Problems for Advanced Linear Algebra

Fall 2012

- Class will be structured around students presenting complete solutions to the problems in this handout. Please only agree to come to the board when you are confident that you can provide a clear, complete presentation of the solution to the class without looking at your notes. I will not look favorably on instances when you agree to come to the blackboard but are not ready to present. On the other hand it will not look good if you have not presented your share of the problems by the end of the semester.
- Although not explicitly listed below, you should do all of the true/false exercises in each section, giving counterexamples to the false statements when possible. We will discuss these exercises – especially the counterexamples – in class.
- When asked to explain something that appears in the book, you are expected to fill in the missing details and provide additional explanation, and not just mimic what is in the book.
- When the handout uses unexplained notation or refers to a "particular" transformation, matrix, etc, you are expected to use the information given in the reference. Unless otherwise stated, V will denote a finite-dimensional vector space, F will denote a field, T will denote a linear transformation, β will denote a basis, and A will denote a matrix.
- Important: The questions below are not always equivalent to the referenced items in the book. The references are only given to you so you can see what part of the book might be relevant to the problems. When there is a discrepancy, please answer the question as written on this handout – with the notation given on the handout – and use the book only as a guide.
- The final exam will be drawn entirely from the problems on this handout, the definitions and theorems in the book, and the true/false questions in the exercises of each section. The majority of your grade for the course will be based on your performance on the final exam. This performance is likely to be higher if you have been active in class throughout the semester.
- This class is not a competition and we will strive for a supportive, friendly atmosphere during student presentations. Please respect your fellow students while they are presenting. Presenters, please present to the class and *not* to me while you are at the board.

Chapter 1: Vector Spaces

1.1 Introduction

1. Explain why $x = u + t(v - u)$ is the equation for the line through the points A and B that are the endpoints of the vectors u and v, respectively. Include a descriptive picture. *(Page 4)*

2. Explain why $x = A + su + tv$ is the equation for a plane through points A, B, and C, where u and v are the vectors that begin at A and end at B and C, respectively. Include a descriptive picture. *(Page 5)*

3. Find the equation of the line through the points $(-2, -1, 5)$ and $(3, 9, 7)$. Then find the equation of the plane through the points $(1, 1, 1)$, $(5, 5, 5)$, and $(-6, 4, 2)$. *(Exercises 2d and 3d)*

1.2 Vector Spaces

4. Prove that $\mathcal{F}(S, F)$ satisfies all eight properties of a vector space. *(Example 3)*

5. Prove that $\mathrm{P}(F)$ satisfies all eight properties of a vector space. *(Example 4)*

6. Each of the following properties follow directly from the properties of a vector space. Prove each one, referring to (VS 1)–(VS 8) when used. (Do them in order; some of them rely on previous ones.)

a) cancellation law for addition *(Theorem 1.1)*

b) the additive identity $\mathbf{0}$ is unique *(Corollary 1)*

c) additive inverses are unique *(Corollary 2)*

d) the zero scalar times any vector is the zero vector *(Theorem 1.2.a)*

e) $(-a)x = -(ax) = a(-x)$ *(Theorem 1.2.b)*

f) every scalar multiple of the zero vector is the zero vector *(Theorem 1.2.c)*

7. Prove that the zero vector space satisfies all eight properties of a vector space. *(Exercise 11)*

8. Exercises 13 and 17 each describe something that is *not* a vector space. List the vector space properties that fail in each case. *(Exercises 13 and 17)*

1.3 Subspaces

9. Prove that $W \subseteq V$ is a subspace if and only if W is closed under addition and scalar multiplication and W contains the zero vector of V. *(Theorem 1.3)*

10. Prove that if U and W are subspaces of V, then $U \cap W$ is a subspace of V. *(Theorem 1.4)*

11. Give a specific example of subspaces U and W of a vector space V such that $U \cup W$ is not a subspace of V, and show explicitly how it fails to be a subspace. *(Page 19)*

12. Prove or disprove that a particular subset is a subspace, in the following vector spaces: $\mathrm{P}(F)$, $M_{m\times n}(F)$, and $C(\mathbb{R})$. *(Exercises 11, 12, 15)*

13. Prove that if $W \subseteq V$ is a subspace, then any finite linear combination of elements of W is also in W. *(Exercise 20)*

14. Show that $M_{m\times n}(F)$ is the direct sum of the subspace of upper triangular matrices and the subspace of strictly lower triangular matrices. *(Exercise 26 and the top of page 22)*

15. Prove that $V = W_1 \oplus W_2$ if and only if every element of V can be uniquely written as the sum of an element from W_1 and an element from W_2. *(Exercise 30 and the top of page 22)*

1.4 Linear Combinations and Systems of Linear Equations

16. Prove that for any nonempty subset $S \subseteq V$, $\mathrm{span}(S)$ is a subspace, and is in fact the smallest subpace containing S. *(Theorem 1.5)*

