Chapter 11 Outline

Math 236 Spring 2001

This outline now covers *all* of Chapter 11 (including Taylor series and power series).

11.1 Infinite Series

- **Sigma notation:** Be able to go from a written-out sum to sigma notation and back. Know how to change indicies and algebraically manipulate a sum in sigma notation.
- **Infinite series as sequences of partial sums:** What does it mean to say that an infinite series converges? See Definition 11.1.1 which describes a series as the limit of its sequence of partial sums. Do not confuse the series with the corresponding sequence!
- **Telescoping series:** Sometimes the terms of a series can be rewritten using partial fractions so that you can find a formula for the n^{th} partial sum for the series. Then you can take the limit of this sequence of partial sums to get the sum of the series (if it exists). Be able to do this and be able to explain what you are doing (including notation) while you do it. Not all series are "telescoping" like this.
- **The geometric series:** What is the general form of a geometric series? When does it converge, and to what? Notice that the formula for the sum of a convergent geometric series only works if you start from $k = 0$. Be able to prove (in a particular case like $x = \frac{1}{4}$ or in general) the formula for the sum of a convergent geometric series. The proof involves finding a formula for the n^{th} partial sum of the series (start by computing $s_n - xs_n$) and then taking the limit of that formula to get the sum of the series.
- **The Divergence Test:** If the terms of a series do not go to zero then the series must diverge. Equivalently (this is the contrapositive statement), if a series converges then its terms must go to zero. What can you say about a series whose terms go to zero? Does it have to converge? What can you say about a divergent series? Can its terms go to zero? Be careful not to confuse the sequence of the terms of a series with the series itself!
- **Properties of infinite series:** If a series starting from $k = 0$ converges, does the same series starting from $k = 5$ converge? What about the other way around? Can you say a similar thing about divergence? Justify your answers.

11.2 The Integral Test; Comparison Theorems

- **The Integral Test:** Be able to give a precise mathematical statement of the Integral Test (including hypotheses on the function defined from the terms of the series). Be able to draw a picture "proving" that the integral test works. Use the integral test to see if various series converge or diverge. If the integral associated to a series converges to 4, does the series converge to 4? Always make sure that the function you are integrating is continuous. Also remember that you are using the techniques of improper integrals here (since the integrals "go out to infinity").

- **The p-series:** What is the general form of a p-series? When does a p-series converge? Prove this using the integral test. Be careful not to confuse the p-series with the geometric series; what is the difference? Notice that we only know whether these p-series converge or diverge; we don't know what they converge *to.* (If you need to know the actual sum of a series, usually it's geometric or telescoping.)
- **The Basic Comparison Test:** Be able to state the BCT, including hypotheses (what has to be true about the terms of the series to apply the BCT?). Remember that if you show that your series is smaller than something divergent or larger than something convergent, you have shown *nothing.* Be sure to justify your inequalities when you use them (remember they only have to be true for "sufficiently large k;" why?). Usually you want to compare your series to something that you know the convergence or divergence of (like a geometric or p-series) or something that you are able to integrate (so you can use the Integral Test).
- **The Limit Comparison Test:** Sometimes when the "obvious" inequality doesn't seem to go the right way you can use the LCT to compare your series with a simpler series. Be able to state the LCT, including hypotheses. What is the LCT telling you when you find that the limit as $k \to \infty$ of the ratio $\frac{a_k}{b_k}$ is a nonzero real number? Try to figure out what the terms of your series are "like" when k is very large; this will help you determine what series you should use to compare with yours. Notice that techniques of limits (including indeterminate forms) will be used here when you try to find the limit in this test. If your series is comparable to a series that converges to 2, does your series necessarily converge to 2?

11.3 The Root Test; The Ratio Test

- **The Root Test:** Be able to state *and prove* the Root Test (including hypotheses). When is it advisable to apply the Root Test? What special limit tends to appear (at least sometimes) when you apply the Root Test? What do you know if you apply the Root Test and the limit you get is equal to 1?
- **The Ratio Test:** Be able to state the Ratio Test, including hypotheses, in mathematical language. You do not need to know how to prove the Ratio Test. When does this test apply? Does it always work? What does it mean if the limit you get is 1?
- **Tips on using convergence tests:** Remember the Root and Ratio Tests are *tests* that involve a limit as $k \to \infty$ and do not tell you what your series is equal to. Be sure it is clear what test you are applying and what the test means about your series, and always use the correct notation (don't drop your limits, don't say things are equal that are not, etc.). This goes for the LCT as well.

