Graph Theory Questions List

Math 353, Spring 2013

The questions below are taken from *Pearls in Graph Theory*, by Hartsfield and Ringel. Most of class will consist of students presenting solutions to these questions at the board, without notes. Course grades will be partially determined by the quality, clarity, and frequency of these presentations. We will proceed through the questions in the order in which they are listed.

Some of these questions are problems to solve, and some are proofs that you must write. For all problems, please remember that your goal is not only to show that you know the answer, but also to make a presentation that shows how you arrived at that answer. Some questions refer to problems and proofs from the reading, which means that their solutions are in the text. You are expected to fill in any clarifying details and illustrations and then present expanded solutions.

Section 1.1: Graphs and Degrees of Vertices

A. Theorem 1.1.1 Prove that the sum of vertex degrees is twice the number of edges.
B. Theorem 1.1.2 Prove that a sequence is graphic if and only if its reduction is graphic.
C. Exercise 1.1.2 Draw families of graphs with given degree sequences.
D. Exercise 1.1.4 Determine if given sequences are graphic.
E. Exercise 1.1.5 Prove that every graph has an even number of odd-degree vertices.
F. Exercise 1.1.6 Prove using induction that a given family of sequences is graphic.
G. Exercise 1.1.9 Draw graphs with given degree sequences.
H. Exercise 1.1.11 Draw a graph with given vertex degrees. *(Bonus: Make it planar.)*

Section 1.2: Subgraphs, Isomorphic Graphs

A. Exercise 1.2.2 Find a 5-vertex graph with 22 cycles, and prove it. *(Hint: Start with K_5.)*
B. Exercise 1.2.3 Find a 5-vertex graph with 13 cycles, and prove it. *(Hint: Start with 1.2.2.)*
C. Exercise 1.2.5 Find a certain type of graph with no K_4 subgraph, and prove it.
D. Exercise 1.2.8 Prove or disprove that given graphs are isomorphic.
E. Exercise 1.2.10 Prove or disprove that given graphs are isomorphic to the Petersen graph.

Section 1.3: Trees

A. Theorem 1.3.1 Prove using induction that connected graphs satisfy $p \leq q + 1$.
B. Theorem 1.3.2 Prove using induction that if a graph is a tree, then $p = q + 1$.
C. Theorem 1.3.3 Prove using contradiction that if $p = q + 1$, then the graph is a tree.
D. Theorem 1.3.5 Prove that a graph is a tree if and only if it has unique connecting paths.
E. Theorem 1.3.6 Prove that every connected graph contains a spanning tree.
F. Exercise 1.3.5 Prove # cycles for average degree sum < 2.
G. Exercise 1.3.6 Prove # cycles for average degree sum $= 2$. *(Hint: Use a spanning tree.)*
H. Exercise 1.3.7 Determine the number of 1's in a certain degree sequence for a tree.
I. Exercise 1.3.13 Prove/disprove there is a disconnected graph with given degree sequence.
J. Exercise 1.3.15 Prove there are only 2 nonisomorphic graphs with given degree sequence.
K. Exercise 1.3.19 Prove or disprove that given graphs are isomorphic.

SECTION 2.1: VERTEX COLORINGS

A.	THEOREM 2.1.1	Use proof by contradiction to show that every critical graph is connected.
B.	THEOREM 2.1.3	Prove that in a critical graph, the degree of each vertex is at least $\chi - 1$.
C.	EXERCISE 2.1.1	Find and prove a formula for $\chi(C_n)$.
D.	EXERCISE 2.1.2	Find and prove a formula for $\chi(W_n)$.
E.	EXERCISE 2.1.3	Prove that if a certain property holds for χ, then G is complete.
F.	EXERCISE 2.1.9	Determine the chromatic number of various graphs.
G.	EXERCISE 2.1.10	Determine critical subgraphs of various graphs.
H.	EXERCISE 2.1.12	Prove by induction that trees have chromatic number at most 2.
I.	EXERCISE 2.1.13	Ladies, gentlemen, and C_6 subgraphs of $K_{3,3}$.
J.	EXERCISE 2.1.15	Find graphs with various vertex, edge, and subgraph properties.
K.	EXERCISE 2.1.16	Find and prove a formula for the number of vertices in diameter 3 trees.
L.	EXERCISE 2.1.18	Find and prove a formula for the number of length n cycles in W_n.
M.	EXERCISE 2.1.19	Count the W_5 subgraphs in the icosahedron graph.
N.	EXERCISE 2.1.22	Count the number of diameter 3 spanning trees of $K_{m,n}$ and K_m.

