Intrinsic Knotting of Bipartite Graphs

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Abstract: We further identify and categorize intrinsically knotted bipartite graphs. We are motivated by a conjecture that a bipartite graph with $E \geq 4V - 17$ is intrinsically knotted. We verify the conjecture for graphs that have exactly 6 vertices in one part and at least 6 in the other. We also provide similar bounds for all bipartite graphs.

1. Introduction

Within the realms of knot theory and graph theory, we find opportunities to profoundly expand our understanding. First, however, we must be aware of some simple concepts and definitions.

A graph consists of edges and vertices. A graph is not a 3-dimensional construct, but we may place it in space as a spatial embedding. In this case, the edges are represented by curves, and the vertices are represented by points. There is generally more than one way to represent a graph as a spatial embedding.

Partite graphs are graphs where the vertices have been partitioned into two or more disjoint sets. A key characteristic of partite graphs is that the vertices from one part do not share any edges; they are connected only to vertices from the other parts. A bipartite graph has two parts. We refer to a bipartite graph with the following notation: $K_{a,b}$\me. K is the symbol for a complete graph, which means that it includes all possible edges. The symbols a and b communicate how many vertices are in each of the two parts of the graph. The letter e stands for edges, and is used only when the graph is a certain number of edges, m, short of being complete. For example, the notation $K_{8,7}$\12e refers to a set of graphs each of which has 15 vertices total, 8 in one part and 7 in the other, and 12 edges missing compared to the complete graph. As the complete graph has 56 edges, a $K_{8,7}$\12e graph will have 44 edges. Notice that there are many ways to remove 12 edges, so this notation does not refer to a single graph. Note also that $K_{a,b}$ and $K_{b,a}$ are two ways of denoting the same graph. We will usually write the larger part first.

A knot is a simple closed curve in space. The unknot is the trivial knot—it can be deformed to look like a circle. Within a graph, there exist many different cycles. These are paths that begin and end at the same vertex. They need not include all vertices and edges. If, in a graph, we follow a cycle of edges and vertices, it may be possible to identify a knotted cycle (i.e., a cycle containing a knot other than the unknot). We say that a graph is intrinsically knotted (IK) if for every spatial embedding, there exists at least one knotted cycle. It is important to understand that intrinsic knottiness is a property of the graph, not of the particular embedding or cycle. The goal of our research is to further identify and categorize the existence of intrinsically knotted bipartite graphs.

There are several methods used to show that a graph is IK. Our favored method is to show that a graph contains an IK subgraph or an IK minor. A subgraph, G, is obtained

from an original graph, G', through any sequence of vertex deletions and edge deletions. Minors are a larger category than subgraphs, as they can also be obtained through edge contractions. If we can show that a graph contains an IK minor or an IK subgraph, then we have shown that it, too, is IK.

In particular, we relied on the existence of a class of graphs described in [KS]. We will refer to them as the KS graphs. These are fourteen graphs that have been obtained from K_7, which was shown to be IK by Conway and Gordon [CG]. The KS graphs have been obtained through a series of $\underline{\Delta Y \text{ transformations}}$. This is a movement that replaces three vertices, connected as a triangle, with four, connected as a 'Y' shape, but maintains three edges. Significantly, this transformation is known to preserve the condition of intrinsic knottiness, so all of the KS graphs obtained from K_7 through ΔY transformations are also IK.
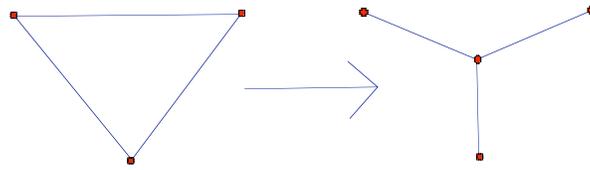
Triangle-Y transformations preserve IK

In our quest to prove that a bipartite graph with $E \geq 4V-17$ is intrinsically knotted, we first came across some smaller, though enlightening, results. For instance, we first studied some specific cases in an attempt to discover a pattern. In the course of these pursuits, we were able to show that $K_{5,5}\backslash 3e$ is one-quarter IK ($E = 4V-18$), that any graph of the form $K_{6,6} \backslash 5e$ is IK ($E = 4V-17$), that a graph of the form $K_{6,6} \backslash 6e$ is IK provided it is not the graph $K_{6,6}\backslash\{(a_1,b_1),(a_2,b_2),(a_3,b_3),(a_4,b_4),(a_5,b_5),(a_6,b_6)\}$, and that any graph of the form $K_{7,7}\backslash 10e$ is IK ($E = 4V-17$). We include the number of edges in terms of vertices to show that we can not do better than $E = 4V - 17$ in the case of $K_{5,5}$ and our results for $K_{6,6}$ and $K_{7,7}$ verify the conjecture for those graphs. We prove these results in section 3 below.