17. Determine whether a particular vector is in the span of two other vectors, in the following vector spaces: $\mathbb{R}^3$, $\mathrm{P}_3(\mathbb{R})$, and $M_{2\times 2}(\mathbb{R})$. *(Exercises 3d, 4d, and 5g)*

18. Prove that the standard sets of generators for the vector spaces F^n, $\mathrm{P}_n(F)$, and $M_{2\times 2}(F)$ are in fact spanning sets. *(Exercises 7, 8, and 9)*

19. Suppose a subset $S \subseteq V$ has the special property that if $a_1v_1 + a_2v_2 + \cdots + a_nv_n = \mathbf{0}$ for $a_i \in F$ and distinct $v_i \in V$, then all of the a_i must be equal to zero. Prove that for such a subset S, every vector in $\mathrm{span}(S)$ can be uniquely written as a linear combination of vectors in S. *(Exercise 16)*

1.5 Linear Dependence and Linear Independence

20. Explain why the existence of a nontrivial representation of $\mathbf{0}$ using vectors from a subset $S \subseteq V$ is equivalent to the existence of at least one vector in S that is a linear combination of other vectors from S. *(Page 36)*

21. Prove that subsets of linear independent sets are linearly independent, and that supersets of linear dependent sets are linearly dependent. *(Theorem 1.6 and its Corollary)*

22. Determine whether certain subsets of $M_{2\times 2}(\mathbb{R})$, $\mathrm{P}_3(\mathbb{R})$, and $\mathbb{R}$ are linearly dependent or linearly independent. *(Exercises 2b, 2d, and 2f)*

23. Prove that the standard sets of generators for the vector spaces F^n, $\mathrm{P}_n(F)$, and $M_{n\times n}(F)$ are linearly independent sets. *(Exercises 4, 5, and 6)*

24. Prove that if u and v are distinct elements of V, then the set $\{u, v\}$ is linearly dependent if and only if u or v is a multiple of the other. *(Exercise 9)*

25. Give an example of three linearly dependent vectors in $\mathbb{R}^3$ such that none of the three is a multiple of another. *(Exercise 10)*

26. Prove that the set of columns of any square upper triangular matrix with nonzero diagonal entries is linearly independent. *(Exercise 17)*

1.6 Bases and Dimension

27. Prove that a subset β of V is a basis if and only if every element of V can be written as a unique linear combination of elements of β, and explain why this means that a vector space with an n-element basis is basically a copy of F^n. *(Theorem 1.8 and page 44)*

28. Prove that if V is dimension n, then any size n spanning subset $S \subseteq V$ is a basis for V (i.e. linear independence comes for free), and any size n linearly independent subset $S \subseteq V$ is a basis for V (i.e. spanning comes for free). *(Corollary 2 on page 47)*

29. Given a set of five vectors, find a subset of that set that is a basis for $\mathbb{R}^3$. *(Exercise 7)*

30. Find the unique representation of an arbitrary vector $(a_1, a_2, a_3, a_4) \in F^4$ as a linear combination of the elements of a given basis of F^4. *(Exercise 9)*

31. Show that if u, v, and w are distinct and $\{u, v, w\}$ is a basis for V, then $\{u + v + w, v + w, w\}$ is also a basis for V. *(Exercise 12)*

32. Find a basis for the set of solutions to a given system of linear equations. *(Exercise 13)*

33. Find bases for certain subspaces of the vector spaces $M_{n\times n}(F)$ and $\mathrm{P}_n(\mathbb{R})$, and find the dimensions of these subspaces. *(Exercises 17 and 26)*

34. Prove that for subspaces W_1, W_2 with bases β_1 and β_2, we have $V = W_1 \oplus W_2$ if and only if $\beta_1 \cap \beta_2 = \emptyset$ and $\beta_1 \cup \beta_2$ is a basis for V. *(Exercise 33)*

Chapter 2: Linear Transformations and Matrices

2.1 Linear Transformations, Null Spaces, and Ranges

35. Prove that if $T\colon V \to W$ is a linear transformation, then $T(\mathbf{0}_V) = \mathbf{0}_W$. *(Page 65)*

36. Suppose $T\colon V \to W$ is a linear transformation. Prove that the kernel $N(T)$ is a subspace of V, and the image $R(T)$ is a subspace of W. *(Theorem 2.1)*

37. Prove the Dimension Theorem: If $T\colon V \to W$ is a linear transformation, then $\mathrm{nullity}(T) + \mathrm{rank}(T) = \dim(V)$. Explain how to construct a basis for V that splits into vectors that span the kernel of T and vectors that span the image of T when mapped to W. *(Theorem 2.3)*

