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The Nash sheaf of a complete resolution

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Abstract. Every three-dimensional complex algebraic variety with isolated singular point has a resolution factoring through the Nash blowup and the blowup of the maximal ideal over which the second Fitting ideal sheaf is locally principal. In such resolutions one can construct Hsiang-Pati coordinates and thus obtain generators for the Nash sheaf that are the differentials of monomial functions.

Key words. Nash sheaf – complete resolution – Hsiang-Pati coordinates – monomial generators – Fitting ideal

1. Introduction

Given a complex analytic three-dimensional variety with isolated singular point, we show that there exists a resolution over which "Hsiang-Pati coordinates" can be constructed. These Hsiang-Pati coordinates induce generators of the Nash sheaf that can be written locally as the differentials of certain monomial functions whose exponents satisfy various ordering and linear independence conditions. We call such generators "monomial generators" for the Nash sheaf.

In fact these results hold for any resolution factoring through the Nash blowup and the blowup of the maximal ideal sheaf over which a certain Fitting ideal is locally principal (we call such resolutions "complete", since no further blowups will be necessary to construct Hsiang-Pati coordinates). The existence of a resolution satisfying the third condition is nontrivial and proved here using a theorem of Hironaka's from [4].

The monomial functions that induce monomial generators for the Nash sheaf are obtained by extracting distinguished monomial components of so-called "Hsiang-Pati" coordinates on a complete resolution. The Hsiang-Pati coordinates are the pullbacks of a generic choice of linear functions that satisfy various minimality conditions involving the Nash sheaf and its exterior powers and the inverse image of the maximal ideal sheaf.

Hsiang and Pati showed in [6] that, in the case where V is an analytic surface with isolated singular point, these coordinates can be obtained by

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repeatedly blowing up V and taking appropriate changes of coordinates as necessary. Pati generalizes this result to the three-dimensional case in [14]. In these papers, however, it is neither clear what resolution will eventually be sufficient for these coordinates to be in the correct form, nor how these coordinates relate to the Nash sheaf.

Pardon and Stern give a more conceptual, geometric view of this process (in the two-dimensional case) in [12], showing that instead of repeated blowups we can take any resolution that factors through the Nash blowup and the blowup of the maximal ideal. In such a resolution a generic choice of linear functions that satisfy certain conditions involving the maximal ideal and the Nash sheaf will pull up to be Hsiang-Pati coordinates.

The results here are thus a generalization of the results of Pardon and Stern to the three-dimensional case. In other words, we provide a more conceptual view of Pati's three-dimensional results (a view which in addition seems to have a clear generalization to the n-dimensional case). In the three-dimensional case a complete resolution must satisfy the same conditions as in Pardon and Stern's two-dimensional case, with the additional property that a particular Fitting ideal must be locally principal. This additional property makes the problem of finding a complete resolution nontrivial in the three-dimensional case.

Note that in [6] and [14] the resolution is obtained by repeated blowups as become necessary in the process of constructing the Hsiang-Pati coordinates. Here (as in [12]) we choose *from the outset* a resolution that is already sufficiently fine for the construction of these coordinates. In fact Hsiang-Pati coordinates exist on a resolution if and only if the resolution is complete.

In Section 4 we set notation and state our main theorems and definitions. Theorem 1 states that a complete resolution always exists. Theorem 2 states that given a complete resolution we can find monomial generators for the Nash sheaf. This second theorem will be a simple corollary of Theorem 3, which states that a generic choice of linear functions will pull up to be Hsiang-Pati coordinates on a complete resolution. These theorems are proved in Sections 5 and 6, respectively. Section 2 below describes possible applications of these monomial generators and future directions in research.

2. Applications

The existence of monomial generators for the Nash sheaf has applications in L_2-cohomology and analysis on singular spaces. In the three-dimensional case Pati uses Hsiang-Pati coordinates to prove that the trace of the heat operator is finite, and in fact satisfies a certain bound in [13] (although this theorem has since been proved in greater generality, by different methods, in [10]). In the two-dimensional case, Pardon and Stern use these generators in [12] to describe the cohomological Hodge structure on the L^2-cohomology of an algebraic surface with isolated singular points in terms of local cohomology groups obtained from a resolution. Pardon also uses

Hsiang-Pati coordinates to examine $L^2 - \bar{\partial}$ cohomology groups (again in the two-dimensional case) in [11]. Perhaps these results can be extended to the three-dimensional case using the Hsiang-Pati coordinates and monomial generators constructed here.

Moreover, the monomial generators for the Nash sheaf allow the construction of an exact sequence that (partially) describes the Nash sheaf in terms of the resolution data (divisors defined from the monomial generators). This exact sequence can be used to describe some of the Chern classes of the Nash sheaf (and thus some of the Mather-Chern classes of the original variety).

The construction of Hsiang-Pati coordinates developed here in the three-dimensional case should be generalizable to the n-dimensional case, where in the more general case a complete resolution will factor through the blowups of a *series* of Fitting ideal sheaves. The exponents of the Hsiang-Pati coordinates in this n-dimensional case will satisfy linear independence and ordering conditions similar to those in the three-dimensional case. The choice of "Nash minimal" functions will involve choosing $\phi_1, \ldots, \phi_n$ so that $d\phi_1 \cdots d\phi_n$ is a minimal generator of $\Lambda^k \mathcal{N}$ for $2 \leq k \leq n-1$. The non-trivial part of the n-dimensional case involves showing that a complete resolution exists. Some of the results above appear in [15], while others are in development and will appear in a future paper.

3. Notation

Let V be a three-dimensional complex algebraic variety with isolated singular point v, and let U be a neighborhood of v in V with an embedding $(U, v) \subseteq (\mathbb{C}^N, 0)$. Let $\pi\colon (\widetilde{U}, E) \to (U, v)$ be a resolution of the singularity v with exceptional divisor E.

The *Nash blowup* $\widehat{U}$ is defined to be the closure of the image of the section $\sigma\colon U - v \to Gr_n(T\mathbb{C}^N)$ of the Grassmann bundle that sends each point of $U - v$ to its tangent space. The Nash blowup comes equipped with a projection $\widehat{\pi}\colon \widehat{U} \to U$ that is the restriction of the projection from the Grassmann bundle. The Nash blowup can also be characterized as the blowup of the sheaf of 1-forms Ω^1_U (see [9] and [2]).

The *Nash bundle* $\nu\colon \mathfrak{N} \to \widehat{U}$ over the Nash blowup $\widehat{U}$ is defined as the restriction of the universal subbundle of $Gr_n(T\mathbb{C}^N)$ to $\widehat{U}$. The *Nash sheaf* $\mathcal{N}$ can then be defined as the sheaf of sections of the dual of the Nash bundle (via the correspondence between vector bundles and locally free sheaves). Note that if $\widehat{E} := \widehat{\pi}^{-1}(v)$, then since as vector bundles we have $\mathfrak{N}|_{\widehat{U}-\widehat{E}} \approx T(\widehat{U} - \widehat{E})$, as locally free sheaves we have $\mathcal{N}|_{\widehat{U}-\widehat{E}} \approx \Omega^1_{\widehat{U}-\widehat{E}}$.

Equivalently, if we think of the Nash blowup as the blowup of the sheaf of 1-forms Ω^1_U, then we can define the Nash sheaf $\mathcal{N}$ to be the locally free sheaf

$$\mathcal{N} := \widehat{\pi}^* \Omega^1_U / \mathrm{Torsion}(\widehat{\pi}^* \Omega^1_U) \approx \gamma^* \mathcal{Q},$$

where $\mathcal{Q}$ is the universal quotient sheaf on $Gr(N-n,N)$ and γ is the canonical map $\gamma\colon \widehat{U} \hookrightarrow Gr(N-n,N)$.

We will call a sheaf $\mathcal{N}$ on a blowup $\pi\colon \widetilde{U} \to U$ a *generalized Nash sheaf* (although we will often say simply "Nash sheaf") if $\widetilde{U}$ factors through the Nash blowup $\widehat{U}$ of U, and if $\mathcal{N}$ is the pullback of the Nash sheaf on $\widehat{U}$. Note that $\mathcal{N}$ is a subsheaf of $\Omega^1_{\widetilde{U}}$ and that the canonical map $\pi^*\Omega^1_U \to \Omega^1_{\widetilde{U}}$ factors through $\mathcal{N}$ (see the Appendix (A3) in [12]).

We now define the Fitting ideal sheaf $\mathcal{F}$ that is integral to our notion of a complete resolution. Given a coherent sheaf $\mathcal{G}$ on a complex algebraic variety V, and an open affine set $U \subset V$, let

$$\mathcal{O}_U^m \xrightarrow{\phi} \mathcal{O}_U^n \to \mathcal{G}|_U \to 0$$

be any local finite presentation of $\mathcal{G}$ over U. Choose bases for the free modules $\mathcal{O}_U^m$ and $\mathcal{O}_U^n$, and consider the matrix of the morphism ϕ with respect to those bases. We define the Fitting ideal $\mathcal{F}_i$ of $\mathcal{G}$ to be the sheaf of ideals locally generated by the $i \times i$ subdeterminants of the matrix for ϕ. This local definition patches together to form a coherent sheaf of ideals on $\mathcal{O}_V$ (see [8]; note that his definition of a Fitting invariant involves the $(n-i)\times(n-i)$ subdeterminants).

Now consider V to be a three-dimensional complex algebraic variety with isolated singular point v, and let U be an affine neighborhood (in some $\mathbb{C}^N$) of v in V. Let $\pi\colon (\widetilde{U},E) \to (U,v)$ be a resolution factoring through the Nash blowup of U. We will be concerned with one of the Fitting ideal sheaves corresponding to the inclusion $\eta\colon \mathcal{N}_{\widetilde{U}} \hookrightarrow \Omega^1_{\widetilde{U}}(\log E)$ of the (generalized) Nash sheaf $\mathcal{N}$ into the sheaf of logarithmic 1-forms (*i.e.* the sheaf of 1-forms with logarithmic singularities along the exceptional set E). Note that this inclusion is the composition of the inclusion $\mathcal{N}_{\widetilde{U}} \hookrightarrow \Omega^1_{\widetilde{U}}$ arising from the construction of the Nash sheaf with the inclusion $\Omega^1_{\widetilde{U}} \hookrightarrow \Omega^1_{\widetilde{U}}(\log E)$ of the sheaf of 1-forms into the sheaf of logarithmic 1-forms. Consider the presentation

$$\mathcal{N}_{\widetilde{U}} \xrightarrow{\eta} \Omega^1_{\widetilde{U}}(\log E) \to \Omega^1_{\widetilde{U}}(\log E)/\mathcal{N}_{\widetilde{U}} \to 0.$$

The second Fitting ideal $\mathcal{F}_2$ of the cokernel sheaf $\Omega^1_{\widetilde{U}}(\log E)/\mathcal{N}_{\widetilde{U}}$ is the sheaf of ideals generated by the 2×2 subdeterminants of the matrix for η (given some bases for $\mathcal{N}_{\widetilde{U}}$ and $\Omega^1_{\widetilde{U}}(\log E)$). We will call this Fitting ideal simply $\mathcal{F}$, and refer to it as the *second Fitting ideal of the Nash sheaf*.

