TEST I

Math 430
February 12, 2004

Name: ________________________________

By writing my name I swear by the honor code.

Read all of the following information before starting the exam:

- When stating definitions, be sure to use careful mathematical notation. Be as clear as possible.
- Your proofs will be graded for clarity as well as for mathematical accuracy. Make sure that your logic is clear and that I can tell what you are proving, and how. Provide reasons for steps whenever possible.
- Don't get hung up on any one problem. If you get stuck, move on and come back to the problem later.
- By writing your name above, you agree to the JMU honor code. In particular, this means that you may not use any notes or crib sheets during this exam, that all work must be your own, and that you may not obtain advance information revealing the problems on this exam.
- This test has 5 problems and is worth 100 points, plus some extra credit at the end. Make sure that you have all of the pages!
- Good luck!

1. (*18 pts*) **True/False.** Determine whether each statement is true (T) or false (F).

(a) **T** **F** Every function $*\colon S \times S \to S$ defines a binary algebraic structure $*$ on S.

(b) **T** **F** $\langle 3\mathbb{Z}, + \rangle$ is isomorphic to $\langle \mathbb{Z}, + \rangle$.

(c) **T** **F** The group $U_6 = \{z \in \mathbb{C} \mid z^6 = 1\}$ (under complex multiplication) is cyclic, and $e^{i\frac{5\pi}{3}}$ is a generator.

(d) **T** **F** Any two groups of three elements are isomorphic.

(e) **T** **F** As subgroups of $\mathbb{Z}_{12}$ under addition, $\langle 8 \rangle \leq \langle 2 \rangle \leq \langle 5 \rangle$.

(f) **T** **F** I would like to have three bonus points, please.

2. (*14 pts*) **Definitions.** Complete the following definitions. Make sure you give complete, accurate, precise mathematical definitions.

a. (*4 pts*) Define what it means for two sets X and Y to have the ***same cardinality***.
(Note: You do NOT know that X and Y are finite sets, so you can NOT just say that X and Y have the "same number of elements.")

b. (*10 pts*) Define what it means for a set G with a binary operation $*$ to be a ***group***.
(Completely define each of the three group conditions; don't just mention them by name.)

3. (*12 pts*) **Un-Proofs.** Clearly explain your answers.

a. (*6 pts*) Consider the relation $\mathcal{R}$ on the set $\mathbb{R}$ defined by $x\mathcal{R}y$ if $x \geq y$. Show that $\mathcal{R}$ is *not* an equivalence relation on $\mathbb{R}$.

b. (*6 pts*) Show that $\langle \mathbb{Q}, \cdot \rangle$ is *not* a group.

4. (*32 pts*) **Proofs.** Make sure your arguments are clear, complete, and presented in a logical order. Provide reasons for steps whenever possible.

a. (*16 pts*) Prove that the map $\phi\colon \langle \mathbb{R}, + \rangle \to \langle \mathbb{R}^+, \cdot \rangle$ defined by $\phi(x) = e^x$ for all $x \in \mathbb{R}$ is an isomorphism.

b. (*16 pts*) Suppose G is a group and $a \in G$ is fixed. Show that $H_a = \{x \in G \mid xa = ax\}$ is a subgroup of G.

5. (*24 pts*) **Examples.** Provide an example of each type of object described below. There are many possible examples for each problem. You do NOT have to prove that your example has the properties described; just describe the example.

(**a**) A group that is not cyclic.

(**b**) A group that is not abelian.

(**c**) An equivalence relation $\sim$ on $\mathbb{Z}$ that is *not* just the "equality" relation.

(**d**) A subset of $\mathbb{Z}_5$ that is not a subgroup of $\mathbb{Z}_5$.

(**e**) A binary operation $*$ on the set $S = \{e, a, b\}$ such that $\langle S, * \rangle$ is a group. (Use a table.)

(**f**) A binary operation $*$ on the set $S = \{e, a, b\}$ such that $\langle S, * \rangle$ is *not* a group. (Use a table.)

Survey Questions: *(2 extra credit points)*

Name a question or topic that could have been on this test, but wasn't.

How do you think you did?

SPACE FOR SCRAP WORK