38. Prove that a linear transformation $T\colon V \to W$ is one-to-one if and only if $N(T) = \{0\}$, and onto if and only if $R(T) = W$. *(Theorem 2.4)*

39. Prove that a linear transformation is completely determined by its action on a basis: If T and U are linear transformations $V \to W$ that agree on all elements of a basis of V, then T and U must agree on *all* elements of V. (Prove from scratch – it's easy!) *(Corollary to Theorem 2.6)*

40. For two particular linear transformations, prove linearity, find bases for the kernel and image subspaces, verify the Dimension Theorem, and determine whether the transformation is one-to-one and/or onto. *(Excercises 2 and 5)*

41. Determine the general form of a linear transformation defined by its values on two particular given vectors, and then determine if the transformation is one-to-one. *(Exercise 10)*

42. Prove that a set of vectors that maps to a linearly independent set under the action of a linear transformation must itself be linearly independent set. *(Exercise 13)*

43. Prove that a linear transformation is one-to-one if and only if it carries linearly independent sets to linearly independent sets, and that a one-to-one and onto linear transformation carries a basis of the domain space to a basis of the target space. *(Exercise 14)*

44. Suppose $T\colon V \to V$ is a linear transformation. Prove that the each of the four subspaces $\{\mathbf{0}\}$, V $R(T)$, and $N(T)$ are T-invariant. Also give an explicit example of a linear transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ with a T-invariant subspace that is not one of the four listed above. *(Exercise 28)*

2.2 The Matrix Representation of a Linear Transformation

45. Suppose V is a vector space with basis β. Show that the map $T\colon V \to F^n$ defined by $T(x) = [x]_\beta$ is a bijective linear transformation. (Hint: Show linearity first.) *(Page 80)*

46. If $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ is defined by $T(a_1, a_2) = (a_1 + 3a_2, 0, 2a_1 - 4a_2)$, find the matrix representation for T in terms of the standard bases for $\mathbb{R}^2$ and $\mathbb{R}^3$, as well as in terms of the alternate bases $\sigma = \{(1,0),(1,1)\}$ for $\mathbb{R}^2$ and $\tau = \{(1,1,0),(0,2,0),(0,0,1)\}$ for $\mathbb{R}^3$. *(Example 3)*

47. If $T\colon V \to W$ is a linear transformation, with bases β for V, and γ for W, then $[T + U]_\beta^\gamma = [T]_\beta^\gamma + [U]_\beta^\gamma$. Explain the significance of this equality. Include a description of what "+" means on each side of the equation. *(Theorem 2.8a)*

48. Compute the matrix representations of various linear transformations between the vector spaces $M_{2\times 2}(F)$, $\mathrm{P}_2(\mathbb{R})$, and F with respect to their standard bases. *(Exercises 4 and 5)*

49. Prove that if $T\colon V \to V$ is linear and $W \subseteq V$ is a T-invariant subspace of dimension k, then there exists a basis β of V such that $[T]_\beta = \left(\begin{smallmatrix} A & B \\ O & C \end{smallmatrix}\right)$ for some matrices A, B, and C where A is size $k \times k$ and O is a matrix of zeros. *(Exercise 11)*

50. Prove that if $T, U\colon V \to W$ are nonzero linear transformations such that $R(T) \cap R(U) = \{0\}$, then $\{T, U\}$ is a linearly independent subset of $\mathcal{L}(V, W)$. *(Exercise 13)*

2.3 Composition of Linear Transformations and Matrix Multiplication

51. Suppose $T, U\colon \mathbb{R}^2 \to \mathbb{R}^2$ are linear transformations defined by $T(v_1, v_2) = (av_1 + bv_2, cv_1 + dv_2)$ and $U(w_1, w_2) = (ew_1 + fw_2, gw_1 + hw_2)$. If β is the standard basis for $\mathbb{R}^2$, find $[T]_\beta$ and $[U]_\beta$. Then find a formula for $(UT)(v_1, v_2)$, where $UT = U \circ T$, and use it to find $[UT]_\beta$. Explain how we would have to define matrix multiplication if we want to have $[UT]_\beta = [U]_\beta [T]_\beta$. *(Page 87)*

52. Matrix multiplication isn't all that nice. It isn't commutative, cancellation laws do not hold, and there are zero divisors (nonzero elements that multiply to zero). Give examples with matrices in $M_{2\times 2}(\mathbb{R})$ to illustrate each failed property. *(Page 90)*

53. Suppose $T\colon V \to W$, where V and W have bases $\beta = \{\beta_1, \dots, \beta_n\}$ and $\gamma = \{\gamma_1, \dots, \gamma_n\}$, respectively. Prove with a direct calculation (with coordinates/coefficients, *not* like in the book) that for all $x \in V$, $[T(x)]_\gamma = [T]_\beta^\gamma [x]_\beta$. *(Page 91)*