In the course of our journey, we were also able to show that all graphs of the form $K_{6+n,\,6}\backslash(2n + 5)e$ are IK where $n \geq 1$ (and, as above, this is also true when $n = 0$). This result proves our conjecture that if $E \geq 4V-17$, then the graph is intrinsically knotted for the case of a graph with exactly 6 vertices in one part and at least 6 in the other. We also found similar general results. For example, we showed that $K_{7+n,\,7}\backslash(2n + 10)e$ is IK where $n \geq 1$. As a Corollary, we see that such graphs are IK when $E \geq 5V-31$. (Although this is far from our conjecture, it is much better than the previous best bound.) For $K_{8+n,8}$, we have, $K_{8+n,8}\backslash(2n+15)$ is IK when $n \geq 1$. Also, any graph of the form $K_{a,\,a}\backslash(6a-34)e$ is IK where $a \geq 9$. Finally, we proved that all graphs of the form $K_{a+n,a}\backslash(3n + 6a-34)e$ are IK when $a \geq 9$ and $n \geq 0$. Together with the work of [CHPS], these give bounds on the number of edges required to ensure intrinsic knottiness for any bipartite graph. We prove these results in section 4.

We begin by presenting two lemmas in the next section.

2. Lemmas

Lemma 1. *Let $(k-1)(a + 1) < m + k$. If all graphs of the form $K_{a,\,b}\backslash m$ edges are IK, then all the graphs with $K_{a+1,\,b}\backslash (m + k)$ are also IK.*

Proof: Assume all $K_{a,b}\backslash me$ graphs are IK. Because $(k-1)(a + 1) < m + k$, more edges are removed than $(k-1)$ times the amount of vertices on the "a" side, so by the Pigeonhole Principle at least one of the $(a + 1)$ vertices in the $K_{a+1,b}\backslash(m + k)e$ has k or more edges removed. If we ignore that vertex we are left with a $K_{a,b}\backslash me$ graph, which is known to be IK because it was our base assumption.

Although we wrote $K_{a,b}$ in the above lemma, we did not assume that $a \geq b$ in proving it.

Lemma 2. *$K_2 + G$ is intrinsically knotted iff G is non-planar.*

We will omit this proof as it is proved in [BBFFHL].

3. Specific Results

To approach our goal of realizing a bound of 4V-17 or more edges for graphs with V vertices, we looked at some specific cases. We began with $K_{5,\,5}\backslash 3e$ for which $E = 4V - 18$, because [CHPS] had already shown that all $K_{5,\,5}\backslash 2e$ *graphs* are IK.

Theorem 1. *Of the four graphs of the form $K_{5,\,5}\backslash 3e$ exactly one is IK.*
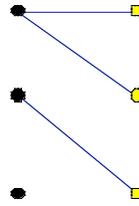
Proof: There are four different ways to remove three edges from a bipartite graph. These complement graphs are shown below:
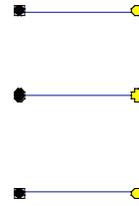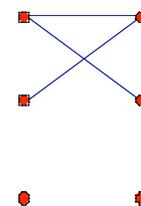
Case 1   Case 2   Case 3   Case 4
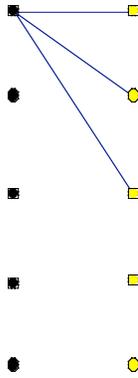
Case 1 was shown to be not IK in [CHPS].

We can show that Cases 2 and 3 are not IK by Lemma 2. According to this lemma, if we take away two vertices, and are able to draw the remaining graph in a plane, then the original graph is not IK. The vertices that will be removed are labeled A - D and will not be shown in the planar graphs below Please note that the graphs below are not complement graphs.
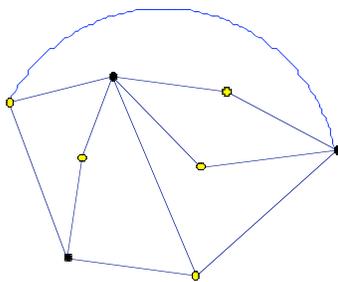
Case 2   Case 3
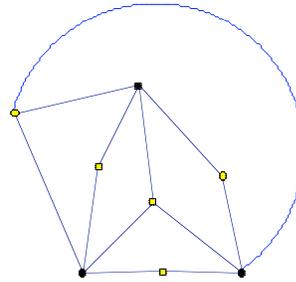
Note that Case 4, only, is IK, as it has the KS graph H_9 as a minor. This was shown in figure 8 of [MOR].