Given V, v, and U as above, and a resolution $\pi\colon (\widetilde{U},E) \to (U,v)$, we use the sheaves $\mathfrak{m}$, $\mathcal{N}$, and $\mathcal{F}$ to determine if the resolution is *complete*, as follows.

Definition 1. $\pi\colon (\widetilde{U},E) \to (U,v)$ *is a* complete *resolution if* $\mathfrak{m}$ *is locally principal,* $\mathcal{N}$ *is locally free, and* $\mathcal{F}$ *is locally principal over* $\widetilde{U}$.

We choose to call such resolutions *complete* because they are blown up just "enough" to allow the construction of so-called Hsiang-Pati coordinates. Once a complete resolution is obtained, no further transforms need be taken to construct these Hsiang-Pati coordinates.

4. Results

The existence of a resolution satisfying the first two conditions of completeness is trivial; simply take any resolution factoring through the Nash blowup and the blowup of the maximal ideal sheaf $\mathfrak{m}_v$ (see [16]). We will call any resolution satisfying these first two conditions a *base resolution*. The existence of a resolution that in addition satisfies the third condition is nontrivial and is our first main theorem (which we will prove in Section 5).

Theorem 1. *Given a three-dimensional complex algebraic variety V with isolated singular point v and a sufficiently small affine neighborhood U of v in V, there exists a complete resolution $\pi\colon (\widetilde{U}, E) \to (U, v)$.*

For our second result we must pass to the analytic category, although we will use the same notation. Given a complete resolution $\pi\colon (\widetilde{U}, E) \to (U, v)$, let $e \in E$ be a point of the exceptional divisor, and let W be an analytic neighborhood of e in $\widetilde{U}$. We say the point e is a *triple point* of E if it lies in the intersection of three components $E_1 \cap E_2 \cap E_3$ of the exceptional divisor. A *double point* $e \in E$ is a point on the intersection of exactly two components $E_1 \cap E_2$, and a *simple point* is a point $e \in E$ that lies in exactly one component E_1 of E.

Given a triple point e we can choose coordinates $\{u, v, w\}$ on W so that the three components of E meeting at e are given by $E_1 = \{u = 0\}$, $E_2 = \{v = 0\}$, and $E_3 = \{w = 0\}$. Similarly if e is a double point we can choose coordinates so that E_1 and E_2 are given by the vanishing of u and v; and if e is a simple point we can choose coordinates so that $E_1 = \{u = 0\}$. We will call coordinates satisfying these conditions *divisor coordinates*.

Before stating the second important theorem we need the following rather technical definition:

Definition 2. *An ordered set of three ordered triples of integers*

$$\{(m_1, m_2, m_3), (n_1, n_2, n_3), (p_1, p_2, p_3)\}$$

is called Hsiang-Pati ordered *for a point $e \in E$ if they satisfy:*

1. *If e is a double point, then either $m_3 = n_3 = 0$ and $p_3 = 1$, or $m_3 = p_3 = 0$ and $n_3 = 1$; and if e is a simple point, then $m_2 = m_3 = 0$, $n_2 = 1$, $p_2 = 0$, $n_3 = 0$, and $p_3 = 1$;*
2. *$0 < m_l \leq n_l \leq p_l$ for $l = 1, 2, 3$ if e is a triple point, for $l = 1, 2$ if e is a double point, or for $l = 1$ if e is a simple point; and*
3. $\begin{vmatrix} m_1 & n_1 & p_1 \\ m_2 & n_2 & p_2 \\ m_3 & n_3 & p_3 \end{vmatrix} \neq 0.$

Our second theorem states that near any point e in the exceptional divisor of a complete resolution one can find local generators for the Nash sheaf that are the differentials of monomial functions whose exponents are Hsiang-Pati ordered for e.

Theorem 2. *Given a three-dimensional complex algebraic variety V with isolated singular point v, let U be an affine neighborhood of v in V with complete resolution $\pi\colon (\widetilde{U}, E) \to (U, v)$. Choose a point $e \in E$ and an analytic neighborhood W of e in $\widetilde{U}$. Then there exists a set of divisor coordinates $\{u, v, w\}$ on W so that the Nash sheaf $\mathcal{N}$ is locally generated by the differentials $d\phi$, $d\psi$, $d\rho$ of monomial functions of the form*

$$\phi = u^{m_1}v^{m_2}w^{m_3}, \quad \psi = u^{n_1}v^{n_2}w^{n_3}, \quad \rho = u^{p_1}v^{p_2}w^{p_3}$$

whose exponents $\{(m_1, m_2, m_3), (n_1, n_2, n_3), (p_1, p_2, p_3)\}$ are a Hsiang-Pati ordered set.

Definition 3. *Generators $d\phi$, $d\psi$, $d\rho$ for the Nash sheaf that satisfy the conditions listed in Theorem 2 will be called* monomial generators.

We will prove this theorem in Section 6. The monomial generators in Theorem 2 will arise directly from the so-called Hsiang-Pati coordinates constructed in Theorem 3 below.

Definition 4. *Given a triple $\{j, k, l\}$ of linear functions on $\mathbb{C}^N \subseteq U$, and a complete resolution $\pi\colon (\widetilde{U}, E) \to (U, v)$, let $\phi := j \circ \pi$, $\psi := k \circ \pi$, and $\rho := l \circ \pi$. Let e be a point in E with analytic neighborhood W. The triple $\{\phi, \psi, \rho\}$ of functions on $\widetilde{U}$ is* Nash-minimal *(with respect to e) if:*

1. *ϕ is a generator for $\mathfrak{m}(W)$;*
2. *$\{d\phi, d\psi, d\rho\}$ is a generating set for $\mathcal{N}(W)$; and*
3. *$d\phi\, d\psi$ is a minimal element of $\Lambda^2\mathcal{N}(W)$.*

The following propositions state that we can generically choose linear functions that pull up to have the Nash-minimality properties above. To get the first two properties we require only a base resolution; the third property requires that we have a complete resolution. We state and prove these two facts separately since the former is needed to *prove* the existence of a complete resolution, and the latter requires that we *have* a complete resolution.

Proposition 1. *Let $\widehat{\pi}\colon (\widehat{U}, \widehat{E}) \to (U, v)$ be any base resolution, and let $\hat{e}$ be a point of the exceptional divisor $\widehat{E}$. Then a generic choice of linear functions j, k, and l on $\mathbb{C}^N \supseteq U$ will pull up by $\widehat{\pi}$ to functions on $\widehat{U}$ that satisfy conditions (1) and (2) of Nash-minimality with respect to $\hat{e}$.*

Proposition 2. *If $\pi\colon (\widetilde{U}, E) \to (U, v)$ is a complete resolution, and e is any point in E, then a generic choice of linear functions j, k, and l on U will pull up to Nash-minimal functions with respect to e on $\widetilde{U}$.*

We will prove Proposition 1 in Section 5.1. Proposition 2 will be proved in Section 6.1. Theorem 3 uses Nash-minimal functions to construct so-called Hsiang-Pati coordinates on the Nash sheaf of a complete resolution, and will be proved in Sections 6.2 and 6.3.

Theorem 3. *Given a three-dimensional complex algebraic variety V with isolated singular point v, let U be an affine neighborhood of v in V with complete resolution $\pi\colon (\widetilde{U}, E) \to (U, v)$. Choose a point $e \in E$ and an analytic neighborhood W of e in $\widetilde{U}$. Let $\{\phi, \psi, \rho\}$ be a Nash-minimal set of functions with respect to the point e. Then there exists a set of divisor coordinates $\{u, v, w\}$ on W and a set $\{(m_1, m_2, m_3), (n_1, n_2, n_3), (p_1, p_2, p_3)\}$ of Hsiang-Pati ordered integers such that:*

1. $\phi = u^{m_1} v^{m_2} w^{m_3}$;
2. $\psi = S + \psi'$, *where*
   (a) $S = \sum s_l \phi^{\epsilon_l}$ *with each* $\epsilon_l \geq 1$ *and rational; and*
   (b) $\psi' = u^{n_1} v^{n_2} w^{n_3}$;
3. $\rho = T + \rho'$, *where*
   (a) $T = \sum t_l \phi^{\delta_l} (\psi')^{\tau_l}$ *with* $\delta_l \geq 1$ *when* $\tau = 0$ *and* $\delta_l m_1 + \tau_l n_1 \geq n_1$ *when* $\tau \neq 0$ *(and similarly for* m_2 *and* n_2 *and for* m_3 *and* n_3 *if* e *is a triple point, or for* m_2 *and* n_2 *if* e *is a double point); and*
   (b) $\rho' = u^{p_1} v^{p_2} w^{p_3}$.

The ϕ, ψ, and ρ are called *Hsiang-Pati coordinates* on $\widetilde{U}$ (see [6], [14], and [12]). The conditions on S and T ensure that if ϕ, ψ, and ρ generate $\mathcal{N}(W)$, then so will ϕ, ψ', and ρ'. Thus the monomials ϕ, ψ', and ρ' in Theorem 3 will be the ϕ, ψ, and ρ for Theorem 2 (so Theorem 2 is in fact a simple corollary of Theorem 3).

5. Constructing a complete resolution

Let $\widehat{\pi}\colon (\widehat{U}, \widehat{E}) \to (U, v)$ be a base resolution for the singularity v. Over such a resolution the generalized Nash sheaf $\widehat{\mathcal{N}}$ is locally free and the inverse image $\widehat{\mathfrak{m}}$ of the maximal ideal $\mathfrak{m}_v$ is locally principal (see [12] and [3]).

Let $\widehat{\mathcal{F}}$ be the second Fitting ideal sheaf for the inclusion $\widehat{\mathcal{N}} \hookrightarrow \widehat{\Omega}^1(\log \widehat{E})$ of the Nash sheaf into the sheaf of logarithmic 1-forms on $(\widehat{U}, \widehat{E})$. It would be trivial to find a further resolution $\widetilde{\pi}\colon \widetilde{U} \to \widehat{U}$ with a locally principal second Fitting ideal $\mathcal{F}$ if it were true that under further resolutions the inverse image $\widetilde{\pi}^{-1}\widehat{\mathcal{F}}$ was equal to the Fitting ideal $\mathcal{F}$ of $\widetilde{U}$ (we could then take any resolution factoring through the blowup of $\widehat{\mathcal{F}}$). Unfortunately this is not generally the case.

