Mancala Matrices

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Abstract

We introduce a new matrix tool for the sowing game Tchoukaillon that enables us to non-iteratively construct an explicit bijection between board vectors and move vectors. This allows us to provide much simpler proofs than currently appear in the literature for two key theorems, as well as a non-iterative method for constructing move vectors. We also explore extensions of our results to Tchoukaillon variants that involve wrapping and chaining.

1 Introduction

Mancala is the most commonly known form of a diverse classification of games known as sowing games. The unifying feature of such games is a game board populated with bins, which are in turn occupied by stones. While the rules and overarching goals of such games vary, they all share an identical sowing mechanic by which players make individual plays. We will focus on the solitaire sowing game Tchoukaillon that models a certain sequence of moves by one player in Mancala.

Tchoukaillon is only considered a game if you consider it a game to walk through a maze with a map. The most challenging aspect of Tchoukaillon is that when playing the 'game,' order is extremely important; much like when using a map, the direction that you turn is very important. The novel, intriguing result of this paper is that taking this orderly 'game' and rewriting it as a matrix equation somehow ignores (or incorporates) order of play. This matrix projection reproduces known Tchoukaillon results in an extremely straightforward manner and also gives much deeper insights into and extensions to similar sowing games. This paper demonstrates a great example of the applicability of linear algebra.

Despite its relative simplicity, Tchoukaillon has been an active area of mathematical research. Tchoukaillon was studied by Gautheron and introduced by Deledicq and Popova [4] in 1977. Initial results yielded many basic mechanical facts about the game, but left open difficult questions concerning the relationship between board length and number of stones on winning boards. A recent paper by Jones, Taalman, and Tongen [6] presents these problems in their historical context and examines extensions of Tchoukaillon that relate to the Chinese Remainder Theorem and graph theory. In this paper we construct a matrix representation of the play/unplay algorithm and show that it is extensible to generalizations of Tchoukaillon.

The outline of this paper is as follows. In Section 2 we review the basic rules of Tchoukaillon and discuss the set of winning Tchoukaillon board vectors, including a non-iterative method for constructing board vectors. In Section 3 we introduce the concept of a move vector, and discuss and prove a well-known relationship between moves and bin quantities in Theorem 2.

In Section 4 we develop our main tool, the unplay matrix. This matrix will allow us to very quickly recover the results of Theorems 1 and 2, and therefore give much more immediate proofs of these theorems than are currently in the literature (see Equations (1) and (3)). In Section 5 we give a complete bijective diagram expressing the correspondence between boards, moves, and positive integers, and give an algorithm for explicitly constructing any move vector given only a positive integer. Finally, in Sections 6 and 7 we extend the notion of an unplay matrix to Tchoukaillon variants that involve wrapping and chaining.

2 Board vectors

Tchoukaillon is played with an arbitrary number of bins indexed $1, 2, 3, \ldots, \ell$ that contain $b_1, \ldots, b_\ell$ stones, and an empty bin called the Ruma. The game is won by sowing all of the stones into the Ruma. To sow, a player chooses a bin to harvest, picks up all stones from that bin, deposits one stone at a time from the harvested bin into each subsequent bin toward the Ruma. To be a legal play, the last stone placed must land in the Ruma. Therefore a bin is *harvestable* if and only if the bin contains the number of stones equal to its index, or $b_i = i$. The game is won by playing harvestable bins until no stones remain on the board, if possible (in which case we say that we began with a *winning board*, or that the board was *winnable*). Figure 1, from [6], demonstrates this concept.

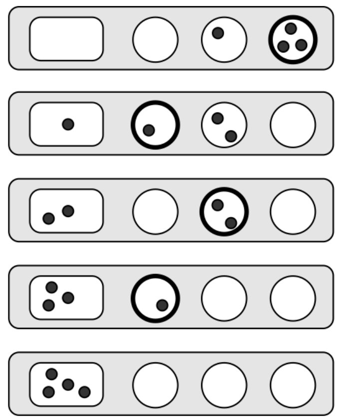

Figure 1: A winning Tchoukaillon board: By playing the harvestable bins shown in bold we can move all stones to the Ruma.

If a bin contains fewer stones than its index, then it is *underfull*, and cannot be played because its last stone would not reach the Ruma. If a bin contains more stones than its index, then it is *overfull*, and cannot be played because its last stone would overshoot the Ruma; see Figure 2. The presence of any overfull bin guarantees a losing Tchoukaillon board.

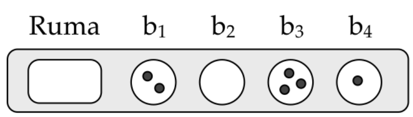

Figure 2: A losing Tchoukaillon board: Since $b_1 = 2$, bin 1 is overfull and this is a losing board; playing bin 1 at any time would overshoot the Ruma and lose the game. Note also that since bin 4 is underfull and there are no nonempty bins to its right, bin 4 would never become harvestable.

On a winnable board there is only one possible winning strategy: at each play one must sow the harvestable bin with the smallest index (which we call the *first harvestable bin*), since to sow a later harvestable bin would overfill the first harvestable bin and lose the game; see Figure 3.