In terms of our conjecture that a bipartite graph with $E \geq 4V-17$ is intrinsically knotted, this result brings us closer by allowing us to show that for $K_{5,5} \backslash 3e$, the bound $E \geq 4V-18$ does not *always* work. We do, however, know that the bound $E \geq 4V-17$ (e.g., $K_{5,5} \backslash 2e$) *does* hold for graphs that have exactly 5 vertices in one part and at least 5 in the other. This was shown in [CHPS].

Theorem 2. *Any graph of the form $K_{6,6} \setminus 5e$ is IK.*

Proof: Let H' (Figure 1) be a K6,6 \ 12e graph that has an H9 minor. Recall that H9 is IK [KS] and that any graph we attain from H9 through vertex expansions and additions is also IK. We will list all ways to remove five edges and demonstrate that every case but one (the first case below which has an F' subgraph) has an H' subgraph. In each such case, these five edges form a subset of edges contained in the complement of H', H'^c. The edges of H'^c are shown in Figure 2 below.
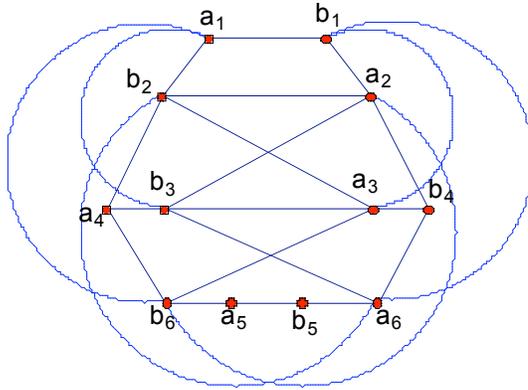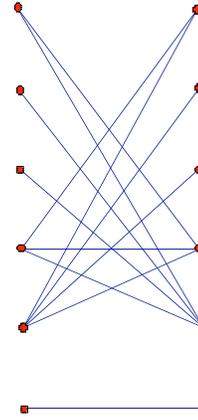

Figure 1. H'   Figure 2. H'^c

To determine all possible ways to remove 5 edges from $K_{6,6}$, first let $a_1, \ldots, a_6$ be the vertices from one part and $b_1, \ldots, b_6$ from the other part of the graph $K_{6,6} \setminus 5e$. Now consider any partition of 5 and take the *i*-th element in a partition to be the number of edges removed from the *i*-th vertex (for convenience, we have used top-down ordering) in the first part of $K_{6,6}$. Likewise, consider another partition and allow its entries to correspond to the number of edges removed from the other part of $K_{6,6}$. Some combinations will yield complement graphs of $K_{6,6} \setminus 5e$ that cannot be realized (for instance, the pairing $\{\{5\}, \{2,2,1\}\}$ cannot be constructed). We have observed that there may be more than one graph that can be constructed given a pairing. Below we have indicated the pairing of partitions of 5 for each of the 20 cases.

In addition, many of these graphs can be shown to be IK using Corollary 2.5 of [CHPS] which states that a $K_{6,5}$ graph with two or fewer edges removed is IK. In those cases, the list below specifies which vertex to remove to arrive at such a $K_{6,5}$. We apply the vertex labeling convention described above:

1. $K_{6,6}$-$\{(a_1,b_4),(a_2,b_5),(a_4,b_1),(a_5,b_2),(a_6,b_6)\}$; $\{\{1,1,1,1,1\},\{1,1,1,1,1\}\}$
2. $K_{6,6}$-$\{(a_1,b_5),(a_2,b_5),(a_4,b_4),(a_5,b_1),(a_6,b_6)\}$; $\{\{2,1,1,1\},\{1,1,1,1,1\}\}$
3. $K_{6,6}$-$\{(a_1,b_5),(a_2,b_5),(a_4,b_1),(a_5,b_1),(a_6,b_6)\}$; $\{\{2,2,1\},\{1,1,1,1,1\}\}$
4. $K_{6,6}$-$\{(a_1,b_5),(a_2,b_5),(a_5,b_1),(a_5,b_2),(a_6,b_6)\}$; $\{\{2,1,1,1\},\{1,1,2,1\}\}$
5. $K_{6,6}$-$\{(a_2,b_5),(a_4,b_1),(a_5,b_1),(a_5,b_2),(a_6,b_6)\}$; $\{\{2,1,1,1\},\{1,2,1,1\}\}$
6. $K_{6,6}$-$\{(a_1,b_5),(a_2,b_5),(a_5,b_1),(a_5,b_2),(a_6,b_6)\}$; $\{\{2,1,2\},\{1,2,1,1\}\}$
7. $K_{6,6}$-$\{(a_1,b_5),(a_2,b_5),(a_5,b_1),(a_5,b_2),(a_5,b_3)\}$; $\{\{2,1,1,1\},\{1,1,3\}\}$

Or remove vertex a_3 to obtain a $K_{6,5} \setminus 2e$ graph which we know to be IK (Corollary 2.5, [CHPS]).