54. Prove that every linear transformation $T\colon F^n \to F^m$ is the same map as the left-multiplication transformation of its matrix representation $[T]_\beta^\gamma$. *(Theorem 2.15d)*

55. Verify for a particular pair of linear transformations $T\colon \mathrm{P}_2(\mathbb{R}) \to \mathrm{P}_2(\mathbb{R})$ and $U\colon \mathrm{P}_2(\mathbb{R}) \to \mathbb{R}^3$, with the standard bases, that the matrix for the composition UT is the product of the matrices for U and T, and that the coordinate vector for $U(y)$ is the product of the matrix for U and the coordinate vector for y. *(Exercise 3)*

56. Prove that $T^2 = T_0$ if and only if $R(T) \subseteq N(T)$. *(Exercise 11)*

57. Prove that if $\mathrm{rank}(T) = \mathrm{rank}(T^2)$, then $R(T) \cap N(T) = \{\mathbf{0}\}$, and therefore $V = R(T) \oplus N(T)$. (Hint: The fact about ranks says something about the null spaces of T and T^2.) Then conclude that for *any* linear T, there exists some positive integer k for which $V = R(T^k) \oplus N(T^k)$. *(Exercise 16)*

2.4 Invertibility and Isomorphisms

58. Suppose $T\colon V \to W$ is invertible. Use the definitions of matrix inverses and map inverses, respectively, to prove that $([T]_\beta^\gamma)^{-1} = [T^{-1}]_\gamma^\beta$, and that $(L_{[T]_\beta^\gamma})^{-1} = L_{([T]_\beta^\gamma)^{-1}}$. *(Pages 101 and 102)*

59. Prove that finite-dimensional vector spaces are isomorphic if and only if they have the same dimension. *(Theorem 2.19)*

60. Prove that if $\dim(V) = n$ and $\dim(W) = m$ then $\mathcal{L}(V, W) \approx M_{m\times n}(F)$. *(Theorem 2.20)*

61. Prove that if β is a basis for V, then the map $\phi_\beta\colon V \to F^n$ defined by $\phi_\beta(x) = [x]_\beta$ is an isomorphism. *(Theorem 2.21)*

62. Verify that $\phi_\gamma T = L_{[T]_\beta^\gamma}\phi_\beta$ in the case where $T\colon P_3(\mathbb{R}) \to P_2(\mathbb{R})$ is defined by $T(f) = f'$ and β and γ are the standard bases. *(Example 7)*

63. Determine whether various pairs of vector spaces are isomorphic, and when they are, construct explicit isomorphisms. *(Exercise 3)*

64. Prove that isomorphisms of vector spaces push subspaces to subspaces, and preserve the dimensions of these subspaces. *(Exercise 17)*

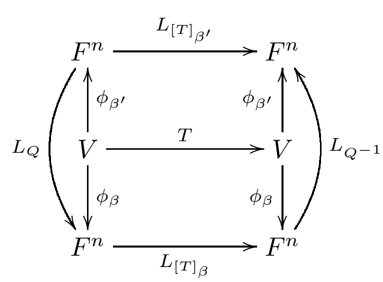
65. Do a commutative diagram chase to argue that if $T\colon V \to W$, then $\operatorname{rank}(T) = \operatorname{rank}(L_{[T]_\beta^\gamma})$ and $\operatorname{nullity}(T) = \operatorname{nullity}(L_{[T]_\beta^\gamma})$. *(Exercise 20)*

2.5 The Change of Coordinate Matrix

66. Consider the vector space $\mathbb{R}^2$ with bases $\beta = \{(1,1),(1,-1)\}$ and $\beta' = \{(2,4),(3,1)\}$. Find the matrix Q that changes β'–coordinates into β–coordinates. Then find the matrix P that changes β–coordinates into β'–coordinates, and verify that Q and P are inverses of each other. *(Example 1)*

67. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $T(a,b) = (3a - b, a + 3b)$, and let β and β' be the same bases of $\mathbb{R}^2$ as in the previous example (note we have already computed the change of coordinate matrices Q and Q^{-1}). Compute $[T]_{\beta'}$ and $[T]_\beta$ directly, and then verify that $[T]_{\beta'} = Q^{-1}[T]_\beta Q$. *(Example 2)*

68. The linear transformation analogue of Theorem 2.23 is: $L_{[T]_{\beta'}} = L_{Q^{-1}}L_{[T]_\beta}L_Q$. Consider the diagram below in the case where $V = P_2(\mathbb{R})$, $T(a + bx + cx^2) = 3a + 5bx - cx^2$, $\beta = \{1, x, x^2\}$, $\beta' = \{1-x, x-x^2, x^2\}$, and Q is the β'–to–β change of coordinate matrix. Verify that all four cycles in the diagram commute, by starting with a general element $a + bx + cx^2$ in the leftmost copy of V and pushing the element through all the arrows of the diagram. *(Pages 112–115)*



