(T_E) + c_1(T_E)\, E \end{aligned}$$

(we will not be needing C_3). We can now confirm that we have Equation 10.1.3 (in what follows, all duality is in $\widetilde{U}$):

$$\begin{aligned} &\text{Dual}(E^3 + C_2 E + E^2 K) \\ &\quad = \text{Dual}(E^3 + (c_2(T_E) + c_1(T_E)\, E)\, E + E^2 K) \\ &\quad = \text{Dual}(E^3 + (c_2(T_E) + (-E - K)\, E)\, E + E^2 K) \\ &\quad = \text{Dual}(E^3 + c_2(T_E)\, E - E^3 - E^2 K + E^2 K) \\ &\quad = \text{Dual}(c_2(T_E)\, E) \\ &\quad = \chi(E). \end{aligned}$$

Thus the conjectured formula (in Conjecture 9.2.7) for the second Chern class of the Nash sheaf over $\widetilde{U}$ is verified (in this example).

10.2 Non-trivial Example

In this section we consider the example where the variety V is the hypersurface in $\mathbb{C}^4\{w, x, y, z\}$ (locally) defined by the function

$$f = wy - x^3 + z^3.$$

Note that $U = \mathcal{V}(f) \subset \mathbb{C}^4$ has an isolated singular point at the origin. As outlined in Section 3.1, we will first obtain a blowup $\check{U}$ factoring though the Nash blowup and the blowup of the maximal ideal; then we will find a resolution $\check{\check{U}} \to \check{U}$. If necessary we will take a further resolution $\widetilde{U}$ of $\check{U}$ to ensure that the appropriate Fitting ideal is locally principal.

10.2.1 Blowing up

We begin by blowing up the maximal and Jacobian ideals (since U is a hypersurface in $\mathbb{C}^4$, the Nash blowup can be achieved by blowing up the Jacobian ideal; see Section 2.2.1.2). We have:

$$\begin{aligned} \mathfrak{m}_v &= \langle w, x, y, z \rangle, \\ Jac(f) &= \langle y, x^2, w, z^2 \rangle. \end{aligned}$$

The blowup of $\mathfrak{m}_v$ and $Jac(f)$ is given by the closure in $\mathbb{C}^4 \times \mathbb{P}^3 \times \mathbb{P}^3$ of the following (see Section 2 of [PP] as well as Section 2.1.2 here):

$$\left\{ ((w, x, y, z), [w, x, y, z], [y, x^2, w, z^2]) \mid (w, x, y, z) \in \mathcal{V}(f) - 0 \right\}.$$

For this blowup alone there will be sixteen patches; here we will work only in the $(2, 1)$ patch:

$$\left((w, x, y, z), \left[\frac{w}{x}, 1, \frac{y}{x}, \frac{z}{x} \right], \left[1, \frac{x^2}{y}, \frac{w}{y}, \frac{z^2}{y} \right] \right).$$

We can use the defining equation $f = wy - x^3 + z^3$ to solve for w:

$$w = \frac{x^3 - z^3}{y}.$$

Putting this into the above we obtain the following rational parameterization for the blowup $\check{U}$ of the maximal and Jacobian ideals:

$$\left(\left(\frac{x^3 - z^3}{y}, x, y, z\right), \left[\frac{x^3 - z^3}{xy}, 1, \frac{y}{x}, \frac{z}{x}\right], \left[1, \frac{x^2}{y}, \frac{x^3 - z^3}{y^2}, \frac{z^2}{y}\right]\right).$$

We will employ the following algorithm, the *Implicitization Algorithm for Rational Parameterizations*, taken directly from [CLO92]:

Algorithm 10.2.1. *Given a rational parameterization* $x_i = \frac{f_i}{g_i}$, *where* $f_1, \ldots, f_n$ *and* $g_1, \ldots, g_n$ *are polynomials in* $\mathbb{C}[t_1, \ldots, t_m]$, *consider the ideal*

$$J := \langle g_1 x_1 - f_1, \ldots, g_n x_n - f_n, 1 - gy \rangle,$$

where y *is a new variable, and* $g := g_1 \cdots g_n$. *Compute a Groebner basis with respect to a lexicographic ordering where* y *and every* t_i *are greater than every* x_i. *Then the elements of the Groebner basis not involving* $y, t_1, \ldots, t_m$ *define the smallest variety in* $\mathbb{C}^n$ *containing the parameterization.*

In our example we will take the t_i to be $((a, b, c, d), [p, 1, q, r], [1, s, t, u])$; then the ideal J is

$$\begin{aligned} J &= \langle (x^2 - z^3) - ay,\, b - z,\, c - y,\, d - z, \\ &\quad (x^3 - z^3) - pxy,\, y - qx,\, z - rx, \\ &\quad x^2 - sy,\, (x^3 - z^3) - ty^2,\, z^2 - uy, \\ &\quad 1 - wy^2 e\rangle. \end{aligned}$$

