Chapter 7 Outline

Math 236 Spring 2001

Note 1: Be sure to read the Disclaimer on Chapter Outlines! I cannot be responsible for misfortunes that may happen to you if you do not.

Note 2: Section 7.9 will not be covered on Test I but is included here for future reference.

7.1 One-to-One Functions; Inverses

- **Definition and properties of one-to-one functions:** Algebraic definition of one-to-one and its contrapositive. Graphical interpretation of one-to-one. Functions that are always increasing or always decreasing are one-to-one (why?). Be able to use all three of these methods to determine if a function is one-to-one.
- **Definition, properties, and graphs of inverse functions:** Algebraic definition of inverse function. Graphical relationship between a function and its inverse. A function has an inverse if and only if it is one-to-one. Properties of one-to-one functions (for example, (a, b) is on the graph of $f(x)$ if and only if (b, a) is on the graph of $f'(x)$). Domains and ranges of inverse functions. Be able to calculate the inverse of a one-to-one function algebraically, and graph the inverse of a function given the graph of the original function.
- **Derivatives of inverse functions:** Know and be able to prove and explain the formula for the derivative of an inverse function. Be able to use this formula to calculate the derivative of the inverse of a function at a given point (even if you are unable to find an equation for the inverse function).

7.2 The Logarithm Function, Part I

- **Definition and properties of general logarithm functions:** Definition of a logarithm function in general. Know and be able to prove the properties that follow from the definition of a logarithm function. Understand the steps of the proof that the derivative of a logarithm function is always a constant times $\frac{1}{x}$.
- **Defintion, properties, and graph of $\ln x$:** Be able to explain how we chose a "natural" logarithm function by setting $f'(1) = 1$. Know the definition of the natural logarithm function (as a definite integral), and know how we got that definite integral. Be able to approximate values of $\ln x$ using this definition (and a Riemman sum). Algebraic and graphical properties of $\ln x$, domain and range of $\ln x$, and arguments why these things are true. Definition of the number e and its relationship to $\ln x$. Solve equations involving $\ln x$.
- **The derivative of $\ln x$:** Use the definition of $\ln x$ and the "Very Important Theorem" to find the derivative of $\ln x$. Be able to differentiate complicated functions that involve $\ln x$.

7.3 The Logarithm Function, Part II

- **The derivative of** $\ln|x|$**:** Split $\ln|x|$ into a piecewise function to see that its derivative is $\frac{1}{x}$. Know the graph of $\ln|x|$ and why we were interested in finding its derivative (rather than just the derivative of $\ln x$). Note that this derivative formula does *not* tell us what the *integral* of $\ln x$ or $\ln|x|$ should be.

- **The integral of** $\frac{1}{x}$**:** Knowing the above tells us how to antidifferentiate $\frac{1}{x}$. Understand the difference between the fact that the antiderivative of $\frac{1}{x}$ is $\ln|x| + C$ and the fact that the definition of $\ln x$ involves a definite integral of $\frac{1}{x}$. Be able to do integrals that involve $\frac{1}{x}$ and $\ln|x|$, including u-substitution problems. As with all sections having to do integration, know how to do associated volume and position/velocity/acceleration problems.

- **Integrating the trigonometric functions:** The information above can be used to integrate the four trig functions whose integrals we did not already know. You do not need to memorize the integrals of $\tan x$, $\cot x$, $\sec x$, and $\csc x$, but you must know how to calculate them using u-substitution and the integral of $\frac{1}{x}$.

- **Graphing review:** Be able to sketch the graph of a function (involving $\ln x$ in particular) by examining its first and second derivatives and its behavior at the "ends" of its domain. This includes identifying intervals of increasing/decreasing and concave up/down as well as finding local and global extrema, inflection points, points of non-differentiability, and asymptotes.

- **Logarithmic differentiation:** Know the formula for differentiating a long product, and be able to prove this formula using logarithmic differentiation. Be able to apply this formula to differentiate long products (or product/quotient combinations).

7.4 The Exponential Function

- **The definition of** e^x**, even for irrational** x**:** Given any x, know the definition of e^x as the unique number whose natural logarithm is x. Why did we need this definition (for irrational numbers x in particular)? Why does this definition imply that e^x is the inverse of $\ln x$?