SECTION 2.2: EDGE COLORINGS

A.	THEOREM 2.2.3	Prove that the edge chromatic number of K_{2n} is $2n-1$.
B.	THEOREM 2.2.4	Prove that the edge chromatic number of K_{2n-1} is $2n-1$.
C.	EXERCISE 2.2.1	Prove that an edge 2-coloring of K_6 contains at least one full triangle.
D.	EXERCISE 2.2.3	Find and prove a formula for the edge chromatic number of $K_{m,n}$.
E.	EXERCISE 2.2.4	Find the edge chromatic number of the Grötzsch graph.
F.	EXERCISE 2.2.6	Determine which complete bipartite graphs $K_{m,n}$ have 1-factors.
G.	EXERCISE 2.2.7	Find proper edge 3-colorings of the cube and dodecahedron graphs.
H.	EXERCISE 2.2.8	Find where $C_x = K_{y,z}$, $C_x = K_y$, $K_x = K_{y,z}$, $K_x = W_y$, and $P_x = K_{y,z}$.
I.	EXERCISE 2.2.9	Find all four nonisomorphic connected graphs with $pq = 20$.

SECTION 2.3: DECOMPOSITIONS AND HAMILTON CYCLES

A.	THEOREM 2.3.1	Prove K_{2n+1} decomposes into n Hamilton cycles.
B.	THEOREM 2.3.2	Prove K_{2n} decomposes into $n-1$ Hamilton cycles and a 1-factor.
C.	THEOREM 2.3.5	Prove that snarks do not have Hamilton cycles.
D.	EXERCISE 2.3.1	Find spanning trees of W_n with every diameter $2 \leq h \leq n$.
E.	EXERCISE 2.3.7	Find Hamiltonian paths and cycles in certain graphs, if possible.
F.	EXERCISE 2.3.8	Thirteen people having dinner various ways...
G.	EXERCISE 2.3.10	Find Hamiltonian paths in certain graphs.
H.	EXERCISE 2.3.11	Why doesn't the "skip k" method work for K_9?
I.	EXERCISE 2.3.12	Find and prove formulas for when $K_{m,n}$ has a Hamilton cycle or path.
J.	EXERCISE 2.3.15	Find and prove a formula for the number of Hamilton cycles in W_n.
K.	EXERCISE 2.3.16	Find spanning trees with diameters $6 \leq h \leq 10$ in a certain graph.
L.	EXERCISE 2.3.17	Decompose K_6 into 3 isomorphic subgraphs with given degree sequence.
M.	EXERCISE 2.3.18	Decompose $K_{2,n}$ into two isomorphic trees for diameters $2 \leq h \leq 4$.

N. EXERCISE 2.3.19 Decompose $K_{3,4}$ into two Hamiltonian paths.
O. EXERCISE 2.3.20 Find and prove the number of Hamilton paths in $K_{m,m+1}$ and $K_{m,m}$.

SECTION 3.1: EULERIAN CIRCUITS

A. THEOREM 3.1.1 Prove $\exists$ Eulerian cycle means every vertex has even degree.
B. THEOREM 3.1.6 Prove $\exists$ Eulerian trail iff connected with exactly 2 odd-degree vertices.
C. THEOREM 3.1.7 Prove Listing's Theorem about subdividing into trails.
D. EXERCISE 3.1.1 Find Euler circuits, when possible.
E. EXERCISE 3.1.2 Decompose $K_{4,6}$ into cycles.
F. EXERCISE 3.1.3 Decompose $K_{4,4}$ into 2-factors.
G. EXERCISE 3.1.5 Decompose the Grotzch graph into three paths.
H. EXERCISE 3.1.9 Decompose the octahedron graph into three length-4 paths.
I. EXERCISE 3.1.11 Nationalities, neighbors, and Euler circuits in K_5.
J. EXERCISE 3.1.13 Prove that an odd number of cycles implies an Euler circuit exists.
K. EXERCISE 3.1.14 Find a non-crossing Euler trail in a graph.

SECTION 3.2: THE OBERWOLFACH PROBLEM

A. NEW EXAMPLE Solve OP for 11 people and 3 tables. Give graphs and seating charts.
B. THEOREM 3.2.1 Prove that a regular graph of even degree has no bridge.
C. EXERCISE 3.2.2 Apply Petersen's Theorem to a given graph.