Taking a Groebner basis with respect to the lexicographic ordering

$$[w, x, y, z, e, a, b, c, d, p, q, r, s, t, u],$$

and eliminating all equations in which the letters w, x, y, z, or e appear, we get the ideal:

$$\begin{aligned} \check{J} &= \langle utbd - as^2 + tba, cp - qa, r^2a - qup, bp - a, qu^2 + ra - sd, \\ &\quad rta + u^2p - sdt, cs^2 - bud - ba, sp - bt, b^2 - cs, br - d, qt - p, \\ &\quad bq - c, p - s + ur, d^2 - cu, -qu + dr, -tbd - u^2b + s^2d, qd - rc, \\ &\quad rcs - bd, -a + tc, r^2c - q^2u, bs - ud - a, -tbd + sra, \\ &\quad -dt - u^2 + rs^2, tap - tas - u^3p + usdt, s^3 - 2udt - u^3 - ta, \\ &\quad -b + qs, -u + r^2s, dp - ra, u^3p^2 - tap^2 - tas^2 + 2t^2ba\rangle. \end{aligned}$$

The ideal $\check{J}$ defines a variety in $\mathbb{C}^{10}\{a, b, c, d, p, q, r, s, t, u\}$. We can eliminate some of these variables to obtain an isomorphic variety in a smaller affine space. Specifically, we eliminate (in this order):

$$\begin{aligned} b &= qs, \\ a &= qsp, \\ c &= q^2 s, \\ d &= qsr, \\ p &= qt, \\ u &= r^2 s. \end{aligned}$$

This process leaves us with the ideal

$$\check{J} = \langle qt - s + r^3 s \rangle,$$

which defines a hypersurface $\check{U}$ in $\mathbb{C}^4\{q, r, s, t\}$; define $F := qt - s + r^3 s$. This hypersurface is singular: the Jacobian for F is

$$Jac(F) = \langle t, r^2 s, r^3 - 1, q \rangle;$$

clearly all the partials of F vanish at points where $t = s = q = 0$, $r^3 = 1$.

To attempt to desingularize $\check{U}$ we will blow up this Jacobian ideal; this gives us the variety $\breve{U}$ given by the closure of

$$\left\{ ((q, r, s, t), [t, r^2 s, r^3 - 1, q]) \mid (q, r, s, t) \in (\mathcal{V}(F))_{\text{smooth}} \right\}.$$

There will be four patches, but we will investigate only the third, which is given by:

$$\left((q, r, s, t)\,, \left[\frac{t}{r^3 - 1}, \frac{r^2 s}{r^3 - 1}, 1, \frac{q}{r^3 - 1} \right] \right).$$

We then use the defining equation $F := qt - s + r^3 s$ to solve for t:

$$t = \frac{-s(r^3 - 1)}{q}.$$

This gives us the following rational parameterization:

$$\left(\left(q, r, s, \frac{-s(r^3-1)}{q}\right), \left[\frac{-s}{q}, \frac{r^2 s}{r^3-1}, 1, \frac{q}{r^3-1}\right]\right).$$

To find the smallest variety in $\mathbb{C}^4\{q, r, s, t\}$ containing this parameterization, we will again employ the *Implicitization Algorithm for Rational Parameterizations* (see Algorithm 10.2.1). In this case the t_i will be $((A, B, C, D), [R, S, 1, T])$, and the ideal J is:

$$\begin{aligned} J \quad = \quad & \langle A - q,\, B - r,\, C - s,\, (s - sr^3) - Dq, \\ & s - qR,\, s - (r^3-1)S,\, q - (r^3-1)T, 1 - eqs\rangle. \end{aligned}$$

Now we take the Groebner basis with respect to the lexicographic ordering

$$[q, r, s, t, e, A, B, C, D, R, S, T],$$

and eliminate all the equations in which q, r, s, t, or e appear,to get the ideal:

$$\begin{aligned} \breve{J} \quad = \quad & \langle B^3T - A - T, AR - C, B^3R + D - R, C + DT, TR - S, \\ & SA - TC, B^3S - C - S, DS + RC, CB^3 + AD - C\rangle. \end{aligned}$$

This defines a variety $\breve{U}$ in $\mathbb{C}^7\{A, B, C, D, R, S, T\}$; by eliminating variables we can obtain an isomorphic variety in a smaller affine space. We eliminate (in order):

$$\begin{aligned} C &= -DT, \\ S &= TR, \\ D &= R - B^3R, \\ A &= B^3T - T. \end{aligned}$$

After eliminating these variables, $\breve{U}$ is simply B,R,T space $\mathbb{C}^3\{B,R,T\}$ (which is clearly smooth). As we will see at the end of the next section, the relevant Fitting ideal is already locally principal on $\breve{U}$, so $\breve{U} = \widetilde{U}$ (in the notation of Section 3.1).