- **Properties and graph of** e^x**:** Be able to prove the functional and algebraic properties of e^x using the definition of e^x and the fact that e^x and $\ln x$ are inverses.

- **The derivative of** e^x**:** Prove that the derivative of e^x is e^x directly (using implicit differentiation) and by using the formula for the derivative of an inverse function. Be able to differenitate complicated functions that involve e^x, and use this information to sketch graphs of such functions. Know how to solve certain limits by recognizing them as derivatives of exponential functions.

- **The integral of** e^x**:** The above information tells us the integral of e^x. Know this and be able to use it in various integration problems, including applications.

7.5 Arbitrary Bases; Other Powers

- **Definition and properties of x^r, even for irrational r:** Know the definition of x^r involving the exponential and logarithmic functions. Why did we need this definition (in particular for irrational x)? Prove properties of x^r using this definition and known properties of logarithmic and exponential functions.

- **The derivative and the integral of x^r:** Prove the power rule by using the definition of x^r above. Be able to use this rule. Know and be able to justify the integral formula for x^r, and be able to integrate functions involving x^r.

- **Definition, properties, and graphs of general exponential functions b^x:** Know definition of b^x (in terms of e^x and logarithms, as we did for x^r). Know how graphs of b^x compare to each other and in particular to the graph of e^x. Why do we assume $b > 0$ and $b \neq 1$? Algebraic properties of b^x are proved the same way we proved those for x^r. Be able to convert from b^x to e^{kx} and vice-versa.

- **The derivative and the integral of b^x:** Know and be able to prove the "exponential rule" derivative formula for b^x (using the definition of b^x). Do not confuse this with the formula for when the exponent is in the base. Know and justify the formula for the integral of b^x. Be able to use the derivative and integral of b^x in various calculations.

- **Derivatives of functions with variables in the base *and* exponent:** Use logarithmic differentiation (take ln of both sides and apply implicit differentiation) to find the derivatives of functions with a variable in the exponent *and* the base. Do not try to apply the power rule or exponential rule to these functions.

- **Definition, properties, and graphs of general logarithmic functions $\log_b x$:** Definition of $\log_b x$ in terms of $\ln x$ and $\ln b$. Know and prove properties of $\log_b x$ using the definition and properties of $\ln x$. Be able to calculate certain values of $\log_b x$ exactly by hand using these properties.

- **The derivative of $\log_b x$:** Know and be able to prove the formula for the derivative of $\log_b x$. Be able to use this both in differentiation problems and integration problems with u-substitution. Note that we do *not* know the *integral* of $\log_b x$.

7.6 Exponential Growth and Decay

- **"The rate of change is proportional to the quantity":** Know and be able to prove that $f'(x) = kf(x)$ if and only if $f(x)$ is an exponential function $f(x) = Ce^{kx}$. One direction is easy (just differentiate any exponential functions). The other direction involves solving the equation $f'(x) - kf(x) = 0$. Be able to do this given the hint that you will need to multiply both sides of the equation by e^{-kx}.

- **Doubling time and half-life:** A function is exponential if and only if it has a *constant* doubling time or half life. Be able to explain what I mean by "constant" here. Be able to find doubling time or half-life given a particular exponential functions. Show that doubling time and half-life depend only on the continuous growth constant k.

- **Yearly percentage growth and the continuous growth constant:** Know the difference between the yearly percentage growth rate and the continuous growth rate. The first involves the b in Cb^x, and the second is equal to the k in Ce^{kx}. Know in a word problem which of these is being discussed.

- **Solving word problems involving exponential growth and decay:** Be able to identify problems where the rate of growth is proportional to the quantity. In these problems the quantity is always an exponential function. Be able to find this function (by finding k and C) and find past or future values, doubling time or half-life of the quantity. This includes in particular population growth, radioactive decay, and (continuously) compounded interest.

7.7 The Inverse Trigonometric Functions

- **Domains and ranges of the six inverse trigonometric functions:** Know how (and why) we restrict the domains of the six trigonometric functions so we can obtain inverses. Define the six inverse trigonometric functions as the inverses of these restricted domain functions. Know domains and ranges of all six inverse trig functions. Be sure you understand the difference between the notation $\sin^{-1} x$ and $\sin^2 x$.

