Very Interesting Stuff **Math 430, Spring 2003**

Here is some fun stuff for you to work on over spring break. These problems investigate the structure of finite groups of small order, and normal subgroups for groups of finite and infinite order. The "fun" part is that you will get to draw some nice pictures. We will roll the dice for this assignment on the Monday when we return. Since there is no POD for today, any roll of 4, 5, or 6 will result in this assignment being collected. Notice that each problem has a footnote with hints or definitions.

1. Make a list of all of the groups of order less than or equal to seven (up to isomorphism). $(\star)$

2. For each group G listed in (1), and *also* for the groups $\mathbb{Z}_{24}$ and D_4, do the following:

- **a)** Make a group table describing the operation in G. $(\spadesuit)$
- **b)** Draw a Cayley digraph that represents G. $(\heartsuit)$
- **c)** Draw the cycle graph for G. $(\clubsuit)$
- **d)** Make the complete subgroup diagram for G. $(\diamondsuit)$
- **e)** Determine which of the subgroups of G are normal subgroups. $(\S)$

3. Do parts (a), (b), and (c) of problem 2 for the group A_4. Then find one proper nontrivial normal subgroup of A_4, and one proper nontrivial subgroup of A_4 that is not normal. $(\dagger)$

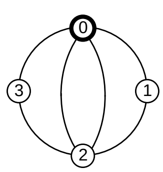
4. So far this assignment has been about finite groups. Don't forget that some groups are infinite! For example, consider the groups $\langle Z, +\rangle$ and $\langle GL(2,\mathbb{R}), \cdot\rangle$. For each of these groups, find one proper nontrivial normal subgroup. Prove that the subgroup you choose is in fact a normal subgroup. $(\ddagger)$

$(\star)$ *Hint: There are only two groups of order six.*

$(\spadesuit)$ *Feel free to skip this one for $\mathbb{Z}_{24}$, for obvious reasons.*

$(\heartsuit)$ *Read Section 7 for more information on Cayley digraphs.*

$(\clubsuit)$ *A **cycle graph** of a finite group G is a graph of vertices and edges such that: (1) there is a vertex for each element $a \in G$, and (2) for $a \in G$ with order r, we connect vertices to show the cycles $e \to a \to a^2 \to \cdots \to a^{r-1} \to a^r = e$. From the cycle graph of G we can easily see each of the cyclic subgroups $\langle a\rangle = \{a, a^2, a^3, \ldots, a^r = e\}$ of G. For example, $\mathbb{Z}_4$ has a cycle taking $0 \to 1 \to 2 \to 3 \to 0$ (generated by $1 \in \mathbb{Z}_4$), and a cycle taking $0 \to 2 \to 0$ (generated by $2 \in \mathbb{Z}_4$). The cycle graph of $\mathbb{Z}_4$ is shown below.*



Technical points: (1) The identity is usually clearly marked so that it stands out, since it is the beginning and end of all cycles. (2) If a cycle has a lot of subcycles (this will happen with $\mathbb{Z}_{24}$), you may want to use different colors to avoid confusion. (3) To avoid cluttering the diagram, if $\langle b\rangle = \langle a\rangle$, we only draw one of the cycle paths, not both; for example in $\mathbb{Z}_6$ we have $\langle 1\rangle = \langle 5\rangle$, and we draw the cycle for $\langle 1\rangle$ but not the one for $\langle 5\rangle$.

$(\diamondsuit)$ *"Complete" means that all of the subgroups have to be in your diagram. For cyclic groups it is easy to find them all. For non-cyclic groups you will have to think a little harder.*

$(\S)$ *A subgroup $H \le G$ is **normal** if the left cosets of H in G are equal to the right cosets of H in G. In other words, H is a normal subgroup if for all $g \in G$ we have $gH = Hg$.*

$(\dagger)$ *Hint: Just hunt around trying different subgroups until you find ones that have the properties you want.*

$(\ddagger)$ *An equivalent condition for H to be a normal subgroup of G is that for all $g \in G$ and all $h \in H$ we have $g^{-1}hg \in H$. This may come in handy for testing whether or not a subgroup of $GL(2,\mathbb{R})$ is normal.*