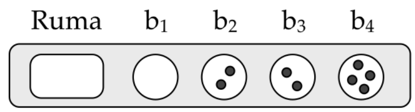

Figure 3: Since bin 2 is the first harvestable bin (note $b_2 = 2$), it must be played first. Although bin 4 is also harvestable, playing bin 4 first would overfill bin 2.

Note that in this paper we are following the convention suggested in [6] that the Ruma is to the left of the bins, and the direction of play is to the left. Although Tchoukaillon and Mancala are traditionally played with the Ruma on the right, a left Ruma placement turns out to be more natural for mathematical study and computer exploration. The winnable Tchoukaillon boards up to length $\ell = 5$ and number $n = 11$ of stones are shown in Figure 4.

n	b_1	b_2	b_3	b_4	b_5
1	1				
2	0	2			
3	1	2			
4	0	1	3		
5	1	1	3		
6	0	0	2	4	
7	1	0	2	4	
8	0	2	2	4	
9	1	2	2	4	
10	0	1	1	3	5
11	1	1	1	3	5

Figure 4: The first eleven winnable Tchoukaillon boards, indexed by number n of stones. Note that playing the first harvestable bin in any row will yield the row immediately above.

It is well known (see [2, 4, 6]) that there is a unique winning Tchoukaillon board for each positive integer n of stones. For any positive integer length ℓ there is a collection of winning Tchoukaillon boards with that given length. In this paper we will be interested in both the set of all possible winning boards and the sets of winning boards of every given length.

Definition 1. *An l-tuple* $(b_1, b_2, \ldots, b_\ell)$ *is a* board vector *if the Tchoukaillon board whose bins contain* $b_1, b_2, \ldots, b_\ell$ *stones is a winnable board. The set* *of all board vectors will be denoted $\mathcal{B}$, and the set of all board vectors with length less than or equal to ℓ will be denoted B_ℓ.*

A obvious question to ask is how we can immediately identify whether or not a given ℓ-tuple is an element of B_ℓ. For example, the vector $(2,0,3,1)$ illustrated in Figure 2 is not in B_4, but the vector $(0,2,2,4)$ illustrated in Figure 3 is in B_4; it is the unique board vector with $n = 8$ stones, as shown in the eighth row of Figure 4. Membership in B_ℓ is determined by the existence of a move vector that describes a winning strategy for the board, as defined in the next section.

The following theorem records two results that explicitly characterize board vectors in terms of the moduli of the bin indices.

Theorem 1 (Jones, Taalman, and Tongen)**.** *An ℓ-tuple $(b_1, b_2, \ldots, b_\ell)$ with $n = \sum_{k=1}^{\ell} b_i$ is a board vector with n stones if and only if $b_i \le i$ for all i and the lower partial sums of the bin values satisfy*

$$\sum_{k=1}^{i} b_k \equiv n \mod (i+1), \text{ for each } 1 \le i \le \ell.$$

This is equivalent to the condition that $b_i \le i$ for all i and the upper partial sums of the bin values satisfy

$$\sum_{k=i}^{\ell} b_k \equiv 0 \mod i, \text{ for each } 1 \le i \le \ell.$$

The first characterization of board vectors in Theorem 1 allows us to quickly construct the unique winning Tchoukaillon board for any number of stones n. The second characterization allows us to quickly construct all the winning Tchoukaillon boards for any length ℓ. We will be able to recover both of these results from the unplay matrix characterization that we will develop in Section 4.

3 Move vectors

If a vector $(b_1, b_2, \ldots, b_\ell)$ represents a winning Tchoukaillon board, then there must be some sequence of Tchoukaillon plays that reduces the vector to zero. These plays must happen in a unique order, since at each stage we are forced to play the first harvestable bin. For example, consider the board vector $(b_1, b_2, b_3, b_4) = (0,2,2,4)$ in the eighth row of Figure 4. Reading the table from bottom to top reveals the unique sequence of plays that must be taken to win this board. We must first harvest the second bin, then the first, then the fourth, and so on, with the complete sequence of eight plays taken from bins $2, 1, 4, 1, 3, 1, 2$, and 1, in that order. Note that since each play adds one stone to the Ruma and thus removes one stone from the board, a winning board with n stones always requires exactly n plays.

Although actually playing a Tchoukaillon board requires that we play in a particular order determined by the first harvestable pit at each stage, the matrix construction in this paper will show that, surprisingly, we do not in fact need to preserve this order of play in mathematical representations of Tchoukaillon. As we will see in Section 4, it suffices to keep track of only the number of times that each bin is played.

For example, looking at the sequence of plays from bins $2, 1, 4, 1, 3, 1, 2, 1$ for the board vector $(b_1, b_2, b_3, b_4) = (0, 2, 2, 4)$, we see that bin 1 is played a total of four times, bin 2 is played a total of two times, and bins 3 and 4 are each played just one time. We can use this information to construct a *move vector* of the form $(m_1, m_2, m_3, m_4) = (4, 2, 1, 1)$. Note that since an n-stone board requires exactly n plays to remove all stones, the sum of the m_i must equal n (in our example $4 + 2 + 1 + 1 = 8$), and that by construction the length of the move vector is the same as the length of the board vector.