Obtaining a resolution $\widetilde{U}$ for which the Fitting ideal is locally principal will consist of three steps. First, we show that the Fitting ideal of any base resolution is locally principal at simple points of the exceptional divisor (Proposition 3). Second, we show that blowups $\widetilde{\pi}\colon \widetilde{U} \to \widehat{U}$ of certain curves and points that do not intersect the simple point set "preserve" the Fitting ideal, in the sense that the inverse image $\widetilde{\pi}^{-1}\widehat{\mathcal{F}}$ is equal to the Fitting ideal $\mathcal{F}$ on $\widetilde{U}$ (Proposition 4). Finally we show that, after a preliminary blowup $\breve{U}$, there exists a finite sequence of such Fitting ideal preserving blowups that results in a resolution $\widetilde{U}$ of $\breve{U}$ that factors through the blowup of the

Fitting ideal $\breve{\mathcal{F}}$ (Propositions 5 and 6). On such a resolution $\widetilde{U}$, the Fitting ideal $\mathcal{F}$ will be equal to the locally principal inverse image of $\breve{\mathcal{F}}$.

5.1. Proof of Proposition 1

Since we will be needing Proposition 1 in the proofs of Propositions 3 and 4, we prove it now.

Proof (Proposition 1). Suppose $\widehat{\pi}\colon (\widehat{U}, \widehat{E}) \to (U, v)$ is any base resolution, and choose any point $\hat{e}$ in $\widehat{E}$. Let W be an analytic neighborhood of $\hat{e}$ in $\widehat{U}$. It suffices to prove that we can make generic choices of linear functions j, k, l, and h so that $h \circ \widehat{\pi}$ generates $\mathfrak{m}(W)$, $\{d(j \circ \widehat{\pi}), d(k \circ \widehat{\pi}), d(l \circ \widehat{\pi})\}$ is a generating set for $\mathcal{N}(W)$, and $D_3 \subset D_1$ where $D_1 := \ker(h)$ and $D_3 := \ker(j) \cap \ker(k) \cap \ker(l)$. The inclusion $D_3 \subset D_1$ will allow us to choose $h = j$. For notational simplicity we will drop the W qualifier from our notation in this proof, although everything will take place in the neighborhood W of $\hat{e}$.

Let $\mathcal{F}$ be a coherent sheaf of rank r, and suppose $\bar{\pi}\colon (\bar{U}, \bar{E}) \to (U, v)$ is the blowup of U relative to $\mathcal{F}$. From the construction of $\bar{U}$ we get a canonical map $\gamma\colon \bar{U} \to \mathrm{Gr}(N-r, N)$. Given a codimension r subspace $D_r \subseteq \mathbb{C}^N$, define the Schubert variety

$$S(D_r) := \{E^r \in \mathrm{Gr}(r, N) \mid \dim(E^r \cap D_r) \geq 1\}.$$

Note that $S(D_r)$ is the codimension 1 subset of $\mathrm{Gr}(r, N)$ consisting of the r-planes in $\mathbb{C}^N$ that are *not* transverse to D_r. Given any linear projection $p\colon \mathbb{C}^N \to \mathbb{C}^r$, define the codimension r plane $D_r := \ker p$. Let Υ be the universal subsheaf over $\mathrm{Gr}(r, N)$. The map p induces (see [12]) a trivialization of Υ over $\mathrm{Gr}(r, N) - S(D_r)$. We can pull this back to a trivialization of $\bar{\pi}^*\mathcal{F}/\mathrm{Tors}(\bar{\pi}^*\mathcal{F})$ over $\bar{U} - \gamma^{-1}S(D_r)$, since this is isomorphic to the pullback of the universal quotient sheaf over $\mathrm{Gr}(r, N)$ (note we are making use of the isomorphism between $\mathrm{Gr}(r, N)$ and $\mathrm{Gr}(N-r, N)$ under which the universal subsheaf is pulled back to the dual of the universal quotient sheaf). Moreover, if $\widehat{\pi}\colon \widehat{U} \to \bar{U}$ is a further blowup of $\bar{U}$, we can pull this trivialization up to a trivialization of $(\bar{\pi} \circ \widehat{\pi})^*\mathcal{F}/\mathrm{Tors}((\bar{\pi} \circ \widehat{\pi})^*\mathcal{F})$ over $\widehat{U} - (\gamma \circ \widehat{\pi})^{-1}S(D_r)$. We will say that a plane D_r is *arranged* relative to a point $\hat{e} \in \widehat{E}$ if $\widehat{\pi}(\hat{e})$ is not in $\gamma^{-1}S(D_r)$. The following lemma, from [12] (a consequence of Kleiman's transversality theorem in [7]), states that "arranged" planes D_r are generic:

Lemma 1. *For generic D_r in $\mathrm{Gr}(N-r, N)$, $\gamma^{-1}S(D_r) \cap \bar{E}$ is either empty or has codimension 1 in $\bar{E}$, and can be arranged to miss any finite set of points in $\bar{E}$.*

Suppose the base resolution $\widehat{\pi}$ factors as $\widehat{\pi} = \pi_0 \circ \widehat{\pi}_0$, where the map $\pi_0\colon (B(U), C) \to (U, v)$ is the blowup of the maximal ideal sheaf $\mathfrak{m}_v$ of the singularity v. Let $\gamma_0\colon B(U) \to \mathrm{Gr}(N-1, N)$ be the canonical map. By the discussion above, any linear projection $h\colon \mathbb{C}^N \to \mathbb{C}$ induces a trivialization of $\pi_0^*\mathfrak{m}_v/\mathrm{Tors}(\pi_0^*\mathfrak{m}_v) = \pi_0^{-1}(\mathfrak{m}_v)$ over $B(U) - \gamma_0^{-1}S(D_1)$. The trivialization in this case is given by the global section $h \circ \pi_0$. Pulling up to $\widehat{U}$, this means that $h \circ \widehat{\pi}$ generates $\widehat{\pi}^*\mathfrak{m}_v/\mathrm{Tors}(\widehat{\pi}^*\mathfrak{m}_v) = \widehat{\pi}^{-1}(\mathfrak{m}_v)$ over $\widehat{U} - (\gamma_0 \circ \widehat{\pi}_0)^{-1}S(D_1)$. Now given an $\hat{e} \in \widehat{E}$, we wish to choose h so that $h \circ \widehat{\pi}$ generates $\pi^{-1}(\mathfrak{m}_v)$ near $\hat{e}$; therefore we must choose h so that $\widehat{\pi}_0(\hat{e})$ is not in $\gamma_0^{-1}S(D_1)$. In other words, given a point $\hat{e}$, we must choose h so that the hyperplane $D_1 = \ker(h)$ is "arranged" relative to $\hat{e}$. Such a choice is possible and generic by Lemma 1.

Since $\widehat{\pi}$ is a base resolution, it also factors as $\widehat{\pi} = \pi_1 \circ \widehat{\pi}_1$, where $\pi_1 \colon (N(U), D) \to (U, v)$ is the Nash blowup of U. Since the Nash blowup can be obtained by blowing up the sheaf Ω_U^1 of 1-forms on U, any linear projection $(j,k,l) \colon \mathbb{C}^N \to \mathbb{C}^3$ induces a trivialization of $\pi_1^*\Omega_U^1/\mathrm{Tors}(\pi_1^*\Omega_U^1) = \mathcal{N}_{N(U)}$ over $N(U) - \gamma_1^{-1}S(D_3)$. Here $\gamma_1 \colon N(U) \to \mathrm{Gr}(N-3, N)$ is the canonical map and $D_3 := \ker(j) \cap \ker(k) \cap \ker(l)$. This pulls up to a trivialization of the generalized Nash sheaf $\widehat{\pi}^*\Omega_U^1/\mathrm{Tors}(\widehat{\pi}^*\Omega_U^1) = \mathcal{N}_{\widehat{U}}$ over $\widehat{U} - (\gamma_1 \circ \widehat{\pi}_1)^{-1}S(D_3)$. In other words, the projection (j,k,l) gives us a system of generators for $\mathcal{N}_{\widehat{U}}$ (namely $\{d(j\circ\pi), d(k\circ\pi), d(l\circ\pi)\}$) near any $\hat{e}$ where $\widehat{\pi}_1(\hat{e})$ is not in $\gamma_1^{-1}S(D_3)$. Given a point $\hat{e}$, by Lemma 1 we can generically choose j, k, and l so that D_3 is an "arranged" codimension three plane.

To show that we can choose j, k, l, and h so that D_3 is contained in D_1 we need the following simple fact concerning open sets of grassmannians:

Lemma 2. *Given any codimension k plane $D_k \in \mathrm{Gr}(N-k, N)$ and any neighborhood $\mathcal{G} \subset \mathrm{Gr}(N-k, N)$ of D_k, let $D_{k-i} \in \mathrm{Gr}(N-(k-i), N)$ be any codimension $(k-i)$ plane containing D_k. Then any codimension $(k-i)$ plane D'_{k-i} sufficiently close to D_{k-i} contains some $D'_k \in \mathcal{G}$.*

Choose any arranged D_3 (by choosing j, k, and l). Since arranged planes are generic there is an open set $\mathcal{G}$ of D_3 in $\mathrm{Gr}(N-3, N)$ consisting entirely of arranged codimension three planes. Now let H be *any* codimension one plane (not necessarily "arranged") containing D_3. Since "arranged" codimension one planes are generic there is a sequence of arranged planes $\{H_i\}$ converging to the chosen H. By Lemma 2, we can choose i sufficiently large so that H_i contains some $D'_3 \in \mathcal{G}$. Choose D_1 to be this H_i (thus choosing h) and then rechoose $j = h$, k, and l so that $\ker(j) \cap \ker(k) \cap \ker(l) = D'_3$ is the new D_3. Now we have "arranged" choices of D_1 and D_3 with $D_3 \subset D_1$. $\square$

5.2. The Fitting ideal at simple points

Our first lemma will show that the second Fitting ideal is always locally principal at simple points of the exceptional divisor. We will be working in the analytic topology until Proposition 4.

Proposition 3. *Let $(\widehat{U}, \widehat{E})$ be a base resolution with second Fitting ideal sheaf $\widehat{\mathcal{F}}$. Let $\hat{e}$ be a simple point of $\widehat{E}$, and let $\widehat{W}$ be an analytic neighborhood of $\hat{e}$ in $\widehat{U}$. Then $\widehat{\mathcal{F}}(\widehat{W})$ is principal.*

Proof. Use Proposition 1 to choose linear functions j, k, and l on $\mathbb{C}^N \supseteq U$ so that $\phi := j \circ \pi$, $\psi := k \circ \pi$, and $\rho := l \circ \pi$ that have properties (1) and (2) of Nash-minimality with respect to the simple point $\hat{e}$. Then $\{d\phi, d\psi, d\rho\}$ is a basis for $\widehat{\mathcal{N}}(\widehat{W})$. Let $\{u, v, w\}$ be a set of divisor coordinates about $\hat{e}$ in $\widehat{W}$, so that $\widehat{E} = \{u = 0\}$ in $\widehat{W}$. The standard basis for the sheaf $\widehat{\Omega}^1(\log \widehat{E})$ of logarithmic 1-forms in these coordinates is then $\{\frac{du}{u}, dv, dw\}$. With respect to these bases, $\widehat{\mathcal{F}}(\widehat{W})$ is generated by the 2×2 subdeterminants of the matrix

$$\begin{pmatrix} u\phi_u & u\psi_u & u\rho_u \\ \phi_v & \psi_v & \rho_v \\ \phi_w & \psi_w & \rho_w \end{pmatrix}.$$

We will rechoose divisor coordinates $\{u, v, w\}$ so that it is obvious that $\widehat{\mathcal{F}}(\widehat{W})$ is principal. This choice of $\{u, v, w\}$ will put ϕ, ψ, and ρ into a form similar to their form in the simple point case of Theorem 3 (and in fact this proof will comprise most of the proof of the simple point case of Theorem 3).