69. Given particular bases β and β' for $P_2(\mathbb{R})$, find the β'–to–β change of coordinates matrix Q by writing the elements of β' as linear combinations of elements of β. Then find the β–to–β' change of coordinates matrix P, and verify directly that Q and P are inverses. *(Exercise 3c)*

70. Given the linear operator $T(p(x)) = p'(x)$ on $P_1(\mathbb{R})$ and two particular bases β and β', use the matrix representation for T in terms of β and a change of coordinates matrix to quickly find the matrix representation for T in terms of β'. *(Exercise 5)*

71. Given a particular matrix A, find the matrix representation for L_A with respect to a given basis β directly. Then find $[L_A]_\beta$ another way, by finding a change of coordinates matrix Q and computing $Q^{-1}AQ$. *(Exercise 6c)*

72. Prove that similarity is an equivalence relation on matrices. *(Exercise 9 and Page 551)*

Chapter 3: Elementary Operations and Systems of Equations

3.1 Elementary Matrix Operations and Elementary Matrices

73. Verify that in $M_{2\times 3}(\mathbb{R})$, each of the following elementary operations is equivalent to multiplication by some elementary matrix on either the left or the right: $R_1 \leftrightarrow R_2$, $R_1 \to 2R_1$, $R_1 \to (R_1 + 3R_2)$, $C_1 \leftrightarrow C_3$, $C_2 \to -C_2$, and $C_1 \to (C_1 - C_3)$. *(Theorem 3.1)*

74. Quickly find the inverses of certain elementary matrices. *(Exercise 3)*

3.2 The Rank of a Matrix and Matrix Inverses

75. Prove that an $n \times n$ matrix is invertible if and only if its rank is n. *(Page 152)*

76. Prove that elementary row and column operations are rank-preserving. *(Theorem 3.4)*

77. Prove that the rank of a matrix is equal to the dimension of the subspace generated by its columns. *(Theorem 3.5)*

78. Prove that every invertible matrix is a product of elementary matrices. Then express a particular invertible matrix as a product of elementary matrices. *(Corollary 3, page 159 and Exercise 7)*

79. Prove directly that if M, A, and B are 2×2 matrices, $M(A \mid B) = (MA \mid MB)$. Is the same fact true for multiplication on the right by M? Use the fact just proved to explain why the algorithm $(A \mid I) \to (I \mid A^{-1})$ for finding inverses of matrices works. *(Page 161)*

80. Prove that for an $m \times n$ matrix A, $\text{rank}(A) = 0$ if and only if A is the zero matrix. *(Exercise 3)*

81. Compute the ranks and find the inverses (if they exist) of particular matrices and linear transformations. *(Exercises 5df, 6ab)*

Chapter 4: Determinants

4.4 Important Facts about Determinants

82. Prove the following three facts by induction, using the recursive definition of determinants (use $n = 2$ as your base case): *(Page 234)*

a) If B is obtained from A by switching two rows of A, then $\det B = -\det A$.

b) If B is obtained from A by multiplying a row of A by a scalar k, then $\det B = k \det A$.

c) If B is obtained from A by adding k times a row of A to antoher row of A, then $\det B = \det A$.

83. Prove the following two properties of the determinant directly, the first one in the 4×4 case, and the second in the 2×2 case. *(Page 235)*

a) If A is upper-triangular, then $\det A$ is the product of the diagonal entries of A.

b) $\det(AB) = (\det A)(\det B)$.

84. Use the properties proved in Questions 82 and 83 above to prove the following two additional properties of the determinant. *(Pages 235–236)*

a) A is invertible iff $\det A \neq 0$.

b) If A and B are similar, then they have the same determinant.

85. As we will see in the next section, the following three properties are crucial, defining properties of the determinant. Prove them in the $n \times n$ case using the recursive definition of determinant and any of the properties you proved in Questions 82–84 above. (Note: Don't worry about induction for the last property; just expand along the key row.) *(Theorem 4.3 on page 212, and page 235)*

a) *normalization*: $\det(I) = 1$.

b) *alternating*: if two rows of A are identical, then $\det A = 0$.

c) *n-linearity*: the determinant is linear on each row if the remaining rows are held fixed.