Definition 2. *An n-tuple $(m_1, m_2, \ldots, m_\ell)$ is the* move vector *corresponding to a winning Tchoukaillon board $(b_1, b_2, \ldots, b_\ell)$ if for each i, m_i is the number of times that one must play from bin b_i to win the board. The set of all move vectors will be denoted $\mathcal{M}$, and the set of all move vectors corresponding to boards of length less than or equal to ℓ will be denoted M_ℓ.*

Constructing a move vector by actually following all steps to win a particular board can be a tedious process when large numbers of stones are involved. Our first result expresses the algebraic relationship between a board and its move vector, providing a much more efficient method for obtaining the move vector of any winning Tchoukaillon board. Notice in particular that we can use the result of this theorem to obtain the move vector that corresponds to a given board vector without ever considering the order in which the plays are made. A similar equation appears in equation (2) of Broline and Loeb's paper [2] in different notation, without proof.

Theorem 2. *The move vector $(m_1, m_2, \ldots, m_\ell)$ corresponding to a winning Tchoukaillon board $(b_1, b_2, \ldots, b_\ell)$ can be recursively defined as follows:*

$$m_i = \begin{cases} 1, & \text{if } i = \ell, \\ \frac{1}{i}\left(b_i + \sum_{j=i+1}^{\ell} m_j\right), & \text{if } 1 \le i \le \ell - 1. \end{cases}$$

Proof. For any winning Tchoukaillon board $(b_1, b_2, \ldots, b_\ell)$ it is obvious that the number of stones eventually played from each bin must be divisible by the index of that bin, since each play from bin i involves picking up i stones. Since no bin in a winning board can be overfull we also have $b_i \leq i$ for all $1 \leq i \leq \ell$. Combining these facts for $i = \ell$, we see that the number of stones in the last bin must be $b_\ell = \ell$, and that this bin must be played exactly $m_\ell = 1$ times.

Now consider any integer i with $1 \leq i \leq \ell - 1$. Since bin i is played m_i times with each play using i stones, the total number of stones played from bin i is equal to im_i. On the other hand, the total number of stones played from bin i is the sum of the number b_i of stones initially in bin i and the number of stones that at some point get added to bin i. Since the number of stones that get added to bin i is equal to the sum of the number of moves from bins with index greater than i, we have

$$b_i + \sum_{j=i+1}^{\ell} m_j = i\, m_i.$$

Simple algebra now gives the desired result. $\square$

For example, with the formula in Theorem 2 we can calculate the move vector corresponding to the winning board $(b_1, b_2, b_3, b_4) = (0, 2, 2, 4)$ without actually playing harvestable bins in sequence. Starting from the index $\ell = 4$ we once again obtain the move vector $(m_1, m_2, m_3, m_4) = (4, 2, 1, 1)$:

$$\begin{aligned}
m_4 &= 1, \\
m_3 &= \tfrac{1}{3}(b_3 + m_4) = \tfrac{1}{3}(2 + 1) = 1, \\
m_2 &= \tfrac{1}{2}(b_2 + m_3 + m_4) = \tfrac{1}{2}(2 + 1 + 1) = 2, \\
m_1 &= \tfrac{1}{1}(b_1 + m_2 + m_3 + m_4) = \tfrac{1}{1}(0 + 2 + 1 + 1) = 4.
\end{aligned}$$

The move vectors corresponding to the eleven winning Tchoukaillon boards in Figure 4 are shown in Figure 5.

n	m_1	m_2	m_3	m_4	m_5
1	1				
2	1	1			
3	2	1			
4	2	1	1		
5	3	1	1		
6	3	1	1	1	
7	4	1	1	1	
8	4	2	1	1	
9	5	2	1	1	
10	5	2	1	1	1
11	6	2	1	1	1

Figure 5: The first eleven move vectors, indexed by number n of stones.

4 The unplay matrix

The simplest, but most tedious, way to construct a list of all of the winning Tchoukaillon boards is to start with the empty board and iterate a sequence of *unplay* moves, as follows. To play a bin on a Tchoukaillon board, one picks up all the stones from the first harvestable pit as sows them one by one towards the Ruma, ending by depositing the last stone into the Ruma. The reverse of this operation is to pick up a stone from the Ruma, and then move away from the Ruma picking up a stone from each nonempty bin along the way. The process ends by depositing all of the stones collected into the first empty bin reached. Each winning Tchoukailon board has exactly one unplay move available, and each unplay move adds one stone to the board.