Since ϕ, ψ, and ρ are Nash-minimal, ϕ generates $\widehat{\mathfrak{m}}$ and thus vanishes only along $\{u = 0\}$. Thus $\phi = u^m \mu$ for some integer m and local unit μ. Change coordinates (by $u \mapsto u\mu^{-1/m}$) to absorb μ into u (note the new coordinates will still be divisor coordinates); then $\phi = u^m$.

A priori we can write ψ as a series $\psi = \sum_{(a,b,c)} s_{(a,b,c)} u^a v^b w^c$. Define S to be the sum of the terms in ψ for which $b = c = 0$, and let n be the minimum of the integers a for which (a, b, c) has some nonzero b or c. Then we have $\psi = S + u^n R$ for some function R that is not divisible by u. Since u does not divide R, the 2-form $du\, dR$ is nowhere-vanishing, and thus R is a coordinate independent of u. Therefore we can change coordinates by setting $v = R$ (note this does not affect ϕ or S), and we have $\psi = S + u^n v$. Note that $n > m$ because ϕ generates $\widehat{\mathfrak{m}}$ and thus ϕ divides ψ.

Similarly, we can write $\rho = \sum_{(\alpha,\beta,\gamma)} t_{(\alpha,\beta,\gamma)} u^\alpha v^\beta w^\gamma = T + u^p Q$, where T is the sum of the terms of ρ that are of the form $\phi^l (u^n v)^k$ for some l and k, and Q is a function not divisible by u. Since u does not divide Q, neither does it divide $du\, dv\, dQ$; hence $du\, dv\, dQ$ is a nowhere-vanishing 3-form on $\widehat{U}$. Thus Q is a coordinate independent of both u and v. Change coordinates once more by setting $w = Q$; we then have $\rho = T + u^p w$. Note that u^m divides both G and u^p.

Since $S_v = S_w = T_w = 0$, the second Fitting ideal $\widehat{\mathcal{N}}(\widehat{W})$ is generated by the 2×2 subdeterminants of the matrix

$$\begin{pmatrix} mu^m & uS_u + nu^n v & uT_u + pu^p w \\ 0 & u^n & T_v \\ 0 & 0 & u^p \end{pmatrix}.$$

By the definition of S and T, u^m divides uS_u and uT_u, and u^n divides T_v. Using these facts, a simple calculation of the ideal generated by the 2×2 subdeterminants of the matrix above reveals that the second Fitting ideal is the principal ideal $\widehat{\mathcal{N}}(\widehat{W}) = \langle u^{m+n} v^{m+p} \rangle$. $\square$

5.3. $\widehat{\mathcal{F}}$-preserving blowups

Our next lemma will show that certain blowups on a base resolution $\widehat{U}$ "preserve" the Fitting ideal; a further resolution $\widetilde{\pi}\colon (\widetilde{U}, E) \to (\widehat{U}, \widehat{E})$ is $\widehat{\mathcal{F}}$-*preserving* (near a point $e \in E$) if the second Fitting ideal $\widehat{\mathcal{F}}$ on $\widehat{U}$ pulls up to the second Fitting ideal $\widetilde{F}$ on $\widetilde{U}$, *i.e.* if $\widetilde{\pi}^{-1}\widehat{\mathcal{F}} = \widetilde{\mathcal{F}}$ (in a local neighborhood of $e \in E$). By Proposition 3 the Fitting ideal $\widehat{\mathcal{F}}$ is always locally principal at simple points of $\widehat{E}$ (and of course is trivial away from $\widehat{E}$); thus we will only be interested in blowups of the double and triple points of $\widehat{E}$.

Proposition 4. *Let $(\widehat{U}, \widehat{E})$ be a base resolution of (U, v). Then:*

1. *If $\widetilde{\pi}\colon (\widetilde{U}, E) \to (\widehat{U}, \widehat{E})$ is a blowup with center a double line $\widehat{E}_1 \cap \widehat{E}_2$ or a triple point $\widehat{E}_1 \cap \widehat{E}_2 \cap \widehat{E}_3$, then $\widetilde{\pi}$ is $\widehat{F}$-preserving near double and triple points of E; and*
2. *If $\widetilde{\pi}\colon (\widetilde{U}, E) \to (\widehat{U}, \widehat{E})$ is the blowup of a single double point of $\widehat{E}$, then $\widetilde{\pi}$ is $\widehat{F}$-preserving near double (but not necessarily triple) points of E.*

Proof. Let $\widetilde{\pi}\colon (\widetilde{U}, E) \to (\widehat{U}, \widehat{E})$ be one of the blowups above. Choose a point $\hat{e} \in \widehat{E}$ in the center of the blowup and a point $e \in E$ in its inverse image $\widetilde{\pi}^{-1}(\hat{e})$. Let $\widehat{W}$ be a neighborhood of $\hat{e}$ in $\widehat{U}$ and choose divisor coordinates $\{\hat{u}, \hat{v}, \hat{w}\}$ on $\widehat{W}$. Let W be a neighborhood of e in $\widetilde{U}$ and choose divisor coordinates $\{u, v, w\}$ on W. Use Proposition 1 to choose linear functions j, k, and l on $\mathbb{C}^N \supseteq U$ and thus functions $\widehat{\phi} := j \circ \widehat{\pi}$, $\widehat{\psi} := k \circ \widehat{\pi}$, and $\widehat{\rho} := l \circ \widehat{\pi}$ with respect to the point $\hat{e}$ so that ϕ generates $\widehat{\mathfrak{m}}$ and $\{d\widehat{\phi}, d\widehat{\psi}, d\widehat{\rho}\}$ is a basis for $\widehat{\mathcal{N}}(\widehat{W})$. In coordinates in $\widehat{W}$ we can write $\widehat{\phi} = \hat{u}^{m_1}\hat{v}^{m_2}\hat{w}^{m_3}$, $\widehat{\psi} = \sum_{(a,b,c)} s_{(a,b,c)}\hat{u}^a\hat{v}^b\hat{w}^c$, and $\widehat{\rho} = \sum_{\alpha,\beta,\gamma} t_{(\alpha,\beta,\gamma)}\hat{u}^\alpha\hat{v}^\beta\hat{w}^\gamma$, where $m_3 = 0$ if $\hat{e}$ is a double point. Note that $\mathcal{N} = \widetilde{\pi}^{-1}\widehat{\mathcal{N}}$ is generated by the functions $\phi := \widehat{\phi} \circ \widetilde{\pi}$, $\psi := \widehat{\psi} \circ \widetilde{\pi}$, and $\rho := \widehat{\rho} \circ \widetilde{\pi}$ on $W \subseteq \widetilde{U}$.

We will need to consider the following six cases:

- $\widetilde{\pi}$ is the blowup of a double line, with (a) $\hat{e}$ a double point, e a double point; (b) $\hat{e}$ a triple point on the double line, e a triple point; or (c) $\hat{e}$ a triple point on the double line, e a double point.
- $\widetilde{\pi}$ is the blowup of a triple point, with (d) $\hat{e}$ a triple point, e a triple point; or (e) $\hat{e}$ a triple point, e a double point.
- $\widetilde{\pi}$ is the blowup of a single double point, with (f) $\hat{e}$ a double point, e a double point.

Case (a). In our divisor coordinates the double line we are blowing up is given by $\hat{u} = \hat{v} = 0$. Consider the patch of $\widetilde{\pi}$ given by $\widetilde{\pi}(u, v, w) = (u, uv, w)$ (the proof for the other patch is entirely analogous with the roles of u and v reversed). Note that in this patch of $\widetilde{\pi}$, monomials on $\widehat{W}$ of the form $\hat{u}^a\hat{v}^b\hat{w}^c$ pull up to monomials $u^{a+b}v^bw^c$ on W. Since the generalized Nash sheaf $\mathcal{N}$ on $\widetilde{U}$ is by definition $\widetilde{\pi}^{-1}\widehat{\mathcal{N}}$, the inclusion $\widehat{\eta}\colon \widehat{\mathcal{N}} \to \widehat{\Omega}^1(\log(\widehat{E}))$ pulls up to a map $\widetilde{\pi}^*\widehat{\eta}\colon \mathcal{N} \to \widetilde{\pi}^{-1}\widehat{\Omega}^1(\log(\widehat{E}))$. Thus the Fitting ideal $\widehat{\mathcal{F}}(\widehat{W})$ pulls up to the the ideal $\widetilde{\pi}^{-1}\widehat{\mathcal{F}}(W)$ generated by the 2×2 subdeterminants of the matrix

$$[\widetilde{\pi}^*\widehat{\eta}] = \begin{pmatrix} m_1 u^{m_1+m_2}v^{m_2} & \sum sau^{a+b}v^b w^c & \sum t\alpha u^{\alpha+\beta}v^\beta w^\gamma \\ m_2 u^{m_1+m_2}v^{m_2} & \sum sbu^{a+b}v^b w^c & \sum t\beta u^{\alpha+\beta}v^\beta w^\gamma \\ 0 & \sum scu^{a+b}v^b w^{c-1} & \sum t\gamma u^{\alpha+\beta}v^\beta w^{\gamma-1} \end{pmatrix}.$$

On the other hand, $\mathcal{F}$ is generated by the 2×2 subdeterminants of the matrix for the map $\eta\colon \mathcal{N}\to\Omega^1(\log E)$:

$$[\eta] = \begin{pmatrix} (m_1+m_2)u^{m_1+m_2}v^{m_2} & \sum s(a+b)u^{a+b}v^b w^c & \sum t(\alpha+\beta)u^{\alpha+\beta}v^\beta w^\gamma \\ m_2 u^{m_1+m_2}v^{m_j} & \sum sbu^{a+b}v^b w^c & \sum t\beta u^{\alpha+\beta}v^\beta w^\gamma \\ 0 & \sum scu^{a+b}v^b w^{c-1} & \sum t\gamma u^{\alpha+\beta}v^\beta w^{\gamma-1} \end{pmatrix}.$$

Notice that

$$\text{if } [\widetilde{\pi}^*\widehat{\eta}] = \begin{pmatrix} A & B & C \\ D & E & F \\ 0 & H & I \end{pmatrix}, \text{ then } [\eta] = \begin{pmatrix} A+D & B+E & C+F \\ D & E & F \\ 0 & H & I \end{pmatrix}; \tag{1}$$

these matrices clearly differ by elementary row operations. Therefore their 2×2 subdeterminants generate the same ideal, *i.e.* $\widetilde{\pi}^{-1}\widehat{\mathcal{F}}(W) = \mathcal{F}(W)$.