10.2.2 Hsiang-Pati Coordinates

Let $\pi\colon (\widetilde{U}, E) \to (U, v)$ be the resolution for $U = \mathcal{V}(f)$ constructed above. We now wish to find Hsiang-Pati coordinates on $\widetilde{U}$. Let us start by examining the map π. To be consistent with previous notation (in previous sections we had local coordinates $\{u, v, w\}$), we define:

$$u := T, \quad v := R, \quad w := B.$$

We also rename our U-coordinates (else we will have two definitions for w):

$$(z_1, z_2, z_3, z_4) := (w, x, y, z).$$

Tracing back through the notation and blowups described above, we see that

$$\begin{aligned} \pi(u,v,w) &= (-u^3v^2(w^3-1)^2,\, u^2v(w^3-1), \\ &\qquad u^3v(w^3-1)(w-1),\, u^2vw(w^3-1)). \end{aligned}$$

In other words,

$$\begin{aligned} z_1 \circ \pi &= -u^3v^2(w^3-1)^2, \\ z_2 \circ \pi &= u^2v(w^3-1), \\ z_3 \circ \pi &= u^3v(w^3-1)(w-1), \\ z_4 \circ \pi &= u^2vw(w^3-1)). \end{aligned}$$

Clearly the maximal ideal $\mathfrak{m}_v = \langle z_1, z_2, z_3, z_4 \rangle$ pulls up to the ideal generated by $z_2 \circ \pi = u^2 v(w^3 - 1)$. Thus the exceptional divisor is $E = E_i \cap E_j = \{u = 0\} \cup \{v = 0\}$; we shall take as our point $e \in E$ the double point $\{u = v = w = 0\}$. Moreover, we now know that ϕ must be:

$$\phi = u^2 v(w^3 - 1).$$

Change coordinates to absorb the unit $(w^3 - 1)$ into the coordinate v (*i.e.* map v to $v(w^3 - 1)^{-1}$). Under these new coordinates we have:

$$\begin{aligned} z_1 \circ \pi &= -u^3 v^2, \\ \phi = z_2 \circ \pi &= u^2 v, \\ z_3 \circ \pi &= u^3 v(w - 1), \\ z_4 \circ \pi &= u^2 vw. \end{aligned}$$

We will now show that z_2, z_4, z_3 can be taken to be j, k, l, respectively (so will pull up to ϕ, ψ, and ρ). In other words we will show that

$$\begin{aligned} \phi &= u^2 v \\ \psi &= u^2 vw \\ \rho &= u^3 v(w - 1) \end{aligned}$$

are Hsiang-Pati coordinates in a neighborhood of e. Clearly this is a double point "case II" situation (see Proposition 5.5.2), and ϕ and ψ are already in the appropriate form. It remains only to absorb the unit $(w - 1)$; do this via the coordinate change:

$$\begin{aligned} u &\mapsto u\,(w-1)^{-1} \\ v &\mapsto v\,(w-1)^{2} \\ w &\mapsto w; \end{aligned}$$

In these new coordinates we have

$$\begin{aligned} \phi &= u^2 v, \\ \psi &= u^2 v w, \\ \rho &= u^3 v. \end{aligned}$$

Note that the fourth coordinate, z_1, now pulls up to the function $z_1 \circ \pi = -u^3 v^2 (w-1)$. The functions ϕ, ψ, and ρ are now clearly Hsiang-Pati coordinates, and define the multiplicities:

$$\begin{aligned} m_i &= 2 \qquad & m_j &= 1 \\ n_i &= 2 \qquad & n_j &= 1 \\ p_i &= 3 \qquad & p_j &= 1. \end{aligned}$$

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Biography

Laura Taalman was born in Norwich, Connecticut on February 14, 1973. She received a Bachelor of Science in Mathematics from the University of Chicago in 1994 and a Master of Arts in Mathematics from Duke University in 1997. She was awarded the L.P. and Barbara Smith Award for Teaching in both 1998 and 1999, and the Dean's Award for Excellence in Teaching in 1997 and 1999. Starting in the fall of 2000, Laura will be an Assistant Professor of Mathematics at James Madison University. When not mathematically occupied, Laura reads too many science fiction novels and spends time in the garden with her husband and their three cats.