Suppose $b = (b_1, b_2, \ldots, b_\ell)$ is a winning Tchoukaillon board and $b' = (b'_1, b'_2, \ldots, b'_\ell)$ is obtained from b by applying the unplay algorithm, where z is the index of the first empty bin of b, and thus the location where the unsowed stones will be deposited. Then the relationship between b and b' is clearly given by the following piecewise function (see also [6]):

$$b'_i = \begin{cases} b_i - 1, & \text{if } i < z \\ i, & \text{if } i = z \\ b_i, & \text{if } i > z. \end{cases}$$

For example, consider the unique winning Tchoukaillon board with seven stones, $b = (b_1, b_2, b_3, b_4) = (1, 0, 2, 4)$. In this example the first nonempty bin from the Ruma is bin $z = 2$, and unplaying b will take away one stone from each bin of index less than z, deposit 2 stones in bin z, and leave unchanged each bin of index greater than z, resulting in the unique winning board with eight stones, $b' = (b'_1, b'_2, b'_3, b'_4) = (b_1 - 1, 2, b_3, b_4) = (0, 2, 2, 4)$.

The main contribution of this paper is to reframe this well-known unplay algorithm in terms of a matrix that transforms move vectors into board vectors. In general, unplaying a board b to the ith empty pit results in a board vector b' that is obtained from the board vector b by adding an *effect vector* of the form

$$\epsilon_i = (-1, \dots, -1, i, 0, \dots, 0).$$

If a board b has corresponding move vector $m = (m_1, m_2, \dots, m_\ell)$, then we obtain b from the empty board by unplaying m_1 times to bin 1, m_2 times to bin 2, and so on; this means that b is in fact the weighted sum $\sum_{i=1}^{\ell} m_i \epsilon_i$ of effect vectors. The order of unplay moves is dictated by first empty pits, but we will not need to consider this order. By defining a matrix whose columns are the effect vectors for each bin, we obtain a matrix that can unplay an entire collection of moves at once:

Definition 3. *Given a positive integer ℓ, the* unplay matrix *of length ℓ is*

$$U_\ell = \begin{bmatrix} 1 & -1 & -1 & \cdots & -1 & -1 \\ 0 & 2 & -1 & \cdots & -1 & -1 \\ 0 & 0 & 3 & \dots & -1 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \ell - 1 & -1 \\ 0 & 0 & 0 & \cdots & 0 & \ell \end{bmatrix}.$$

Theorem 3. *If a winning board $b = (b_1, b_2, \dots, b_\ell)$ can be obtained from the empty board by unplaying according to move vector $m = (m_1, m_2, \dots, m_\ell)$, then*

$$U_\ell\, m = b.$$

Proof. Let U_ℓ and ϵ_i be as above, and let $e_i = (0, 0, ..., 1, ..., 0)$ be the ith coordinate vector, with a 1 in the ith coordinate and 0s elsewhere. By construction, applying U_ℓ to a valid move vector m will return the winning Tchoukaillon board b corresponding to that move vector, since

$$U_\ell\, m = U_\ell \left(\sum_{i=1}^{\ell} m_i e_i \right) = \sum_{i=1}^{\ell} m_i U_\ell e_i = \sum_{i=1}^{\ell} m_i \epsilon_i = b.$$

□

For example, we have seen that the board $b = (0, 2, 2, 4)$ has corresponding move vector $m = (4, 2, 1, 1)$, and we can verify that

$$U_\ell\, m = \begin{bmatrix} 1 & -1 & -1 & -1 \\ 0 & 2 & -1 & -1 \\ 0 & 0 & 3 & -1 \\ 0 & 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 2 \\ 4 \end{bmatrix} = b.$$

Unplay matrices provide a very efficient framework for answering questions about Tchoukaillon. For example, the unplay matrix U_ℓ allows us to recover Theorem 2 immediately, as follows. If $b = U_\ell\, m$ we have $b = \sum_{k=1}^{\ell} m_k \epsilon_k$. Let $(\epsilon_k)_i$ denote the ith coordinate of the kth effect vector, which is equal to 0 if $i > k$, equal to i if $i = k$, and equal to -1 if $i < k$. Then for $1 \le i \le \ell - 1$ we have

$$b_i = \sum_{k=1}^{\ell} m_k (\epsilon_k)_i = i m_i - \sum_{k=i+1}^{\ell} m_k, \tag{1}$$

and for $i = \ell$ we have $b_\ell = \ell m_\ell$ and thus $m_\ell = 1$. This is exactly the result from Theorem 2. Compare the efficiency of this argument to the proof of Theorem 2 presented earlier.

It is straightforward to verify that the inverse of the unplay matrix U_ℓ is the matrix

$$U_\ell^{-1} = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{6} & \cdots & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & \frac{1}{3} & \cdots & \frac{1}{12} & \frac{1}{12} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{\ell-1} & \frac{1}{\ell(\ell-1)} \\ 0 & 0 & 0 & \cdots & 0 & \frac{1}{\ell} \end{bmatrix}.$$

The matrix U_ℓ^{-1} can be applied to any winning Tchoukaillon board vector b to produce the move vector $m = U_\ell^{-1}\, b$ corresponding to that board.