4.5 A Characterization of the Determinant

86. Prove that the determinant is the unique function $M_{n\times n}(F) \to F$ that is alternating, n-linear, and evaluates to 1 on the identity matrix. You may use any earlier results, but as usual, state the results you use rather than just referring to a previous theorem number. *(Theorem 4.12)*

Chapter 5: Diagonalization

5.1 Eigenvalues and Eigenvectors

87. Suppose $T: V \to V$ is a linear transformation, and β is a basis for V. Explain in detail why $[T]_\beta$ is diagonal if and only if T acts on each vector in β by scalar multiplication. *(Page 246)*

88. Prove that λ is an eigenvalue of $A \in M_{n\times n}(F)$ if and only if $\det(A - \lambda I_n) = 0$. *(Theorem 5.2)*

89. For a given linear operator T on $\mathrm{P}_2(\mathbb{R})$, calculate the characteristic polynomial of T in terms of the standard basis β of $\mathrm{P}_2(\mathbb{R})$ and then again in terms of the basis $\gamma = \{1 + x, 2x, x^2 - 1\}$, to verify that they are the same. *(Example 5)*

90. Suppose $T: V \to V$ is a linear operator and λ is an eigenvalue for T. Prove that $v \in V$ is an eigenvector of T with eigenvalue λ if and only if $v \neq 0$ and $v \in \mathrm{N}(T - \lambda I)$. *(Theorem 5.4)*

91. Use the commutative diagram on page 252 to prove that $v \in V$ is an eigenvector for T with eigenvalue λ if and only if $[v]_\beta \in F^n$ is an eigenvector for $L_{[T]_\beta}$ with eigenvalue λ. *(Pages 251–252)*

92. Prove that a square matrix $A \in M_{n\times n}(F)$ is similar to a diagonal matrix if and only if there exists a basis for F^n consisting of eigenvectors of A. *(Page 251 and Exercise 1h)*

93. Prove that similar matrices always have the same eigenvalues. *(Exercise 1i)*

94. For a particular matrix A, find the eigenvalues, the corresponding sets of eigenvectors, a basis consisting of eigenvectors (if possible) of A. Then find an invertible matrix Q and a diagonal matrix D such that $D = Q^{-1}AQ$, and verify directly that this equation holds. *(Exercise 3ad)*

95. For a particular linear operator T on $\mathrm{P}_1(\mathbb{R})$, find the eigenvalues of T and a basis β of eigenvectors for T. Then verify directly that $[T]_\beta$ is diagonal. *(Exercise 4d)*

96. Make a commutative diagram like the one shown in Question 68 on this handout to illustrate what is happening in the worked out example in the problem previous to this one. One basis will be the standard basis α and the other will be the basis β found in the problem. *(Question 68)*

97. Prove that a linear operator $T: V \to V$ is invertible if and only if it does not have 0 as an eigenvalue. Then prove that if T is invertible, then λ is an eigenvalue of T if and only if λ^{-1} is an eigenvalue of T^{-1}. *(Exericse 8)*

5.2 Diagonalizability

98. Prove that a set of eigenvectors corresponding to distinct eigenvalues must be linearly independent. What does this imply about an $n \times n$ matrix with n distinct eigenvalues? *(Theorem 5.5)*

99. Prove that if a linear operator T is diagonalizable, then its characteristic polynomial splits into (not necessarily distinct) linear factors. *(Theorem 5.6)*

100. Prove that the geometric multiplicity of an eigenvalue is always greater than or equal to one, and always less than or equal to the algebraic multiplicity of that eigenvalue. *(Theorem 5.7)*

101. Let $T\colon V \to V$ be a linear operator whose characteristic polynomial splits. Prove that T is diagonalizable if and only if each of its eigenvalues has full geometric algebraic multiplicity (equal to its algebraic multiplicity). Also prove that if T is diagonalizable, then the union of bases for the eigenspaces of T forms a basis for V. *(Theorem 5.9)*

102. Consider the matrices $A = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $B = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)$, $C = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$, and $D = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. For each matrix, find the characteristic polynomial, eigenvalues, and corresponding eigenspaces. Determine if the matrix is diagonalizable, and if so, find a basis of its eigenvectors for $M_{2\times 2}(\mathbb{R})$. Finally, illustrate geometrically on the Cartesian plane what the matrix does to each element of that basis of eigenvectors (or to the standard basis, if the matrix is not diagonalizable). *(Examples 3 and 4)*

103. Prove that $T\colon V \to V$ is diagonalizable if and only if its characteristic polynomial splits and for each eigenvalue λ of T, $\text{mult}(\lambda) = \dim(V) - \text{rank}(T - \lambda I)$. *(Page 269)*

104. Prove that a linear operator $T\colon V \to V$ is diagonalizable if and only if V is the direct sum of the eigenspaces of T. *(Theorem 5.11)*

105. Test a particular matrix A for diagonalizability by examining $\text{rank}(A - \lambda I)$ for each eigenvalue λ of A. *(Exercise 2ef)*

106. Show that a particular linear operator T on $\text{P}_2(\mathbb{R})$ is diagonalizable, and find a basis β such that $[T]_\beta$ is diagonal. Verify directly that T acts on each element of this basis by scalar multiplication by an eigenvalue. *(Exercise 3b)*