Case (b). In this case a computation similar to the above shows that $[\widetilde{\pi}^*\widehat{\eta}]$ and $[\eta]$ are again in the form (1) and thus that $\widetilde{\pi}^{-1}\widehat{\mathcal{F}}(W) = \mathcal{F}(W)$.

Case (c). Let $\hat{e}$ be a triple point on the double line $\hat{u} = \hat{v} = 0$, and consider the patch $\widetilde{\pi}(u,v,w) = (u,uv,w)$ of the blowup of that double line. Suppose e is a double point in $\widetilde{\pi}^{-1}(\hat{e})$; for example, suppose e is on the double line $u = w = 0$. A computation as in Case (a) shows that $[\widetilde{\pi}^*\widehat{\eta}]$ and $[\eta]$ are related as follows:

$$\text{if } [\widetilde{\pi}^*\widehat{\eta}] = \begin{pmatrix} A & B & C \\ D & E & F \\ 0 & H & I \end{pmatrix}, \text{ then } [\eta] = \begin{pmatrix} A+D & B+E & C+F \\ D/v & E/v & F/v \\ 0 & H & I \end{pmatrix}.$$

Since $1/v$ is a local unit at e, once again $[\widetilde{\pi}^*\widehat{\eta}]$ and $[\eta]$ differ by elementary row operations.

Case (d). The matrices $[\widetilde{\pi}^*\widehat{\eta}]$ and $[\eta]$ are of the form:

$$[\widetilde{\pi}^*\widehat{\eta}] = \begin{pmatrix} A & B & C \\ D & E & F \\ G & H & I \end{pmatrix}, \quad [\eta] = \begin{pmatrix} A+D+G & B+E+H & C+F+I \\ D & E & F \\ G & H & I \end{pmatrix}.$$

Case (e). In this case we have:

$$[\widetilde{\pi}^*\widehat{\eta}] = \begin{pmatrix} A & B & C \\ D & E & F \\ G & H & I \end{pmatrix}, \quad [\eta] = \begin{pmatrix} A+D+G & B+E+H & C+F+I \\ D & E & F \\ G/w & H/w & I/w \end{pmatrix},$$

where $1/w$ is a local unit near e.

Case (f). In the final case we have:

$$[\widetilde{\pi}^*\widehat{\eta}] = \begin{pmatrix} A & B & C \\ D & E & F \\ 0 & H & I \end{pmatrix}, \quad [\eta] = \begin{pmatrix} A+D & B+E+wH & C+F+wI \\ D & E & F \\ 0 & H & I \end{pmatrix},$$

where w is a local unit near e. $\square$

5.4. Results from Hironaka

The remaining two Lemmas require some notation and results from [4]. Let $\widehat{F}$ be the second Fitting ideal sheaf for the Nash sheaf over a base resolution $\widehat{\pi}\colon (\widehat{U}, \widehat{E}) \to (U, v)$. Note that $\widehat{\mathcal{F}}$ is a sheaf of ideals supported on $\widehat{E}$ and locally principal at simple points of $\widehat{E}$ (by Proposition 3).

Definition 5. *Let x be any point in $\widehat{U}$. The* order $\nu(\widehat{\mathcal{F}}_x)$ *of $\widehat{\mathcal{F}}$ at x is the highest power p for which the p-th power of the maximal ideal $\mathfrak{m}_x$ of $\mathcal{O}_{X,x}$ contains $\widehat{\mathcal{F}}_x$, i.e.*

$$\nu(\widehat{\mathcal{F}}_x) := \max\{p \in \mathbb{Z} \mid \mathfrak{m}_x^p \supseteq \widehat{\mathcal{F}}_x\}.$$

The function $x \mapsto \nu(\widehat{\mathcal{F}}_x)$ is upper-semicontinuous by Corollary 1 to Theorem 2 in [5]. In other words, for every integer r, the set $U_r := \{x \in \widehat{U} \mid \nu(\widehat{\mathcal{F}}_x) \le r\}$ is open in $\widehat{U}$. For each component $\widehat{E}_i$ of $\widehat{E}$, let g_i be the generic point of $\widehat{E}_i$, and define $\nu_i := \nu(\widehat{F}_{g_i})$. Note that ν_i is also the generic value of $\nu(\widehat{\mathcal{F}}_x)$ over all points $x \in \widehat{E}_i$, as follows. In a neighborhood of a simple point, $\widehat{\mathcal{F}}$ is locally principal, say $\langle u^a \rangle$. Then given a simple point $x \in \widehat{E}$ we have $\nu(\widehat{\mathcal{F}}) = \max\{m \mid \langle u, v, w\rangle_x^m \supseteq \langle u\rangle_x^a\} = a$. Thus for a dense open set of points of $\widehat{E}_i$ the order $\widehat{\mathcal{F}}$ is equal to ν_i.

Stratify the exceptional divisor $\widehat{E}$ as follows: $S_0 := \{\text{triple points of } \widehat{E}\}$, $S_1 := \{\text{double points of } \widehat{E}\}$, and $S_2 := \{\text{simple points of } \widehat{E}\}$. We will call these strata the "multiplicity" strata of the exceptional divisor $\widehat{E}$.

Proposition 5. *Given a base resolution $(\widehat{U}, \widehat{E})$ of (U, v), there exists a further blowup $\breve{\pi}\colon (\breve{U}, \breve{E}) \to (\widehat{U}, \widehat{E})$ over which the order $\nu(\breve{\mathcal{F}}_x)$ is locally constant on the multiplicity strata of $\breve{E}$.*

The proof of Proposition 5 will require Theorem 4 below.

Suppose $\widetilde{\pi}\colon \widetilde{U} \to \widehat{U}$ is a monoidal transform with nonempty irreducible nonsingular center $B \subseteq \widehat{U}$. Define $B' := \widetilde{\pi}^{-1}(B)$ and let $\mathcal{I}_{B'}$ be the sheaf of ideals on $\widetilde{U}$ that defines B'. Let b be the generic point of B and define $\nu_b := \nu(\widehat{\mathcal{F}}_b)$.

Definition 6. *Suppose $\widetilde{\pi}\colon \widetilde{U} \to \widehat{U}$ is a monoidal transform with center B. The* weak transform $\widetilde{\pi}^w(\widehat{\mathcal{F}})$ *of $\widehat{\mathcal{F}}$ by $\widetilde{\pi}$ is defined to be*

$$\widetilde{\pi}^w(\widehat{\mathcal{F}}) := \widetilde{\pi}^{-1}(\widehat{\mathcal{F}})\, \mathcal{I}_{B'}^{-\nu_b}.$$

Now suppose we have a finite sequence of monoidal transforms

$$\widetilde{U}_r \xrightarrow{\widetilde{\pi}_{r-1}} \widetilde{U}_{r-1} \xrightarrow{\widetilde{\pi}_{r-2}} \cdots \xrightarrow{\widetilde{\pi}_1} \widetilde{U}_1 \xrightarrow{\widetilde{\pi}_0} \widetilde{U}_0 := \widehat{U}$$

with centers $B_i \subseteq \widetilde{U}_i$. Denote the weak transform at the i^{th} level by $\mathcal{F}_i := \widetilde{\pi}_{i-1}^w(\mathcal{F}_{i-1})$ for $0 < i \le r$, with $\mathcal{F}_0 := \widehat{\mathcal{F}}$. Given a divisor with normal crossings $D = D_0$ on $\widehat{U}$, define $D_i := \operatorname{red}(\widetilde{\pi}_{i-1}^{-1}(D_{i-1}) \cup \widetilde{\pi}_{i-1}^{-1}(B_{i-1})) \subseteq \widetilde{U}_i$

for $0 < i \leq r$. Let d_i be the maximum order of $\mathcal{F}_i$ over all of $\widetilde{U}_i$. The following theorem is due to Hironaka (a more general version of this theorem appears as Main Theorem II in [4]).

Theorem 4. *Let $\mathcal{F}$ be a coherent sheaf of nonzero ideals on a nonsingular space $\widetilde{U}_0$. Then there exists a finite sequence of monoidal transforms $\{\widetilde{\pi}_i \colon \widetilde{U}_{i+1} \to \widetilde{U}_i\}$ with centers $B_i \subseteq \widetilde{U}_i$ such that:*

1. *B_i is nonsingular and irreducible;*
2. *$\nu((\mathcal{F}_i)_{x_i}) \geq d_i$ for all points $x_i \in B_i$;*
3. *D_i has only normal crossings with B_i.*
4. *D_r has only normal crossings, and $\nu((\mathcal{F}_r)_x) < d_0$ for every point $x \in \widetilde{U}_r$.*

Note that by repeated application of Theorem 4 we can obtain a space $\widetilde{U}_R$ with the property that $\nu((\mathcal{F}_R)_x) = 0$ for every point $x \in \widetilde{U}_R$. This means that we can "trivialize" the sheaf $\widehat{\mathcal{F}} = \mathcal{F}_0$ in the sense that we can find a finite sequence of monoidal transforms so that its (iterated) weak transform $\mathcal{F}_R$ is the trivial sheaf.

It is very important to note that in the sequence of transformations obtained in Theorem 4, the centers B_i are strata of D_i where the order $\nu((\mathcal{F}_i)_x)$ is locally at a maximum. This fact is shown in [1], where a constructive proof of Theorem 4 is given (in particular, see Remark 1.8 and Theorem 1.10 there): at each stage of the resolution, the center of blowing-up is a locally maximum stratum of the local invariant ν. This method of constructive resolution is also used in [17] and [18]. In particular, the order of $\mathcal{F}_i$ at each point x in B_i will be greater than or equal to the order of $\mathcal{F}_i$ at any neighboring point in D_i. This fact will be useful in the proof of Proposition 5 as well as at the end of the proof of Proposition 6 below. We are now in a position to prove Proposition 5.

Proof (Proposition 5). The order $\nu(\widehat{\mathcal{F}}_x)$ must be locally constant on S_0 (which consists only of isolated points) and on S_2 (since $\widehat{\mathcal{F}}$ is locally principal at these points). Since $\nu(\widehat{\mathcal{F}}_x)$ is upper-semicontinuous, the only way that $\nu(\widehat{\mathcal{F}}_x)$ can fail to be locally constant on S_1 is if it jumps to a higher value at isolated double points.

Suppose b is a double point of $\widehat{E}$ with the property that $\nu(\widehat{\mathcal{F}}_b) > \nu(\widehat{\mathcal{F}}_x)$ for nearby points x on the same double line. Then the order of $\widehat{\mathcal{F}}$ at b is at a local maximum, and thus the point b is a permissible center B_0 for blowing up. Blowing up with center b will result in a new component of the exceptional divisor; above b we will have new simple, double, and triple points.