Just as we used U_ℓ to write the b_i in terms of the m_i, we can use U_ℓ^{-1} to very quickly get an expression for the m_i in terms of the b_i. A similar calculation to the one above gives

$$m_i = \frac{(i+1)b_i + \sum_{k=i+1}^{\ell} b_k}{i(i+1)}. \tag{2}$$

By combining Equations (1) and (2) we can now recover Theorem 1. The two equations together imply that

$$\sum_{k=i+1}^{\ell} b_k = (i+1) \sum_{k=i+1}^{\ell} m_k, \tag{3}$$

which means that the upper sum of the b_k from $i+1$ to l are all congruent to 0 modulo $i+1$. This is equivalent to the second formulation of Theorem 1. Note that the unplay matrix has allowed us to make a simpler proof than the one presented in [6].

5 Constructing board vectors and move vectors

In [6], Jones, Taalman, and Tongen describe an explicit way to construct the unique winning Tchoukaillon board for any positive integer n, without resorting to iterative unplay of first harvestable bins, using the first part of the result we quoted in Theorem 1. Given any $n \in \mathbb{Z}^+$, they construct $b = (b_1, b_2, \ldots, b_\ell)$ by defining each b_i as shown below.

$$\begin{aligned}
b_1 &= n \mod 2 \\
b_2 &= n - b_1 \mod 3 \\
b_3 &= n - (b_1 + b_2) \mod 4 \\
\vdots \quad &\quad \vdots \\
b_\ell &= n - (b_1 + b_2 + \cdots + b_{\ell-1}) \mod \ell
\end{aligned}$$

Note that this method of constructing the b_i automatically finds ℓ, since when the sum of the b_i is equal to n this algorithm will set all subsequent values of b_i equal to zero. Note that in this algorithm we are using the notation $a = b \bmod c$ to mean that $b = qc + a$ for some unique $q \in \mathbb{N}$ and $0 \leq a < c$.

If we denote the process described above as a map $\beta \colon \mathbb{Z}^+ \to \mathcal{B}$, let Σ denote the map that adds the components of a vector, and let U be an unplay matrix either of infinite dimension or sufficiently high dimension to accommodate the Tchoukaillon boards we are interested in, we obtain the commutative diagram shown in Figure 6.

$$
\begin{tikzcd}
\mathcal{M} \arrow[rr, "U"] \arrow[dr, "\Sigma"] & & \mathcal{B} \arrow[ll, bend right, "U^{-1}"'] \arrow[dl, "\Sigma"'] \\
& \mathbb{Z}^+ \arrow[ul, bend left, "\mu"] \arrow[ur, bend right, "\beta"'] &
\end{tikzcd}
$$

Figure 6: Maps between moves, boards, and positive integers

Every map in this diagram is a bijection. One of the key results in [6] was to make the map β explicit, to obtain a non-iterative construction of Tchoukaillon boards for any number of stones. We now prove an analogous result, making explicit the map $\mu\colon \mathbb{Z}^+ \to \mathcal{M}$ in Figure 6, to obtain a non-iterative construction of move vectors for any positive integer size.

Of course one could easily define μ to be the composition $U^{-1}\beta$, but our goal is to produce a straightforward algorithm that mimics the one for β described above. Interestingly, we can obtain this result with a clever use of ceiling functions.

Theorem 4. *Given any $n \in \mathbb{Z}^+$, the move vector for solving the unique Tchoukaillon board with n stones is $m = (m_1, m_2, \ldots, m_\ell)$, where*

$$
\begin{aligned}
m_1 &= \left\lceil \frac{n}{2} \right\rceil \\
m_2 &= \left\lceil \frac{n - m_1}{3} \right\rceil \\
m_3 &= \left\lceil \frac{n - (m_1 + m_2)}{4} \right\rceil \\
\vdots &= \qquad \vdots \\
m_\ell &= \left\lceil \frac{n - (m_1 + m_2 + \cdots + m_{\ell-1})}{\ell + 1} \right\rceil .
\end{aligned}
$$

Proof. Given $n \in \mathbb{Z}^+$ and m as defined in the theorem, it suffices to show that the vector $b = Um$ is the unique board vector with n stones. To do that, we invoke the second part of Theorem 1, and show that $\sum_{i=r+1}^{\ell} b_i$ is congruent to 0 modulo $r + 1$ for each $1 \leq r \leq \ell$.

For notational convenience we will set $A_i = n - (m_1 + m_2 + \cdots + m_{i-1})$ for $2 \leq i \leq \ell$, and $A_1 = n$. Combining this with Equation (1) and our construction of the m_i we have

$$\begin{aligned}
\sum_{i=r+1}^{\ell} b_i &= \sum_{i=r+1}^{\ell}\left(im_i - \sum_{k=i+1}^{\ell} m_k\right) \\
&= \sum_{i=r+1}^{\ell}\left(i\left\lceil\frac{A_{i-1}}{i+1}\right\rceil - \sum_{k=i+1}^{\ell}\left\lceil\frac{A_{k-1}}{k+1}\right\rceil\right) \\
&= \sum_{i=r+1}^{\ell}\left(i\left\lceil\frac{A_{i-1}}{i+1}\right\rceil - \left\lceil\frac{A_{i}}{i+2}\right\rceil - \cdots - \left\lceil\frac{A_{\ell-1}}{l+1}\right\rceil\right) \\
&= (r+1)\left\lceil\frac{A_r}{r+2}\right\rceil + (r+1)\left\lceil\frac{A_{r+1}}{r+3}\right\rceil + \cdots + (r+1)\left\lceil\frac{A_{\ell-1}}{l+1}\right\rceil.
\end{aligned}$$