- **Properties and graphs of the six inverse trigonometric functions:** Properties of the inverse trig functions follow from their definition as the inverses of the (restricted) trig functions. Be able to use these and know when they apply and when they do not. Be able to sketch the graphs of these inverse trig functions by flipping the graphs of the restricted trig functions over the $y = x$ line.

- **Calculating exact values of trig and inverse trig functions:** Use the unit circle to calculate exact values of the trig functions and the inverse trig functions for values that involve angles or side lengths (respectively) of the $30° - 60° - 90°$ or $45° - 45° - 90°$. Obviously you need to memorize the side lengths of these triangles to do this. Keep in mind the domains and ranges when calculating values of inverse trig functions. Know how to convert degrees into radians and vice-versa.

- **Derivatives of the six inverse trigonometric functions:** Use implicit differentiation to find the derivatives of the inverse trig functions. Use triangles to rewrite these derivatives in an algebraic form. Memorize the algebraic forms of these derivatives and be able to use them.

- **Integrals involving inverse trig functions and their derivatives:** Be able to do integrals involving u-substitutions with inverse trig functions and integrals that you can recognize as the derivatives of inverse trig functions. Be able to convert an integrand like, for example, $\frac{1}{\sqrt{5-4x^2}}$ by algebra and u-substitution into an integrand that is the derivative of an inverse trig function.

7.8 The Hyperbolic Sine and Cosine

- **Definitions of hyperbolic sine and cosine:** Know the definitions and be able to pronounce the names of $\sinh x$ and $\cosh x$.

- **Derivatives of hyperbolic sine and cosine:** Know and be able to prove the derivative formulae for $\sinh x$ and $\cosh x$. How does the derivative relationship between $\sinh x$ and $\cosh x$ motivate their names? (In other words, why give these functions these trig-sounding names when they involve e^x?) Differentiate functions involving hyperbolic sine and cosine using either their definitions or their derivative formulae.

- **Properties and graphs of hyperbolic sine and cosine:** Graph $\sinh x$ and $\cosh x$ using their derivatives and properties. How do their graphs compare to the graph of $\frac{1}{2}e^x$ and why?

- **Applications, Identities, and Hyperbolae:** Be able to do word problems and prove identities involving $\sinh x$ and $\cosh x$. Know how $\sinh x$ and $\cosh x$ are related to a hyperbola (just as $\sin x$ and $\cos x$ are related to a circle).

- **Integrals involving hyperbolic sine and cosine:** Use the definitions or properties or derivatives of $\sinh x$ and $\cosh x$ to solve integrals involving $\sinh x$ and $\cosh x$. Sometimes it is best to use the definition to convert the hyperbolic sine and/or cosine into an expression into e^x's, and sometimes it is best to use the derivatives of $\sinh x$ and $\cosh x$.

7.9 Other Hyperbolic Functions

- **Definitions and properties of the remaining four hyperbolic trig functions:** Define the remaining four hyperbolic trig functions in terms of $\sinh x$ and $\cosh x$. Be able to write these four functions in terms of e^x's using these definitions. Prove identities involving these functions using these definitions.

- **Derivatives of the remaining four hyperbolic trig functions:** Find the derivatives of these four functions using their definitions and the derivatives of $\sinh x$ and $\cosh x$, or by using their definitions in terms of e^x's. Sketch graphs of these functions using this derivative information.

- **Integrals involving hyperbolic trig functions:** As with $\sinh x$ and $\cosh x$, be able to solve integrals involving the other four hyperbolic trig functions.

- **Inverse hyperbolic trig functions:** Find formulas for the inverses of the six hyperbolic trig functions by using their definitions in terms of e^x's.

- **Derivatives of inverse hyperbolic trig functions:** Use these expressions for the inverse hyperbolic trig functions to calculate their derivatives. See in particular exercises 19, 20, and 21 in 7.9. Memorize and be able to use these derivatives.

- **Integrals involving inverse hyperbolic trig functions and their derivatives:** Use the derivatives of the inverse hyperbolic trig functions to solve integrals, either by u-substitution or by recognizing an integrand as the derivative of one of the inverse hyperbolic trig functions.