1. Use polynomial long division to write \( \frac{x^2}{(x+3)(x-2)} \) as the sum of a polynomial and a proper rational function.

\[
\begin{array}{c|cc|cc}
& x^2 & + & 0 & + & 0 \\
\hline
x^2 & x & - & 6 & \Theta & x^2 & \Theta & 0 & \Theta & 6 \\
- & x & + & 6 & & & 1 & + & \frac{-x+6}{x^2+x-6} \\
\hline
& 1 & & & & & 4 \text{ pts}
\end{array}
\]

(To check if you factor, check \( x^2 = (x^2+x-6)+(x+6) \).)

2. Write \( \sin^{-1} \left( \frac{x}{2} \right) \) as an algebraic function. Show your work with a triangle.

\[
\sin^{-1} \left( \frac{x}{2} \right) = \theta \\
\frac{x}{2} = \sin \theta \\
\frac{2}{\sqrt{4-x^2}}
\]

So \( \tan \left( \sin^{-1} \left( \frac{x}{2} \right) \right) = \tan \theta = \frac{x}{\sqrt{4-x^2}} \)

3. Suppose you want to use the formula \( M(b-a)^3 \frac{1}{24n^2} \) to find the smallest value of \( n \) for which we can guarantee that an \( n \)-rectangle Midpoint Sum for \( \int_0^3 x^3 \, dx \) will be within 0.1 of the exact answer. What is the numerical value of \( M \) in this example, and why?

\[
M = \text{max val. of } |f''(x)| \text{ on } [0,3] \\
f(x) = x^3 \\
f'(x) = 3x^2 \\
f''(x) = 6x \\
\frac{1}{2} \text{ has max value of 18 on } [0,3] \\
\text{so } M = 18
\]
4. For each integral below, describe a method that will work but DO NOT SOLVE THE INTEGRAL HERE. Here are just a few examples of proper descriptions:

- substitution with \( u = \ldots \) and \( du = \ldots \)
- parts with \( u = \ldots \), \( du = \ldots \), \( v = \ldots \), and \( dv = \ldots \)
- partial fractions decomposition of the form \( \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1} \) (do not solve for coefficients)
- trig substitution with \( x = \ldots \) and \( dx = \ldots \)
- algebra/identity to rewrite as \( \ldots \) and then (describe method)

a) \[ \int \sqrt{x} \ln x \, dx \]
   - parts with \( u = \ln x \), \( du = \frac{1}{x} \, dx \)
   - \( v = \frac{2}{3} x^{3/2} \), \( dv = \sqrt{x} \, dx \)

b) \[ \int \frac{x^2 + 4x + 1}{x^3 + x^2} \, dx \]
   - partial fractions of form \( \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1} \)
   - \( \frac{A}{x^2} \)

\[ \int (1 - x^2)^{-\frac{3}{2}} \, dx \]
   - \( \int \frac{1}{(\sqrt{1-x^2})^3} \) \( \text{trig sub with } x = \sin u, \, dx = \cos u \, du \)

\[ \int \tan x \cos^5 x \, dx \]
   - algebra to rewrite \( \sin x \cos^5 x = \sin x \cos^4 x \) as \( \sin x \cos^4 x \) and then \( \text{sub with } u = \cos x, \, du = -\sin x \, dx \)

\[ \int \frac{x}{\sqrt{3x^2 + 1}} \, dx \]
   - \( \text{sub with } u = 3x^2 + 1, \, du = 6x \, dx \)
   - \( (\sqrt{3})^2 \)
   - \( \text{or trig sub with } x = \frac{1}{\sqrt{3}} \tan u, \, dx = \frac{1}{\sqrt{3}} \sec^2 u \, du \)

\[ \int \sin^5 x \cos^2 x \, dx \]
   - \( \sin^4 x \cos^2 x \sin x = (1 - \cos^2 x)^2 \cos^3 x \sin x \)
   - \( \text{as shown and then sub with } u = \cos x, \, du = \sin x \, dx \)

\[ \int \frac{\sec^2 x}{\tan x + 1} \, dx \]
   - \( \text{sub with } u = \tan x + 1, \, du = \sec^2 x \, dx \)
5. Solve ONE of the integrals on the previous page, showing all work very clearly from start to finish. Choose the hardest one that you can solve correctly; you will get more points for solving something difficult than for solving something easy.

a) \[ \int \sqrt{x} \ln x \, dx \]

\[ = \frac{3}{2} \frac{3}{2} \frac{1}{2} x^{\frac{3}{2}} \ln x - \frac{1}{2} x^{\frac{1}{2}} + C \]

b) \[ \int \frac{x^2 + 4x + 1}{x^2(x+1)} \, dx \]

\[ = \int \left( \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1} \right) \, dx \]

\[ \Rightarrow A = 3, B = 1, C = -2 \]

\[ = 3 \ln x - x^{-1} - 2 \ln x + x + 1 + C \]

c) \[ \int \frac{1}{\sqrt{1-x^2}} \, dx \]

\[ = \int \sec^2 u \, du = \tan(\sin^{-1} x) + C \]

\[ = \sqrt{1-x^2} + C \]

d) \[ \int \tan x \cos^5 x \, dx \]

\[ = \int \sin x \cos^4 x \, dx \]

\[ = \frac{1}{u} \, du = -\frac{1}{u} \cos^4 x + C \]

\[ = -\frac{1}{2} \cos^2 x + C \]

e) \[ \int \frac{x}{\sqrt{3x^2+1}} \, dx \]

\[ = \int \frac{3x^2 + 1}{6} \, du = \frac{1}{6} \cdot 2u^{\frac{3}{2}} + C \]

\[ = \frac{1}{3} \sqrt{3x^2+1} + C \]

f) \[ \int \sin^5 x \cos^2 x \, dx \]

\[ = \int (1-\cos^2 x)^2 \sin x \, dx \]

\[ = \int (1-u^2)^2 u \, du = -\int (u^2 - 2u + 1) \, du \]

\[ = \frac{1}{3} \cos^3 x - \frac{1}{2} \sin x + \frac{1}{5} \cos^5 x + C \]

g) \[ \int \sec^2 x \, dx \]

\[ = \int \tan x + 1 \, dx \]

\[ = \int \ln |\tan x| + 1 + C \]

\[ = 16 \text{ pts: (c), (g)} \]

\[ 15 \text{ pts: (a)} \]

\[ 20 \text{ pts: (b), (c), (d), (f)} \]

Survey for 2 bonus points: How do you think you did? What is a question or topic that could have been on this exam, but wasn’t?

definite integrals
simpson's rule
actually calculating a Riemann sum (left, mid, trap, simp, etc)
powers of sec x and tan x
more integrals of tan x, sec x, ln x

up to \[ 20 \]