Since this is clearly divisible by $r+1$, the proof is complete. □

6 Extensions to wrapping

In Mancala and Tchuka Ruma, a legal move may "wrap" around the board, sowing stones one or more times around the entire board and then ending in the Ruma. As a first step to generalizing unplay matrix methods to these games, let us consider the game of *Tchoukaillon-with-wrapping*. Analysis of this game is significantly more complicated than analysis of Tchoukaillon. In this section we generalize the idea of an unplay matrix to a pair of matrices that together represent unplaying-with-unwrapping.

In Tchoukaillon-with-wrapping we connect the two ends of a Tchoukaillon board to make a circular board and then sow seeds counterclockwise, allowing wrapping, as illustrated in Figure 7.

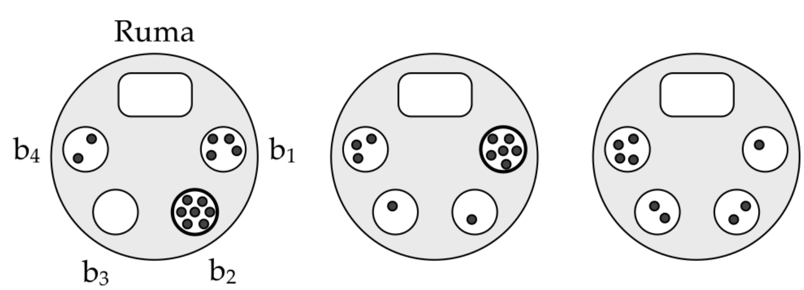

Figure 7: The winning Tchoukaillon-with-wrapping board $(4, 7, 0, 2)$: Playing bin 2 results in the board $(6, 1, 1, 3)$. Then playing bin 1 results in the board $(1, 2, 2, 4)$, which is winnable because it is a winning (non-wrapping) Tchoukaillon board.

In this example there was only one harvestable bin at each stage, but in general there can be more than one harvestable bin in a Tchoukaillon-with-wrapping board, and there is no known strategy for choosing which harvestable bin to play first.

The notion of unplaying in Tchoukaillon-with-wrapping is much like our previous notion of unwrapping, where we pick up a stone from the Ruma and then from each subsequent pit until we arrive at an empty pit and deposit the stones in our hand. The only differences are that the board is of a fixed length and that the unplay process might result in (un-)wrapping around the board one or more times before being able to deposit the stones in an empty bin. In general, unplaying on a length ℓ wrapping board b to the ith empty bin after k wraps results in a board b' that is obtained from b by adding both the ith effect vector

$$\epsilon_i = (-1, \ldots, -1, i, 0, \ldots, 0)$$

and the (k,i)th *unwrapping effect vector*

$$\omega_{k,i} = (-k, \ldots, -k, k\ell, -k, \ldots, -k),$$

where the i in the ϵ_i effect vector and the $k\ell$ in the $\omega_{k,i}$ unwrapping effect vector occur in the ith coordinate. Note that these vectors depend on ℓ but we choose not to include that in the notation since we will always have l fixed at the outset.

For example, the middle board $(6,1,1,3)$ shown in Figure 7 unwraps to the leftmost board $(4,7,0,2)$ after $k=1$ wrap that ends in bin $i=2$. This unwrapping move is equivalent to addition by $\epsilon_2 = (-1,2,0,0)$ and $\omega_{1,2} = (-1,4,-1,-1)$:

$$(6,1,1,3) + (-1,2,0,0) + (-1,4,-1,-1) = (4,7,0,2).$$

This clearly suggests that we will require two matrices to represent Tchoukaillon-with-wrapping unplay moves. Let U_ℓ be the unplay matrix defined earlier, and define the *unwrap matrix*

$$W_\ell = \begin{bmatrix} \ell & -1 & -1 & \cdots & -1 & -1 \\ -1 & \ell & -1 & \cdots & -1 & -1 \\ -1 & -1 & \ell & \ldots & -1 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & -1 & \ldots & \ell & -1 \\ -1 & -1 & -1 & \cdots & -1 & \ell \end{bmatrix}.$$

Now suppose that a winning Tchoukaillon-with-wrapping board $b = (b_1, ..., b_\ell)$ can be won by playing a total of m_i times from each bin b_i, and that these plays involve a total of w_i wraps from each bin b_i. This gives us a move vector $m = (m_1, \ldots, m_\ell)$ and a wrap vector $w = (w_1, \ldots, w_\ell)$ for the board b, and we have

$$U_\ell m + W_\ell w = b.$$

Writing Tchoukaillon-with-wrapping boards using the above construction could give much insight into this new game, but the purpose of this section is to show the power of the general matrix techniques derived in this paper.

