TEST III

Math 430
April 23, 2004

Name: ______________________________
By writing my name I swear by the honor code.

Read all of the following information before starting the exam:

- When stating definitions, be sure to use careful mathematical notation. Be as clear as possible.
- Your proofs will be graded for clarity as well as for mathematical accuracy. Make sure that your logic is clear and that I can tell what you are proving, and how. Provide reasons for steps whenever possible.
- Don't get hung up on any one problem. If you get stuck, move on and come back to the problem later.
- By writing your name above, you agree to the JMU honor code. In particular, this means that you may not use any notes or crib sheets during this exam, that all work must be your own, and that you may not obtain advance information revealing the problems on this exam.
- This test has 5 problems and is worth 100 points, plus some extra credit at the end. Make sure that you have all of the pages!
- Good luck!

1. (*36 pts*) Determine whether each statement is true (T) or false (F).

(a) **T** **F** Every subgroup of an abelian group G is a normal subgroup.

(b) **T** **F** Every factor group of an abelian group is abelian.

(c) **T** **F** Every factor group of a nonabelian group is nonabelian.

(d) **T** **F** In the factor group $\mathbb{Z}_{12}/\langle 4\rangle$, the element $5 + \langle 4\rangle$ has order 4.

(e) **T** **F** $(\mathbb{Z}_4 \times \mathbb{Z}_8)/\langle(1, 2)\rangle$ is isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}_4$.

(f) **T** **F** There are no group homomorphisms $\phi\colon \mathbb{Z}_3 \to \mathbb{Z}$.

(g) **T** **F** The image of a group of 6 elements under some group homomorphism may have 4 elements.

(h) **T** **F** If the commutator subgroup $C(G)$ of a group G is $\{e\}$, then G is abelian.

(i) **T** **F** Suppose $\phi\colon G \to G'$ is a group homorphism with kernel $\ker\phi$ and image $\operatorname{im}\phi$. Then $G/\ker\phi \simeq \operatorname{im}\phi$.

(j) **T** **F** Suppose $\phi\colon G \to G'$ is a group homomorphism with $H = \ker\phi$. Then for every $a \in G$, we have $aH = \phi^{-1}[\{\phi(a)\}]$.

(k) **T** **F** The map $\phi\colon \mathbb{Z} \to \mathbb{Z}$ defined by $\phi(1) = 5$ is a ring homomorphism.

(l) **T** **F** Suppose D is an integral domain with field of quotients F. Then every element of D is a unit when considered as an element of F.

2. (*16 pts*) Let H be a normal subgroup of G. Prove that the cosets of H form a group G/H under the binary operation $(aH)(bH) = (ab)H$.

3. (*16 pts*) Prove that if the cancellation laws hold in a ring R, then R has no zero divisors.

4. (*12 pts*) Give examples of each of the following. (Don't prove anything, just clearly define your example.)

(**a**) A nontrivial additive group homomorphism $\phi\colon \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ such that $\ker(\phi) = 3\mathbb{Z}$.

(**b**) A nontrivial ring homomorphism $\phi\colon \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$.

5. (*20 pts*) Fill in the table below, with yes/no entries or, if the first entry is blank, with the name of a ring with the given properties. (If no such ring exists, don't leave the first entry blank; instead write "DNE".) In each case you can assume the "usual" addition and multiplication for that ring is being used. The first row is filled in as an example.

Ring	Multiplication is commutative	Has unity	Has at least one zero divisor	Every nonzero element is a unit
$\mathbb{R}$				
$\mathbb{Z}_6$				
$\mathbb{Z}_7$				
$\mathbb{Z} \times \mathbb{Z}$				
$5\mathbb{Z} \times \mathbb{Z}_5$				
$M_3(\mathbb{Z}_2)$				
$\mathbb{Z}_4[x]$				
	yes	no	no	no
	yes	yes	no	no
	yes	yes	yes	yes

Survey Questions: *(2 extra credit points)*

Name a question or topic that could have been on this test, but wasn't.

How do you think you did?

SPACE FOR SCRAP WORK