107. Suppose $W_1, W_2, \ldots, W_k$ are subspaces of V such that $V = \sum W_i$. Prove that $V = \oplus W_i$ if and only if $\dim V = \sum \dim W_i$. *(Exercise 20)*

5.4 Invariant Subspaces and the Cayley-Hamilton Theorem

108. For some linear operator T on some vector space V, give an example of a subspace $W \subseteq V$ that is T-invariant, and an example of a subspace $U \subseteq V$ that is not. Then prove that if λ is an eigenvalue of $T\colon V \to V$, then the eigenspace E_λ is a T-invariant subspace of V. *(Example 1)*

109. Fix some $x \in V$. Prove that the T-cyclic subspace of V generated by x is the smallest T-invariant subspace of V that contains x. *(Page 313)*

110. Suppose $W \subseteq V$ is a T-invariant subspace. Explain why this is exactly what is needed to ensure that the restriction $T_W\colon W \to W$ is well-defined. Then prove that in this situation, the characteristic polynomial of T_W divides the characteristic polynomial of T. *(Theorem 5.21)*

111. Prove that if the T-cyclic subspace of V generated by a nonzero $v \in V$ has dimension k, then it is generated by the basis $\{v, T(v), T^2(v), \ldots, T^{k-1}(v)\}$. *(Theorem 5.22a)*

112. Suppose $f(t)$ is the characteristic polynomial of a matrix A. Explain why the sentence "$f(A) = \det(A - AI) = \det(O) = 0$" does not make any sense (despite how tempting it looks as an explanation for the Cayley-Hamilton theorem). *(Theorem 5.23)*

113. Given a particular linear transformation on $\mathbb{R}^2$, verify that its matrix under the standard basis satisfies its characteristic polynomial. Then verify that the transformation itself satisfies its characteristic polynomial. *(Example 7)*

114. Suppose $T\colon V \to V$ is linear, and let W_1 and W_2 be T-invariant subspaces with bases β_1 and β_2. Define $\beta = \beta_1 \cup \beta_2$. Show that if $V = W_1 \oplus W_2$, then $[T]_\beta = [T_{W_1}]_{\beta_1} \oplus [T_{W_2}]_{\beta_2}$. *(Theorem 5.25)*

115. Let T be the linear operator on $\text{P}_4(\mathbb{R})$ defined by $T(a + bx + cx^2 + dx^3 + ex^4) = e + (a-b)x + dx^2 + (c+d)x^3 + ax^4$. By inspection, find T-invariant subspaces W_1 and W_2 so that $V = W_1 \oplus W_2$. Then verify that $[T]_\beta = [T_{W_1}]_{\beta_1} \oplus [T_{W_2}]_{\beta_2}$, where β_1 and β_2 are bases for W_1 and W_2 and $\beta = \beta_1 \cup \beta_2$. Finally, verify that the characteristic polynomial of T is the product of the characteristic polynomials for T_{W_1} and T_{W_2}. *(Example 8)*

116. For particular linear operators on $\mathbb{R}^4$ and $P_3(\mathbb{R})$, find an ordered basis for the T-cyclic subspace generated by a certain vector. *(Exercise 6ab)*

Chapter 7: Canonical Forms

7.1 The Jordan Canonical Form I

117. Suppose $T\colon V \to V$ is a linear operator and $\beta = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ is a basis of V so that $[T]_\beta$ has the form below.

$$[T]_\beta = \begin{pmatrix} 3 & & & & & \\ & 3 & & & & \\ & & 2 & 1 & & \\ & & & 2 & 1 & \\ & & & & 2 & \\ & & & & & 2 \end{pmatrix}$$

List all the eigenvalues of T and their multiplicities. Find $T(v_i)$ for each $1 \le i \le 6$, and determine which of the basis vectors v_i are eigenvectors for T. Identify the remaining basis vectors as generalized eigenvectors for some eigenvalue, and explain your answer. Find all of the eigenspaces of T and the sum of their dimensions, and then find all of the generalized eigenspaces of T and the sum of their dimensions. What can you conclude about $\oplus E_\lambda$? About $\oplus K_\lambda$? *(Example 1 and pages 484–485)*

118. Suppose T is linear and λ is an eigenvalue of T. Prove that the generalized eigenspace K_λ is a subspace, is T-invariant, and contains E_λ. *(Theorem 7.1a)*

119. Suppose that the characteristic polynomial of T splits, and that λ is an eigenvalue of T with multiplicity m, and $\dim K_\lambda = d$. Prove that the characteristic polynomial of T_{K_λ} is $(-1)^d(t-\lambda)^d$, and use this fact to prove that $\dim(K_\lambda) \le m$ and $K_\lambda = \mathrm{N}((T-\lambda I)^m)$. *(Theorem 7.2)*