7 Extensions to chaining

The game of Tchuka Ruma also allows "chaining" moves, where a player can sow a stone to a nonempty bin that is not the Ruma, and then pick up all the stones in that bin and continue to sow. If repeated chaining ultimately ends in the Ruma then the move is valid, but if at any point the player sows to an empty bin, the game is lost. As another step toward generalizing our methods to Tchuka Ruma and other games, we conclude this paper by constructing the unplay matrices for the game *Tchoukaillon-with-chaining* and presenting a basic preliminary result.

In Tchoukaillon-with-chaining we use a linear Tchoukaillon board and sow towards the Ruma allowing chains as needed. Note that once a chain is begun, it must be followed until the stones are sown to the Ruma; for example, see Figure 8.

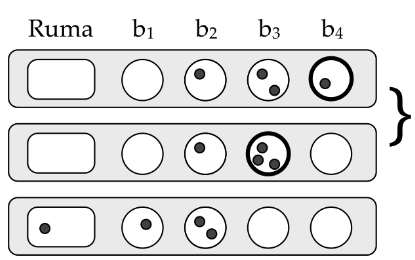

Figure 8: The winning Tchoukaillon-with-chaining board $(0, 1, 2, 1)$: Sowing from bin 4 ends in the nonempty bin 3, and results in the board $(0, 1, 3, 0)$; at this point we are obligated to continue the chain and sow the three stones in bin 3. This sowing lands in the Ruma and thus finishes the chain, obtaining the board (1,2,0,0) which is a winning (non-chaining) Tchoukaillon board.

In general there may be more than one sequence of moves that successfully clears a winning Tchoukaillon-with-chaining board. In particular, notice that the criterion of harvestability is weaker in this game, since bins can be harvestable even when underfull.

Each chaining move is a sequence of sowings of bins $a_1, \ldots, a_k$, where we first sow from bin a_1 to bin a_2, then bin a_2 to bin a_3, and so on. We denote such a chaining move as $c_{a_1,\ldots,a_k}$. Note that $a_1 > a_2 > \cdots > a_k$ and for $1 \leq j \leq k$, and that a board is in *chainable position* for a chain through $a_1, \ldots, a_k$ if and only if for the jth sowing we have $a_j - a_{j+1}$ stones in bin a_j, so as to land in bin a_{j+1}.

By following through a chaining move backwards, it can be shown that the *unchaining effect vector* $u_{a_1,\ldots,a_k}$ for the chain $c_{a_1,\ldots,a_k}$ has ith coordinate

$$(u_{a_1,\ldots,a_k})_i = \begin{cases} a_k - 1, & \text{if } i = a_k \text{ and } k > 1 \\ a_j - a_{j+1} - 1, & \text{if } i = a_j \text{ for some } 1 < j < k \\ a_1 - a_2, & \text{if } i = a_1 \\ -1, & \text{if } i < a_1 \text{ and the above cases do not apply} \\ 0, & \text{if } i > a_1. \end{cases}$$

For example, to unchain from the third Tchoukaillon-with-chaining board $(1, 2, 0, 0)$ in Figure 8 to the top board $(0, 1, 2, 1)$ in that figure, we have to add the unchaining effect vector

$$u_{4,3} = (-1, -1, (3-1), (4-3)) = (-1, -1, 2, 1).$$

As a more complicated example, consider the chain that sows bins $a_1 = 7$, $a_2 = 5$, and $a_3 = 3$ on a length $\ell = 8$ Tchoukaillon-with-chaining board. The corresponding unchaining effect vector is

$$\begin{aligned} u_{7,5,3} &= (-1, -1, (3-1), -1, (5-3-1), -1, (7-5), 0) \\ &= (-1, -1, 2, -1, 1, -1, 2, 0). \end{aligned}$$

It is important to note that not all decreasing sequences of bins can be part of a chaining sequence, and that by our chainable position criteria, an unchaining move is only possible on very specific board configurations. For example, in order to be able to apply the unchaining move $u_{7,5,3}$ just described we must start with the Tchoukaillon-with-chaining board $(1, 1, 0, 1, 0, 1, 0, 0)$. Adding $u_{7,5,3}$ to this vector obtains the new board $(0, 0, 2, 0, 1, 0, 2)$. The reader can verify that the new board becomes the original board if a chain from a_7 to a_5 to a_3 to the Ruma is applied, and that this configuration is necessary.

Given a fixed board length ℓ we can enumearate all possible chains $c_{a_1,\ldots,a_k}$. For example, if $\ell = 5$ then the possible 2-step chains are $c_{5,4}$, $c_{5,3}$, $c_{5,2}$, $c_{4,3}$, $c_{4,2}$, and $c_{3,2}$, and there is only one 3-step chain, namely, $c_{5,4,2}$. Note that not all chains of the form $c_{i,j}$ or $c_{i,j,k}$ are possible here, due to the chainable position criteria. We also have the simple one-move unchains c_5, c_4, c_3, c_2, and c_1. Thus there are 12 possible chaining moves for a board of length $\ell = 5$. It happens that 12 is the sum of the first five Fibonacci numbers, and, in fact, this relationship between Fibonacci numbers and chaining moves is true in general:

Theorem 5. *The number of possible chaining moves for a board of length ℓ is the sum of the first ℓ Fibonacci numbers.*

Proof. Let S_i be the number of chains originating at bin i, that is, the chains of the form $c_{i,a_2,\ldots,a_k}$. To prove the theorem, it suffices to prove that $|S_i| = f_i$, the ith Fibonacci number. We will do so by (strong) induction.