By the discussion above we are only concerned with making the order of the ideal sheaf locally constant along the double point set. If there are more "bad" double points in the new double point set, they are once again permissible centers for blowing up in the Hironaka argument and we can repeat the process. By part (d) of Theorem 4, this process of blowing up the "bad" double points will eventually result in a space for which the maximum order of the (iterated) weak transform $\widehat{\mathcal{F}}^w$ of $\widehat{\mathcal{F}}$ has been reduced, *i.e.* has been made strictly less than $\nu(\widehat{\mathcal{F}}_b)$ at every new double point. By repeating *this* process we can obtain a space $\breve{U}$ for which the weak transform $\widehat{\mathcal{F}}^w$ is locally constant along the strata of $\breve{E}$.

Since the weak transform of $\widehat{\mathcal{F}}$ and the inverse image of $\widehat{\mathcal{F}}$ by $\breve{\pi}$ differ by a locally principal ideal, we now know that the the inverse image $\breve{\pi}^{-1}(\widehat{\mathcal{F}})$ has locally constant order along the strata of $\breve{E}$. By Part (2) of Lemma 4 we know that each of the blowups described above is $\widehat{\mathcal{F}}$-preserving at the new double points. Therefore we will have $\breve{\mathcal{F}}_y = \breve{\pi}^{-1}(\widehat{\mathcal{F}}_b)$ for each new double point y above b, and thus $\breve{\mathcal{F}}$ will have locally constant order along the strata of $\breve{E}$. □

Proposition 6. *Let $(\breve{U},\breve{E})$ be a resolution of (U,v) over which $\nu(\breve{\mathcal{F}}_x)$ is locally constant. Then there exists a $\breve{\mathcal{F}}$-preserving map $\widetilde{\pi}\colon (\widetilde{U},E) \to (\breve{U},\breve{E})$ such that $\widetilde{\pi}^{-1}(\breve{\mathcal{F}})$ is locally principal on $\widetilde{U}$.*

To prove Proposition 6 we need to show that there exists a further resolution over which the inverse image of the Fitting ideal sheaf $\breve{\mathcal{F}}$ is locally principal. By Theorem 4 we can find a resolution so that the weak transform of a given sheaf of ideals is locally principal. We will define an ideal sheaf $\breve{\mathcal{G}}$ associated to $\breve{\mathcal{F}}$ in such a way that the inverse image of $\breve{\mathcal{F}}$ will be locally principal whenever the weak transform of $\breve{\mathcal{G}}$ is the trivial sheaf, and apply Theorem 4 to $\breve{\mathcal{G}}$.

Let $\breve{E}_i$ denote a component of the exceptional divisor $\breve{E}$. Define the divisor $L := \sum \nu_i \breve{E}_i$, and let $\mathcal{I}_L$ be the sheaf of ideals on $\breve{U}$ defining L. Let $\breve{\mathcal{F}}$ be the Fitting ideal sheaf for the Nash sheaf on $\breve{U}$.

Lemma 3. *With notation as above, $\breve{\mathcal{F}} \subseteq \mathcal{I}_L$.*

Proof. It suffices to prove that $\breve{\mathcal{F}}_x \subseteq (\mathcal{I}_L)_x$ for all $x \in \breve{E}$. In the case where x is a simple point in $\breve{E}_i = \{u = 0\}$ the inclusion is in fact an equality: $\breve{\mathcal{F}}_x = \langle u^{\nu_i}\rangle_x$ in this case, and $(\mathcal{I}_L)_x = (\mathcal{I}_{\nu_j E_i})_x = \langle u^{\nu_i}\rangle_x$.

Suppose that x is a double point in $\breve{E}_i \cap \breve{E}_j = \{u = 0\} \cap \{v = 0\}$, and choose an element $\sigma \in \breve{\mathcal{F}}_x$. Since $\breve{\mathcal{F}}$ is supported along $\breve{E}$, σ vanishes along E_i and E_j; thus we have $\sigma = u^a f$ with $u \nmid f$ and $\sigma = v^b g$ with $v \nmid g$. Since $\breve{\mathcal{F}}_y = \langle u^{\nu_i}\rangle_y$ at all nearby simple points $y \in E_i$, we have $a \geq \nu_i$; similarly we have $b \geq \nu_j$. Thus we can write $\sigma = u^{\nu_i} A$ and $\sigma = v^{\nu_i} B$ for some holomorphic A and B; since $(\mathcal{O}_{\breve{U}})_x$ is a unique factorization domain we thus have $\sigma = u^{\nu_i} v^{\nu_j} C$ for some holomorphic C. In other words, we have $\sigma \in \langle u^{\nu_i} v^{\nu_j}\rangle_x = (\mathcal{I}_L)_x$. The proof in the triple point case is analogous. □

By the Claim above we can now define $\breve{\mathcal{G}}$ to be $\breve{\mathcal{G}} := \breve{\mathcal{F}}\mathcal{I}_L^{-1}$, the ideal quotient of $\breve{\mathcal{F}}$ by $\mathcal{I}_L$. Note that $\breve{\mathcal{G}} = \mathcal{O}_{\breve{U}}$ if and only if $\breve{\mathcal{F}}$ is locally principal. Moreover, since $\breve{\mathcal{F}}$ is locally principal at the simple points of $\breve{E}$ (by Proposition 3), $\breve{\mathcal{G}}$ is supported on the set of double and triple points of $\breve{E}$. By applying Theorem 4 to the ideal sheaf $\breve{\mathcal{G}}$ we will now prove Proposition 6.

Proof. By repeated application of Theorem 4 to the coherent sheaf of ideals $\breve{\mathcal{G}}$ and divisor with normal crossings $\breve{E}$ on $\breve{U}$ we can trivialize $\breve{\mathcal{G}}$, as follows. Applying Theorem 4 to $\breve{\mathcal{G}}$ gives us a sequence of monoidal transforms

$$\widetilde{U}_r \xrightarrow{\widetilde{\pi}_{r-1}} \widetilde{U}_{r-1} \xrightarrow{\widetilde{\pi}_{r-2}} \cdots \xrightarrow{\widetilde{\pi}_1} \widetilde{U}_1 \xrightarrow{\widetilde{\pi}_0} \widetilde{U}_0 := \breve{U}$$

with centers $B_i \in \widetilde{U}_i$ satisfying (1)–(3) of Theorem 4. By part (d) of Theorem 4, $\nu((\breve{\mathcal{G}}_r)_x) < d_0$ for every point $x \in \widetilde{U}_r$; thus the maximum order of the ideal sheaf has strictly decreased after this sequence of monoidal transformations. By repeating this process we can obtain a sequence

$$\widetilde{U} := \widetilde{U}_R \xrightarrow{\widetilde{\pi}_{R-1}} \widetilde{U}_{R-1} \xrightarrow{\widetilde{\pi}_{R-2}} \cdots \xrightarrow{\widetilde{\pi}_1} \widetilde{U}_1 \xrightarrow{\widetilde{\pi}_0} \widetilde{U}_0 := \breve{U}$$

with the property that the weak transform $\breve{\mathcal{G}}_R = (\widetilde{\pi}^w_{R-1} \circ \cdots \circ \widetilde{\pi}^w_0)(\breve{\mathcal{G}})$ has a maximum order of zero; in other words $\nu((\breve{\mathcal{G}}_R)_x) = 0$ for every $x \in \widetilde{U}$. Thus we have $\breve{\mathcal{G}}_R = \mathcal{O}_{\widetilde{U}}$.

Let y_i be the generic point of B_i and define $b_i := \nu((\breve{\mathcal{G}}_i)_{y_i})$. Let B'_i denote the inverse image $B'_i := \widetilde{\pi}^{-1}(B_i) \subseteq \widetilde{U}_{i+1}$. Define $\widetilde{\pi} := \widetilde{\pi}_{R-1} \circ \cdots \circ \widetilde{\pi}_0$. Then by Definition 6 we have

$$\breve{\mathcal{G}}_R = \widetilde{\pi}^{-1}(\widehat{\breve{\mathcal{G}}})\,((\widetilde{\pi}^{-1}_{R-1} \circ \cdots \circ \widetilde{\pi}^{-1}_1)\mathcal{I}_{B'_0})^{-b_0} \cdots (\widetilde{\pi}_{R-1}\mathcal{I}_{B'_{R-2}})^{-b_{R-2}}\mathcal{I}^{-b_{R-1}}_{B'_{R-1}}.$$

By the definition of $\breve{\mathcal{G}}$ we have $\widetilde{\pi}^{-1}(\breve{\mathcal{G}}) = \widetilde{\pi}^{-1}(\breve{\mathcal{F}})(\widetilde{\pi}^{-1}(\mathcal{I}_L))^{-1}$. Then since $\breve{\mathcal{G}}_R = \mathcal{O}_{\widetilde{U}}$ we can write $\widetilde{\pi}^{-1}(\breve{\mathcal{F}})$ as the locally principal ideal

$$(\widetilde{\pi}^{-1}(\mathcal{I}_L))\,((\widetilde{\pi}^{-1}_{R-1} \circ \cdots \circ \widetilde{\pi}^{-1}_0)\mathcal{I}_{B_0})^{d_0} \cdots (\widetilde{\pi}_{R-1}\mathcal{I}_{B'_{R-2}})^{-b_{R-2}}\mathcal{I}^{-b_{R-1}}_{B'_{R-1}}.$$

It now remains only to show that $\widetilde{\pi}$ is a $\breve{\mathcal{F}}$-preserving map. By Proposition 4 it suffices to show that each center B_i is a double line or triple point of D_i. Since by Proposition 5 the order of $\breve{\mathcal{G}}_i$ is locally constant along the strata of D_i, it suffices to prove that B_i is supported away from the simple point set of D_i and that the order of $\breve{\mathcal{G}}_i$ is greater at points of B_i than at any neighboring points of $\widetilde{U}_i$. Note in particular that this excludes the possibility that B_i is a single double point of D_i.

We begin by showing that the first center B_0 satisfies these conditions. If x is a simple point of $\breve{E} = D_0$ then $\breve{\mathcal{F}}$ is principal at x, so $\breve{\mathcal{G}}_x \approx \mathcal{O}_{\widetilde{U},x}$, and thus $\nu(\breve{\mathcal{G}}_x) = 0$. But part (2) of Theorem 4 ensures that for $x \in B_0$, the order $\nu(\breve{\mathcal{G}}_x)$ is greater than or equal to the maximum order d_0 on $\widetilde{U}_0$; thus B_0 is supported away from the simple point set of D_0. By Proposition 5 the order of $\breve{\mathcal{G}}$ is locally constant along the strata of D_0, and by the remarks following Theorem 4 the center B_0 must be a maximal component of that stratification. Thus B_0 must be either a double line or triple point of D_0.