120. Suppose that the characteristic polynomial of T splits as $(-1)^n(t-\lambda_1)^{m_1}\cdots(t-\lambda_k)^{m_k}$. Let $\beta_1, \ldots, \beta_k$ be bases for the generalized eigenspaces $K_{\lambda_1}, \ldots, K_{\lambda_k}$. Theorem 7.3 says that $V = \sum K_{\lambda_i}$, and Theorem 7.2 says that $\dim(K_{\lambda_i}) \le m_i$. Prove that in fact the following stronger assertions are true: $V = \oplus K_{\lambda_i}$ and $\dim(K_{\lambda_i}) = m_i$. *(Theorem 7.4)*

121. Suppose λ is an eigenvalue of $T\colon V \to V$, x is a generalized eigenvector of T for the eigenvalue λ, and γ is the cycle of generalized eigenvectors for T corresponding to λ and x. Show that $W = \mathrm{span}(\gamma)$ is T-invariant, and that the matrix $[T_W]_\gamma$ has the form of a Jordan block. Conclude that if β is a basis for V that is a disjoint union of cycles as above, then $[T]_\beta$ is a matrix in Jordan canonical form. *(Theorem 7.5)*

122. Suppose T is a linear operator and $\beta = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$ is a Jordan canoical basis for V such that $[T]_\beta$ is of the form shown below.

$$[T]_\beta = \begin{pmatrix} -2 & 1 & & & & & \\ & -2 & & & & & \\ & & -2 & & & & \\ & & & -2 & & & \\ & & & & 4 & 1 & \\ & & & & & 4 & 1 \\ & & & & & & 4 \end{pmatrix}$$

Write the Jordan canonical basis β in such a form that the cycles of generalized eigenvectors are clear; that is, express each v_i that is not the base of a cycle in the form $(T-\lambda I)^k(v_j)$ for some λ, k, and j. Also, for each element v_i with corresponding eigenvalue λ, determine the smallest positive integer k so that $v_i \in N((T-\lambda I)^k)$. *(Pages 488-489)*

123. Prove that if the characteristic polynomial of a linear operator T splits, then T has a Jordan canonical form. *(Corollary 1 on page 491 and Exercise 13)*

124. For two particular matrices and one linear transformation, find the Jordan canonical form and a corresponding basis of cycles of generalized eigenvectors. For each matrix A with Jordan canonical form J, verify directly that $Q^{-1}AQ = J$. *(Exercises 2a, 2d, 3a)*

7.2 The Jordan Canonical Form II

125. Write down labeled dot diagrams for the matrices in Jordan canonical form shown in Questions 117 and 122 above. *(Page 498)*

126. Suppose T is a linear transformation with eigenvalue λ, and that $\text{nullity}(T - \lambda I) = 5$, $\text{nullity}((T-\lambda I)^2) = 9$, $\text{nullity}((T-\lambda I)^3) = 11$, $\text{nullity}((T-\lambda I)^4) = 13$, and $\text{nullity}((T-\lambda I)^5) = 14$. Make the dot diagram for λ, and the Jordan canonical form for the blocks of the matrix for T that are associated to λ. *(Theorem 7.9)*

127. Prove that two matrices A and B are similar if and only if they have the same Jordan canonical form. *(Theorem 7.11)*

128. Find the Jordan canonical form that corresponds to a particular dot diagram. As part of your work, list the nullities of $N((T - \lambda I)^k)$ for all relevant λ and k. *(Exercise 2)*

129. Given only the Jordan canonical form of a particular linear operator T, find the characteristic polynomial, dot diagrams, and various other information about T. *(Exercise 3)*

130. For a particular matrix A, find a Jordan canonical form J and an invertible matrix Q such that $J = Q^{-1}AQ$. Verify directly that $J = Q^{-1}AQ$ to check your answer. *(Exercise 4bd)*

131. For a particular linear operator T, find a Jordan canonical form J and a Jordan canonical basis β so that $[T]_\beta = J$. Verify directly that $[T]_\beta = J$ to check your answer. *(Exercise 5b)*

132. Find the Jordan canonical form J of the matrix A given below, as well as a basis of cycles of generalized eigenvectors that corresponds to a diagonal matrix Q such that $Q^{-1}AQ = J$. Be sure to check your answer by verifying this matrix equation (for which you can use a computer algebra program if you like). *(Example 2)*

$$A = \begin{pmatrix} 2 & 1 & 0 & 0 & 2 & 0 & 0 & 0 & -2 & 0 \\ 0 & 2 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & -1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

You may find that you need the following matrices, provided here so that you do not have to compute them:

$$A - 2I = \begin{pmatrix} 0 & 1 & 0 & 0 & 2 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad (A - 2I)^2 = \begin{pmatrix} 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$(A - 2I)^3 = \begin{pmatrix} 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad (A - 2I)^4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$