TEST II

Math 430
March 19, 2004

Name: ______________________________
By writing my name I swear by the honor code.

Read all of the following information before starting the exam:

- When stating definitions, be sure to use careful mathematical notation. Be as clear as possible.
- Your proofs will be graded for clarity as well as for mathematical accuracy. Make sure that your logic is clear and that I can tell what you are proving, and how. Provide reasons for steps whenever possible.
- Don't get hung up on any one problem. If you get stuck, move on and come back to the problem later.
- By writing your name above, you agree to the JMU honor code. In particular, this means that you may not use any notes or crib sheets during this exam, that all work must be your own, and that you may not obtain advance information revealing the problems on this exam.
- This test has 4 problems and is worth 100 points, plus some extra credit at the end. Make sure that you have all of the pages!
- Good luck!

1. (*12 pts*) Determine whether each statement is true (T) or false (F).

(**a**) **T** **F** Every infinite cyclic group is isomorphic to $\langle \mathbb{Z}, + \rangle$.

(**b**) **T** **F** All generators of $\mathbb{Z}_{20}$ are prime numbrs.

(**c**) **T** **F** Suppose G and G' are any groups. If $\phi\colon G \to G'$ is a one-to-one homomorphism, then G is isomorphic to $\phi[G]$.

(**d**) **T** **F** Every group G has only one possible Cayley graph.

(**e**) **T** **F** Every group is isomorphic to some group of permutations.

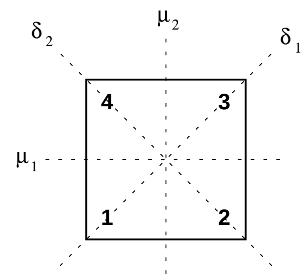
(**f**) **T** **F** The odd permutations in S_8 form a subgroup of S_8.

2. (*20 pts*) Make sure your arguments are clear, complete, CONCISE, and presented logically. (You may want to collect your thoughts on the scrap page before writing your final version.)

(**a**) Prove that every group of prime order is cyclic. (Hint: Use the Theorem of Lagrange.)

(**b**) A positive integer m is ***square-free*** if it is not divisible by the square of any prime. Prove that every abelian group with square-free order is cyclic. (Hint: Use the Fundamental Theorem of Finitely Generated Abelian Groups.)

3. (*36 pts*) $D_4 = \{\rho_0, \rho_1, \rho_2, \rho_3, \mu_1, \mu_2, \delta_1, \delta_2\}$ is the group of symmetries of a square. Here ρ_1 represents a rotation counterclockwise of 90°, and the four flips are indicated in the figure.



(a) Which element of D_4 is $\rho_1\delta_1\rho_2$?

(b) D_4 is a subgroup of S_4. What is the index of D_4 in S_4?

(c) Considered as an element of S_4, is ρ_3 an element of the alternating group A_4?

(d) Write the following elements of D_4 as products of disjoint cycles in S_4.

$\rho_3 =$ ______________ $\mu_1 =$ ______________ $\delta_2 =$ ______________

(e) Write the following elements of D_4 as products of transpositions in S_4.

$\rho_3 =$ ______________ $\mu_1 =$ ______________ $\delta_2 =$ ______________

(f) List all of the elements of the subgroup $\langle \rho_2, \delta_1 \rangle$ of D_4. (In other words, the subgroup generated by ρ_2 and δ_1 together.)

(g) Consider the subgroup $H = \langle \mu_1 \rangle$ of D_4. Describe all of the left cosets of H in D_4.

4. (*32 pts*) Consider the group $G = \mathbb{Z}_6 \times \mathbb{Z}_{12} \times \mathbb{Z}_{20}$.

(**a**) What is the order of the group G?

(**b**) What is the order of the element $(4, 5, 6)$ in the group G?

(**c**) List all of the elements in the subgroup generated by the set $\{(3, 8, 10), (0, 6, 0)\}$

(**d**) Circle <u>all</u> of the following that describe the group G:

abelian cyclic finite finitely generated

(**e**) Circle <u>all</u> of the following groups that are isomorphic to G:

$$\mathbb{Z}_3 \times \mathbb{Z}_4 \times \mathbb{Z}_{10} \times \mathbb{Z}_{12}$$

$$\mathbb{Z}_5 \times \mathbb{Z}_9 \times \mathbb{Z}_{32}$$

$$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5$$

(**f**) Write G in the form $\mathbb{Z}_{p_1^{r_1}} \times \mathbb{Z}_{p_2^{r_2}} \times \cdots \times \mathbb{Z}_{p_n^{r_n}}$, where the p_i are prime (not necessarily distinct) and the r_i are in $\mathbb{Z}^+$.

(**g**) Write G in the form $\mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2} \times \cdot \times \mathbb{Z}_{m_r}$, where $m_i \in \mathbb{Z}^+$ and m_i divides m_{i+1}.

Survey Questions: *(2 extra credit points)*

Name a question or topic that could have been on this test, but wasn't.

How do you think you did?

SPACE FOR SCRAP WORK