We must now show that blowing up with center B_1 is a $\breve{\mathcal{F}}$-preserving operation. We first show that B_1 can not contain any simple points of D_1. Suppose $x \in D_1$ is a simple point. By the definition of $\breve{\mathcal{G}}_1$ we have

$$\breve{\mathcal{G}}_1 = (\pi_0^{-1}\breve{\mathcal{G}})_x\,(\mathcal{I}_{B'_0})^{-b_0}_x = (\widetilde{\pi}_0^{-1}\breve{\mathcal{F}})_x\,(\widetilde{\pi}_0^{-1}\mathcal{I}_L)^{-1}_x\,(\mathcal{I}_{B'_0})^{-b_0}_x. \qquad (2)$$

By Proposition 4, since B_0 is a double line or a triple point of D_0, the map $\widetilde{\pi}_0$ is a $\breve{\mathcal{F}}$-preserving map. Thus $(\widetilde{\pi}_0^{-1}\breve{\mathcal{F}})_x$ is isomorphic to the Fitting ideal on $\widetilde{U}_1$ at x, which by Proposition 3 is principal. Thus in particular $\pi_0^{-1}\breve{\mathcal{G}}_0$ is principal at x. If y is the generic point of B_0', then $\nu(\widetilde{\pi}^{-1}\breve{\mathcal{G}}_y) = b_0$; thus b_0 is the largest integer for which the locally principal ideal $\mathcal{I}_{B_0'}$ divides $\widetilde{\pi}_0^{-1}\breve{\mathcal{G}}$ (see the remarks above and page 142 of [4]). Therefore by (2) we must have $(\breve{\mathcal{G}})_x \approx \mathcal{O}_{\widetilde{U},x}$, and thus $\nu((\breve{\mathcal{G}}_1)_x) = 0$. By part (2) of Theorem 4 the order of $\breve{\mathcal{G}}_1$ cannot be zero at any point of B_1. Therefore the intersection of B_1 with the simple point set of D_1 is empty.

As we saw with B_0, since B_1 is supported away from the simple points of D_1 and must be a maximal component of D_1 according the the stratification of D_1 by the order of $\breve{\mathcal{G}}_1$, B_1 must be a double line or a triple point of D_1. Repeating these arguments for $B_2, \ldots, B_{R-1}$ completes the proof. □

5.5. Proof of Theorem 1

Proof (Theorem 1). Let $\widehat{\pi}\colon (\widehat{U}, \widehat{E}) \to (U, v)$ be a base resolution. If the associated Fitting ideal $\widehat{\mathcal{F}}$ is locally principal, we are done. If not, by Proposition 5 we can obtain a further resolution $\breve{\pi}\colon (\breve{U}, \breve{E}) \to (\widehat{U}, \widehat{E})$ over which the order of $\breve{\mathcal{F}}$ is locally constant along the multiplicity strata of $\breve{E}$. Then by Proposition 6 we can obtain a further resolution $\widetilde{\pi}\colon (\widetilde{U}, \widetilde{E}) \to (\breve{U}, \breve{E})$ by a sequence of $\breve{\mathcal{F}}$-preserving blowups such that $\widetilde{\pi}^{-1}\breve{\mathcal{F}} = \mathcal{F}$ is locally principal. The resolution $\pi := \widehat{\pi} \circ \breve{\pi} \circ \widetilde{\pi}$ is then a complete resolution of (U, v). □

6. Hsiang-Pati coordinates and monomial generators

In this section we prove that in a complete resolution $\widetilde{U}$, the Nash sheaf has local generators that are the differentials of monomial functions (Theorem 2). These monomial generators will be obtained by constructing Hsiang-Pati coordinates on $\widetilde{U}$ from a generic choice of Nash-minimal functions (Proposition 2 and Theorem 3) and then extracting the distinguished monomial components of these coordinates.

We begin by proving Proposition 2 so that we can choose Nash-minimal functions. In Section 6.2 we prove Theorem 3 in the case where e is a triple point of E. The double and simple point cases are discussed in Section 6.3. In Section 6.4 we prove Theorem 2 as a simple corollary of Theorem 3.

6.1. Nash-minimal functions

In Section 5.1 we proved that given a base resolution $\widetilde{U}$, a generic choice of linear functions on $\mathbb{C}^N \supseteq U$ pulls up to functions on $\widetilde{U}$ that satisfy conditions (1) and (2) of Nash-minimality. In this section we prove that in a complete resolution, a generic choice of linear functions induce functions on $\widetilde{U}$ that satisfy all three conditions of Nash-minimality.

Let $\pi\colon (\widetilde{U}, E) \to (U, v)$ be any complete resolution, and choose any point $e \in E$. Although we will not always indicate so in our notation, we will be working in an analytic neighborhood W of e in $\widetilde{U}$; let $\{u, v, w\}$ be divisor coordinates for W centered at e.

Choose j, k, and l as in Proposition 1. Then $\{d(j \circ \pi)\, d(k \circ \pi),\ d(k \circ \pi)\, d(l \circ \pi),\ d(j \circ \pi)\, d(l \circ \pi)\}$ is a set of generators for $\Lambda^2\mathcal{N}$. We must show that one of these generators is a minimal element of $\Lambda^2\mathcal{N}$. The following definition describes what we mean by "minimal".

Definition 7. *A two-form $\omega \in \Lambda^2\mathcal{N} \subset \Omega^2(\log E)$ is* minimal *at e if one of its coefficients in the standard basis for $\Omega^2(\log E)$ vanishes to the minimum possible order at the point e.*

In other words, ω vanishes to the minimum order if one of its coefficients in the logarithmic basis is μg, where μ is a local unit and g is the generator of the locally principal Fitting ideal sheaf $\mathcal{F}$. Such an ω will vanish to the minimum order along each of the components of E. A generic 2-form $\omega \in \Lambda^2\mathcal{N}$ will vanish to the minimum order (that of g) at e. We are now in a position to prove Proposition 2.

Proof (Proposition 2). Choose j, k, and l as above. These induce generators of $\Lambda^2\mathcal{N}$, one of which must vanish to the minimum order (since minimality is generic). Without loss of generality suppose that $d(j \circ \pi)\, d(k \circ \pi)$ is this minimum generator, and define $D_2 := \ker(j) \cap \ker(k)$. We will say that a codimension two plane D_2 is *arranged* if it comes from a arranged codimension three plane D_3. The set of arranged D_2 planes is generic since minimality is generic.

Let D_3 be as in the proof of Proposition 1, and choose h (and $D_1 := \ker(h)$) so that $h \circ \widehat{\pi}$ generates $\mathfrak{m}$. It remains only to prove that we have a flag of arranged planes $D_3 \subset D_2 \subset D_1$ (after possibly rechoosing j, k, l, and h).

Choose any arranged D_3 and use the minimal generator of $\Lambda^2\mathcal{N}$ to choose D_2. By definition D_2 will be "arranged" and contain D_3. Now choose *any* codimension one plane H containing D_2. As we did in Proposition 1, use Lemma 2 to find a nearby codimension one plane that is arranged and contains some arranged D_2' in a neighborhood of D_2. Rechoose $j = h$, k, and l accordingly. □

6.2. Hsiang-Pati coordinates at triple points

The proof of Theorem 3 in the triple point case will depend on the following key computational lemma.

Lemma 4. *Let $(\widetilde{U}, E)$ be a complete resolution of (U, v) and choose a triple point $e \in E_1 \cap E_2 \cap E_3$. Given Nash-minimal ϕ, ψ, and ρ for e, and divisor coordinates $\{u, v, w\}$ in a neighborhood W of e, we have:*

$$\begin{aligned} d\phi\, d\psi &= u^{g_1} v^{g_2} w^{g_3} \left(A \frac{dudv}{uv} + B \frac{dvdw}{vw} + C \frac{dudw}{uw} \right) \\ d\phi\, d\psi\, d\rho &= u^{d_1} v^{d_2} w^{d_3} (\mu\; dudvdw) \end{aligned}$$

where the g_l and d_l are positive integers, μ is a local unit, and at least one of A, B, or C is a local unit.

Proof. We give an idea of the proof and leave the details to the reader. That one of A, B, or C is a local unit follows from the fact that $\mathcal{F}$ is locally principal and $d\phi\, d\psi$ is a minimal element in $\Lambda^2\mathcal{N}$. The exponents g_l are positive integers because $u^{g_1} v^{g_2} w^{g_3}$ differs from the generator of $\mathcal{F}$ by a local unit. Finally, since $\Lambda^3\mathcal{N} \hookrightarrow \Omega^3$ is an inclusion of locally free rank one sheaves that is an isomorphism everywhere away from E we have $\Lambda^3\mathcal{N} \approx \Omega^3 \otimes \mathcal{O}(-D)$ for some positive divisor D supported on E. Then $u^{d_1} v^{d_2} w^{d_3}$ is the local defining function for D in W. $\square$

Proof (Theorem 3, triple point case). Let $\pi\colon (\widetilde{U}, E) \to (U, v)$ be a complete resolution, and choose a triple point $e \in E_1 \cap E_2 \cap E_3 \subset E$. Let $\{\phi, \psi, \rho\}$ be a Nash-minimal set of function with respect to e, and choose divisor coordinates $\{u, v, w\}$ in an analytic neighborhood W of e in $\widetilde{U}$. All computations that follow are assumed to be taking place in W.

By Nash-minimality, ϕ generates $\mathfrak{m}$; thus (after a simple change of coordinates) we can write $\phi = u^{m_1} v^{m_2} w^{m_3}$, where $Z = \sum m_i E_i$ is the divisor determined by the sheaf of locally principal ideals $\mathfrak{m}$. Note that by definition ϕ will divide both ψ and ρ.

A priori we can write ψ as a convergent power series $\sum s_{(a,b,c)} u^a v^b w^c$. For notational simplicity we will drop the subscript on the coefficients and write each $s_{(a,b,c)}$ simply as s. Define ψ' to be the sum of the terms in ψ for which (a, b, c) is linearly independent of (m_1, m_2, m_3). We will use an asterisk to denote that only these "independent" terms are to be considered; thus $\psi' = \sum_* s u^a v^b w^c$. Then $\psi = S + \psi'$ where $S = \sum s_l \phi^{\epsilon_l}$. Each ϵ_i is ≥ 1 since ϕ divides ψ.

To show that ϕ' can be written in the form $u^{n_1} v^{n_2} w^{n_3}$ we calculate $d\phi\, d\psi$ and apply Lemma 4. Define n_1 to be the minimum exponent a appearing in ψ'. Similarly define n_2 and n_3 to be the minimum such b and c. A simple calculation shows that

$$d\phi\, d\psi = u^{m_1+n_1} v^{m_2+n_2} w^{m_3+n_3} \left(A \frac{dudv}{uv} + B \frac{dvdw}{vw} + C \frac{dudw}{uw} \right) \qquad (3)$$

where $A = \sum_* s u^{a-n_1} v^b w^c \left| \begin{smallmatrix} m_1 & a \\ m_2 & b \end{smallmatrix} \right|$ (and B and C are similar expressions). By Lemma 4 and the definition of the n_l, at least one of A, B, or C must be a local unit. Thus (n_1, n_2, n_3) must be one of the triples (a, b, c) appearing in ψ'; in other words, the minimum powers of u, v, and w in

ψ' all appear in the *same* monomial. Therefore $\psi' = u^{n_1}v^{n_2}w^{n_3}R$ where $R = \sum_* su^{a-n_1}v^{b-n_2}w^{c-n_3}$ is a local unit.