SECTION 3.3: INFINITE LATTICE GRAPHS

A. EXERCISE 3.3.1 Find a spanning tree of L_2 with no degree 2 vertices.
B. EXERCISE 3.3.2 Find a spanning subgraph of L_2 with all degree 3 vertices.
C. EXERCISE 3.3.3 Find a finite connected subgraph of L_2 with average degree 3.
D. EXERCISE 3.3.4 Find the chromatic number of L_2.
E. EXERCISE 3.3.5 Find the girth of L_2.
F. EXERCISE 3.3.6 Decompose L_2 into four 1-factors.
G. EXERCISE 3.3.7 Decompose L_2 into subgraphs isomorphic to C_{4k}.
H. EXERCISE 3.3.8 Prove that L_2 can't be decomposed into C_6 subgraphs.
I. EXERCISE 3.3.10 Decompose L_2 into various connected subgraphs.
J. EXERCISE 3.3.11 Decompose L_2 into subgraphs isomorphic to $K_{r,s}$ for all possible r, s.
K. EXERCISE 3.3.13 Is the $\sqrt{2}$-diagonal integer lattice M_2 connected?
L. EXERCISE 3.3.14 Is the $\sqrt{5}$-diagonal integer lattice N_2 connected?
M. EXERCISE 3.3.15 Find a Hamiltonian line in $L_2 - v$.
N. EXERCISE 3.3.16 Find a Hamiltonian lines decomposition of the triangular lattice graph.
O. EXERCISE 3.3.17 Show $\exists$ an infinite graph with all degrees even but no Eulerian line.
P. EXERCISE 3.3.19 Find a one-way Eulerian trail in the spiderweb graph.
Q. EXERCISE 3.3.20 Find a one-way Eulerian path in the triangle lattice.

Section 8.1: Planar Graphs

A.	Theorem 8.1.1	Use induction on cycles to prove Euler's polyhedral formula.
B.	Theorem 8.1.2	Prove that in a maximal planar graph we have $q = 3p - 6$.
C.	Theorem 8.1.4	Prove that K_5 is not planar.
D.	Theorem 8.1.7	Prove that every planar graph contains at least one vertex of degree ≤ 5.
E.	Exercise 8.1.2	Find a planar, straight line drawing of a certain graph.
F.	Exercise 8.1.5	Determine if certain graphs are planar, or maximal planar.
G.	Exercise 8.1.6	Prove that $K_{3,3}$ is not planar.
H.	Exercise 8.1.7	Find the smallest planar graph that is regular of degree 4.
I.	Exercise 8.1.8	Find a planar graph besides the icosohedron that is regular of degree 5.
J.	Exercise 8.1.9	Show that a given graph is planar and then add edge until maximal planar.
K.	Exercise 8.1.11	Find planar and non-planar graphs with degree sequence $(4, 4, 4, 4, 3, 3)$.
L.	Exercise 8.1.12	Prove that a certain type of graph must have 8 squares and 10 triangles.
M.	Exercise 8.1.15	What is the max # of edges you can add to W_5 and preserve planarity?
N.	Exercise 8.1.17	Find planar and non-planar grpahs with degree sequence $(6, 5, 5, 5, 3, 3, 3)$.

Section 8.2: The Four Color Theorem

A.	Theorem 8.2.1	Prove that countries 4-colorable implies edges are 3-colorable.
B.	Lemma 8.2.2	Prove that a certain type of set of regions is 2-colorable.
C.	Theorem 8.2.3	Prove that edges 3-colorable implies countries are 4-colorable.
D.	Theorem 8.2.4	Prove that edges 3-colorable implies vertices are special 2-colorable.
E.	Theorem 8.2.5	Prove that vertices special 2-colorable implies edges are 3-colorable.
F.	Theorem 8.2.9	Explain how this theorem relates to the Four Color Theorem.
G.	Exercise 8.2.1	Find a country 4-coloring, edges 3-coloring, and vertices special 2-coloring.
H.	Exercise 8.2.2	Find and identify the duals of the five regular Platonic solids.
I.	Exercise 8.2.3	Why is the US map not normal? Give as many reasons as possible.
J.	Exercise 8.2.4	Use a dual to prove that the U.S. states are not 3-colorable.