11.4 Absolute and Conditional Convergence; Alternating Series

- **Absolute and conditional convergence:** Define what it means for a series to be absolutely convergent. What does this mean about the convergence of the series? What is the definition of "conditionally convergent"? Give an example of a series that is conditionally convergent.

- **Alternating series:** What is an alternating series? Be able to use $(-1)^k$ and $(-1)^{k+1}$ to express an alternating sum as a series. What are two ways of writing the alternating harmonic series? Are there other ways? Why can't we use the Integral Test, BCT, LCT, Root Test, or Ratio Test on an alternating series?

- **The Alternating Series Test:** If a series is alternating then the Divergence Test works "both ways" and is called the AST. Be able to state and correctly use this test to see if an alternating series converges. Note that after you discover by AST that a series converges, you then have to determine if the series converges absolutely or conditionally.

- **Error in approximating an alternating series:** Know the formula (11.4.5) bounding the error in approximating an alternating series by calculating the n^{th} partial sum. Be able to state and describe the meaning of this formula as well as use it to bound error. Can you find the smallest n so that the partial sum s_n is within 0.01 of the actual sum of a convergent series?

- **Rearrangements:** Remember that you cannot rearrange the terms of an alternating series without potentially changing the sum! Strange but true.

11.5 Taylor Polynomials and Taylor Series in x

- **The basics:** Know the formula for the n^{th} degree Taylor polynomial $P_n(x)$ of a function $f(x)$ and the Lagrange formula for its remainder $R_{n+1}(x)$. Be able to explain how a function is approximated by its Taylor polynomials. When does a function equal its Taylor series? (Hint: think about the remainder.)

- **Computations:** Be able to calculate Taylor polynomials (written out or in sigma notation) and Taylor series from the formulae in Taylor's Theorem. Note that you don't have to find the "general term" to write out a low-degree Taylor polynomial, but you do need to find the general term to write the Taylor series or the general formula for the Taylor polynomial. Be able to bound the error in using a Taylor polynomial to approximate a function (this involves the Lagrange formula for the remainder).

- **Important Taylor series:** Know by heart the Taylor series for e^x, $\sin x$, $\cos x$, and $\ln(1+x)$, and know for which values of x these series are valid. Don't forget that the geometric series is the Taylor series for the function $\frac{1}{1-x}$. You should also be able to derive the formulas for these series.

11.6 Taylor Polynomials and Taylor Series in $x - a$

- **The basics and computations:** Same as in 11.5 but for Taylor polynomials and series in powers of $(x-a)$. Be able to calculate Taylor series (and the intervals on which they are valid) in $(x-a)$ by using what you already know abount Taylor series in x. Also be able to calculate "from scratch", but make sure that you can also calculate Taylor series from other Taylor series. Know what it means to write a function "in powers of $(x-a)$."

- **Important Taylor series:** Know by heart the Taylor series for $\ln x$ centered at $x = a$ (and the values of x for which this series is valid). You should also be able to derive the formula for this series from scratch or from the series for $\ln(1+x)$.

11.7 Power Series

- **Basic definitions:** What is a power series? What does it mean for a power series to converge at a point c? On a set S?
- **Convergence of power series:** Know and understand Theorem 11.7.2 and what it means about the convergence of power series. This includes the fact that every power function has a radius of convergence (what does that mean?). What can you say about the absolute/conditional convergence of a power series?
- **Finding the radius and interval of convergence:** Given a power series, find its radius of convergence r. Then find its interval of convergence by checking the convergence of the series at $x = r$ and $x = -r$ (if centered around 0) or at $x = a - r$ and $x = a + r$ (if centered around a). Can you give examples of power series that converge on $(-\infty, \infty)$ or just at $\{0\}$ or $\{a\}$?

11.8 Differentiation and Integration of Power Series

- **Theorems about differentiating and integrating power series:** Know and understand the theorems in this section and what they mean about the differentiation and integration of power series. In particular, can you differentiate or integrate a series "term by term"? How is the radius of convergence of the resulting series related to the radius of convergence of the original series? Be able to explain what Abel's Theorem means (see the box on pg. 697).
- **Power series and Taylor series:** On its interval of convergence a power series is the Taylor series of its sum. What does that mean?
- **Calculations:** Look in particular at the calculation we did in class for the Taylor series of $\tan^{-1}(x)$. Also be able to find the Taylor series of a function by differentiating, integrating, or manipulating a known Taylor series. Note that now we can find the *exact* sum of a large number of infinite series (those that are the Taylor series of known functions). Don't worry about the error approximations for definite integrals.