We now must change coordinates to absorb R into u, v, and w without disrupting the form of $\phi = u^{m_1}v^{m_2}w^{m_3}$. Since (n_1, n_2, n_3) is a triple (a, b, c) that is independent of (m_1, m_2, m_3), at least one of $m_1n_2 - m_2n_1$, $m_2n_3 - m_3n_2$, and $m_1n_3 - m_3n_1$ is nonzero. Without loss of generality say $\Delta := m_1n_2 - m_2n_1 \neq 0$. Rechoose coordinates by mapping $u \mapsto uR^{m_2/\Delta}$ and $v \mapsto vR^{-m_1/\Delta}$. This fixes ψ and S while sending ψ' to $u^{n_1}v^{n_2}w^{n_3}$. Note that since ϕ divides ψ we have $m_l \leq n_l$ for each $l = 1, 2, 3$.

Now write ρ as the series $\sum s_{(\alpha,\beta,\gamma)}u^\alpha v^\beta w^\gamma$. Define T to be the sum of the terms in ρ where the triple (α, β, γ) is a rational linear combination of (m_1, m_2, m_3) and (n_1, n_2, n_3). Then we have $\rho = T + \rho'$ where $T = \sum t_l \phi^{\delta_l}(\psi')^{\tau_l}$ and $\rho' = \sum_{**} su^\alpha v^\beta w^\gamma$.

Define p_1 to be the minimum α appearing in ρ'. Similarly define p_2 and p_3 to be th minimum β and γ in ρ'. With this notation we calculate:

$$d\phi\, d\psi\, d\rho = u^{m_1+n_1+p_1-1}v^{m_2+n_2+p_2-1}w^{m_3+n_3+p_3-1}\,\mu\, dudvdw,$$

where $\mu = \sum_{**} tu^{\alpha-p_1}v^{\beta-p_2}w^{\gamma-p_3} \begin{vmatrix} m_1 & n_1 & \alpha \\ m_2 & n_2 & \beta \\ m_3 & n_3 & \gamma \end{vmatrix}$. By Lemma 4, μ must be a local unit. Thus (p_1, p_2, p_3) is one of the triples (α, β, γ) in ρ', and hence $\rho' = u^{p_1}v^{p_2}w^{p_3}\,Q$ where Q is a local unit. Moreover, since (p_1, p_2, p_3) is a triple satisfying $(**)$ and (n_1, n_2, n_3) is a triple satisfying $(*)$, the matrix $M := \begin{pmatrix} m_1 & n_1 & p_1 \\ m_2 & n_2 & p_2 \\ m_3 & n_3 & p_3 \end{pmatrix}$ has nonzero determinant.

We now absorb the local unit Q into the coordinates $\{u, v, w\}$ while preserving the forms of ϕ and ψ constructed above. Change coordinates by mapping $u \mapsto uQ^A$, $v \mapsto vQ^B$, and $w \mapsto wQ^C$ where A, B, and C are defined by $\begin{pmatrix} A \\ B \\ C \end{pmatrix} := M^{-1}\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$. Under these new divisor coordinates we have $\rho' = u^{p_1}v^{p_2}w^{p_3}$ while ϕ, ψ, and T are fixed.

It remains to show that $\delta_l \geq 1$ for terms of T where $\tau = 0$, $\delta_l m_1 + \tau_l n_1 \geq n_1$ when $\tau \neq 0$ (and similarly for m_2, n_2 and m_3, m_3), and that (n_1, n_2, n_3) is piecewise less than or equal to (p_1, p_2, p_3). We will use two different notations for T, namely $T = \sum s_l \phi^{\delta_l}(\psi')^{\tau_l}$ and $T = \sum_{\mathrm{not}**} s_{(\alpha,\beta,\gamma)}u^\alpha v^\beta w^\gamma$. With this notation, to prove the three remaining facts it suffices to prove that $\delta_l \geq 1$ when $\tau = 0$ and that (α, β, γ) is piecewise greater than or equal to (n_1, n_2, n_3) for *all* triples (α, β, γ) appearing in ρ that are linearly independent of (m_1, m_2, m_3).

For the terms in T where $\tau = 0$ we have $(\alpha, \beta, \gamma) = (m_1\delta_l, m_2\delta_l, m_3\delta_l)$. Since ρ vanishes to at least order ϕ, each (α, β, γ) in T is piecewise greater than or equal to (m_1, m_2, m_3). Thus when $\tau = 0$ we have $\delta_l \geq 1$.

The final fact will be proved by comparing $d\phi\, d\psi$ and $d\phi\, d\rho$ and invoking the third condition of Nash-minimality. We calculate that $d\phi\, d\psi$ is equal to:

$$u^{m_1+n_1}v^{m_2+n_2}w^{m_3+n_3}\left(\begin{vmatrix} m_1 & n_1 \\ m_2 & n_2 \end{vmatrix}\frac{dudv}{uv} + \begin{vmatrix} m_2 & n_2 \\ m_3 & n_3 \end{vmatrix}\frac{dvdw}{vw} + \begin{vmatrix} m_1 & n_1 \\ m_3 & n_3 \end{vmatrix}\frac{dudw}{uw}\right)$$

and $d\phi\, d\rho$ is equal to:

$$u^{m_1}v^{m_2}w^{m_3}\sum\nolimits_* tu^\alpha v^\beta w^\gamma\left(\begin{vmatrix} m_1 & \alpha \\ m_2 & \beta \end{vmatrix}\frac{dudv}{uv} + \begin{vmatrix} m_2 & \beta \\ m_3 & \gamma \end{vmatrix}\frac{dvdw}{vw} + \begin{vmatrix} m_1 & \alpha \\ m_3 & \gamma \end{vmatrix}\frac{dudw}{uw}\right).$$

Note that in the second equation we are taking the sum over all triples (α, β, γ) appearing in ρ that are linearly independent of (m_1, m_2, m_3). By Nash-minimality, $d\phi\, d\psi$ vanishes to the least possible order at the point e and thus (n_1, n_2, n_3) must be less than or equal to (α, β, γ) for each triple (α, β, γ) satisfying $(*)$. $\square$

6.3. Hsiang-Pati coordinates at double and simple points

The proof of Theorem 3 in the double and simple point cases is similar to the above. We briefly discuss these cases here.

During the proof of Proposition 3 we put ϕ, ψ and ρ into the desired forms for Theorem 3 (in the simple point case), with the exception that we did not show that $n_1 < p_1$. In Proposition 3 we assumed only that ϕ, ψ, and ρ satisfied the first two conditions of Nash-minimality. Since we are now working in a complete resolution we can assume all three conditions of Nash-minimality hold. In a manner entirely similar to the triple point case proved above we can then show that $n_1 < p_1$.

The computations involved in proving the double point case of Theorem 3 are essentially the same as those used in the proof of the triple point case above, with one exception. The double point case version of Equation (3) is

$$d\phi\, d\psi = u^{m_1+n_1} v^{m_2+n_2} \left(A\frac{dudv}{uv} + B\frac{dvdw}{v} + C\frac{dudw}{u} \right).$$

By Lemma 4 at least one of A, B, or C is a local unit. This leads to two possible cases: we can write $\psi' = u^{n_1} v^{n_1} R$ where R is either a local unit (if A is a local unit) or R is the product of a local unit with w (if A is not a local unit). This produces the two cases in part (1) of Definition 2 for Hsiang-pati ordered exponents at double points. Although the proof of Theorem 3 in the double point case must now be split into two separate cases, the calculations are again analogous to those in the proof of the triple point case.

It is worth remarking that in Pati's double-point statement of the three dimensional case (see [14]), the functions playing the roles of ψ' and ρ' are always of the form of the first case, namely $\psi' = u^{n_1} v^{n_2}$ and $\rho' = u^{p_1} v^{p_2} w$. Pati blows up as much as necessary to achieve this result, while here we only want to blow up enough so that we have a complete resolution. A complete resolution guarantees that we are in one of the two cases described above, but cannot ensure that we are in the "first" case as in [14]. This does not, however, make the Hsiang-Pati coordinates constructed here any less useful.

6.4. Proof of Theorem 2

Now that we have proved Theorem 3 we can prove Theorem 2 as a simple corollary. As the simple, double, and triple point cases are entirely analogous we focus only on the triple point case.

Proof (Theorem 2). Let $\pi\colon(\widetilde{U},E)\to(U,v)$ be a complete resolution and choose a triple point $e\in E$. Use Theorem 3 to construct Hsiang-Pati coordinates ϕ, $\psi=S+\psi'$, and $\rho=T+\rho'$ in an analytic neighborhood W on $\widetilde{U}$ with divisor coordinates $\{u,v,w\}$ centered at e. We will show that the distinguished monomial parts ϕ, ψ', and ρ' of these Hsiang-Pati coordinates induce monomial generators of $\mathcal{N}$. By construction the exponents of ϕ, ψ', and ρ' form a Hsiang-Pati ordered set, and the differentials of the functions ϕ, ψ, and ρ generate the Nash sheaf $\mathcal{N}$ on $\widetilde{U}$. It remains to show that $d\phi$, $d\psi'$, and $d\rho'$ also generate $\mathcal{N}$.

We will show that the basis $\{d\phi,d\psi,d\rho\}$ can be written in terms of $d\phi$, $d\psi'$, and $d\rho'$. It suffices to show that dS and dT can be written in terms of $d\phi$, $d\psi'$, and $d\rho'$. It is simple to calculate:

$$dS=\Big(\sum s_l\epsilon_l\phi^{\epsilon_l-1}\Big)\,d\phi.$$

Thus since each ϵ_l is greater than or equal to one by part (2a) of Theorem 3, dS is a holomorphic multiple of $d\phi$.

For the second calculation we will separate the terms of T according to whether τ_l is equal to zero, so that $T=\sum_{\tau_l=0}t_l\phi^{\delta_l}+\sum_{\tau_l\neq0}t_l\phi^{\delta_l}(\psi')^{\tau_l}$. We can then calculate:

$$dT=\Big(\sum_{\tau_l=0}t_l\delta_l\phi^{\delta_l-1}+\sum_{\tau_l\neq0}t_l\delta_l\phi^{\delta_l-1}(\psi')^{\tau_l}\Big)d\phi+\Big(\sum_{\tau_l\neq0}t_l\tau_l\phi^{\delta_l}(\psi')^{\tau_l-1}\Big)d\psi'.$$

By part (3a) of Theorem 3 we have $\delta_l\geq1$ when $\tau_l=0$ and $\delta_l m_1+\tau_l n_1$ greater than or equal to n_1 (and thus also m_1) when $\tau_l\neq0$ (similarly for m_2 and m_3). Thus dT is a holomorphic linear combination of $d\phi$ and $d\psi'$.

The double and simple point cases are similar. $\square$

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