Because the only chain originating at $i = 1$ is c_1 and the only chain originating at $i = 2$ is c_2, we have $|S_1| = 1$ and $|S_2| = 1$. Now suppose that for some integer $i > 3$ we have $|S_j| = f_j$ for all $1 \leq j < i$, and consider the chains in S_i. We say a chain $c_{i,a_2,\ldots,a_k}$ in S_i is *terminal* if $a_2 = i - 1$, and non-terminal otherwise. Note that terminal chains cannot be extended to longer chains; for example, $c_{5,4,2}$ cannot be part of a longer chain $c_{r,5,4,2}$ because sowing from bin r to bin 5 would leave at least two stones in bin 5, and therefore it would not be possible to continue chaining to bin 4. Let T_i be the set of terminal chains in S_i and N_i be the set of non-terminal chains, so that $|S_i| = |T_i| + |N_i|$.

Since a terminal chain $c_{i,i-1,a_3,\ldots,a_k} \in T_i$ must have $a_3 \neq i - 2$, we can associate it by extension to a non-terminal chain $c_{i-1,a_3,\ldots,a_k} \in N_{i-1}$, and therefore $|T_i| = |N_{i-1}|$.

Now consider the non-terminal chains $N_i \subset S_i$. We can construct a bijection $\phi\colon N_i \to T_{i-1} \cup S_{i-2}$, as follows. Given $c = c_{i,a_2,\ldots,a_k} \in N_i$ we have $a_2 \neq i - 1$. Define

$$\phi(c) = \begin{cases} c_{i-1,i-2,a_3,\ldots,a_k} & \text{if } a_2 = i - 2 \\ c_{a_2,\ldots,a_k} & \text{if } a_2 \neq i - 2. \end{cases}$$

With this map, each non-terminal chain $c \in N_i$ can be obtained either by changing the originating bin of a terminating chain in T_{i-1} from $i - 1$ to i, or by extending a chain in S_{i-2} by appending an i to the start of the chain. We can now conclude that $|N_i| = |T_{i-1}| + |S_{i-2}|$.

We have thus shown that $|S_i| = |T_i| + |N_i| = |N_{i-1}| + |T_{i-1}| + |S_{i-2}|$, and therefore that $|S_i| = |S_{i-1}| + |S_{i-2}| = f_{i-1} + f_{i-2} = f_i$. By induction we therefore have $|S_i| = f_i$ for all positive integers i. $\square$

There are $\sum_{j=1}^{\ell} f_j = f_{\ell+2}$ possible chains of length less than or equal to ℓ, and thus $f_{\ell+2}$ possible unchaining vectors. We can form an unchaining matrix of size $\ell \times f_{\ell+2}$ by taking the unchaining effect vectors as columns. For example, when $\ell = 5$ we have the twelve chains c_1, c_2, c_3, $c_{3,2}$, c_4, $c_{4,3}$, $c_{4,2}$, c_5, $c_{5,4}$, $c_{5,3}$, $c_{5,2}$, and $c_{5,4,2}$. These in turn have unchaining vectors as shown, respectively, in the columns of the following unchaining matrix:

$$C_5 = \begin{bmatrix} 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 0 & 2 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 \\ 0 & 0 & 3 & 1 & -1 & 2 & -1 & -1 & -1 & 2 & -1 & -1 \\ 0 & 0 & 0 & 0 & 4 & 1 & 2 & -1 & 3 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & 1 & 2 & 3 & 1 \end{bmatrix}.$$

Now if $b = (b_1, \ldots, b_\ell)$ is a winning Tchoukaillon-with-chaining board with a winning sequence of moves that uses the chains $c_1, c_2, \ldots, c_{5,4,2}$ in the order listed above a total of $t_1, \ldots, t_{12}$ times, respectively, then

$$C_5 t = b.$$

For instance, $t_1 = (5, 2, 1, 0, 1, 0, 0, 1, 0, 0, 0, 0)$ and $t_2 = (5, 0, 0, 3, 1, 0, 0, 1, 0, 0, 0, 0)$ both give the board vector $b = (0, 1, 1, 3, 5)$. This result means that the board $b = (0, 1, 1, 3, 5)$ has two different ways to win in Tchoukaillon-with-chaining, one with and one without chaining. This result also implies that there may be a relationship between playing moves and chaining moves.

Now that these matrix tools for Tchoukaillon have been extended to include wrapping and chaining separately, the next step is to use this framework to study the primary single-player sowing game Tchuka Ruma, which is a game that includes both wrapping and chaining. We believe that the matrix tools introduced in this paper will serve as a mathematical key to unlocking some of the secrets of Tchuka Ruma!

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