8. $K_{6,6}-\{(a_1,b_4),(a_1,b_5),(a_5,b_1),(a_5,b_2),(a_5,b_3)\}$; $\{\{2,3\},\{1,1,1,1,1\}\}$
      Or remove vertex a_3 [CHPS].
9. $K_{6,6}-\{(a_1,b_5),(a_4,b_1),(a_4,b_4),(a_4,b_5),(a_5,b_1)\}$; $\{\{3,1,1\},\{1,2,2\}\}$
      Or remove vertex a_1 [CHPS].
10. $K_{6,6}-\{(a_4,b_1),(a_4,b_4),(a_4,b_5),(a_5,b_1),(a_6,b_6)\}$; $\{\{3,1,1\},\{1,1,2,1\}\}$
      Or remove vertex a_1 [CHPS].
11. $K_{6,6}-\{(a_4,b_4),(a_5,b_1),(a_5,b_2),(a_5,b_3),(a_6,b_6)\}$; $\{\{3,1,1\},\{1,1,1,1,1\}\}$
      Or remove vertex a_1 [CHPS].
12. $K_{6,6}-\{(a_1,b_4),(a_1,b_5),(a_5,b_1),(a_5,b_2),(a_5,b_4)\}$; $\{\{2,3\},\{1,2,1,1\}\}$
      Or remove vertex a_3 [CHPS].
13. $K_{6,6}-\{(a_1,b_4),(a_5,b_1),(a_5,b_2),(a_5,b_3),(a_5,b_4)\}$; $\{\{1,4\},\{2,1,1,1\}\}$
      Or remove vertex a_2 [CHPS].
14. $K_{6,6}-\{(a_5,b_1),(a_5,b_2),(a_5,b_3),(a_5,b_4),(a_6,b_6)\}$; $\{\{4,1\},\{1,1,1,1,1\}\}$
      Or remove vertex a_1 [CHPS].
15. $K_{6,6}-\{(a_1,b_1),(a_1,b_2),(a_1,b_3),(a_1,b_4),(a_1,b_5)\}$: $\{\{5\},\{1,1,1,1,1\}\}$
      Remove the vertex a_1 from original graph to obtain a complete $K_{6,5}$ graph. This
      graph has a $K_{5,5}$ minor, which is known to be IK [S].
16. $K_{6,6}-\{(a_1,b_4),(a_4,b_1),(a_4,b_5),(a_5,b_1),(a_5,b_4)\}$; $\{\{1,2,2\},\{2,2,1\}\}$
17. $K_{6,6}-\{(a_1,b_4),(a_4,b_1),(a_5,b_1),(a_5,b_4),(a_6,b_6)\}$; $\{\{1,2,1,1\},\{2,2,1\}\}$
18. $K_{6,6}-\{(a_1,b_4),(a_1,b_5),(a_2,b_5),(a_4,b_4),(a_4,b_5)\}$; $\{\{2,2,1\},\{2,3\}\}$
      Or remove vertex b_2 [CHPS].
19. $K_{6,6}-\{(a_1,b_4),(a_1,b_5),(a_4,b_4),(a_4,b_5),(a_6,b_6)\}$; $\{\{2,2,1\},\{2,2,1\}\}$
20. $K_{6,6}-\{(a_1,b_4),(a_4,b_1),(a_4,b_4),(a_4,b_5),(a_5,b_4)\}$; $\{\{1,3,1\},\{1,3,1\}\}$
      Or remove vertex a_2 [CHPS].

Theorem 3. *A graph of the form $K_{6,6} \setminus 6e$ is IK provided it is not the graph*
$K_{6,6} \setminus \{(a_1,b_1),(a_2,b_2),(a_3,b_3),(a_4,b_4),(a_5,b_5),(a_6,b_6)\}$.

Proof: Let F' be a graph of the form $K_{6,6} \setminus 12e$ having an F_9 minor. Similar to the method used for showing that $K_{6,6} \setminus 5e$ was IK, we use graphs H' and F', obtained by performing vertex expansions and additions on H_9 and F_9.
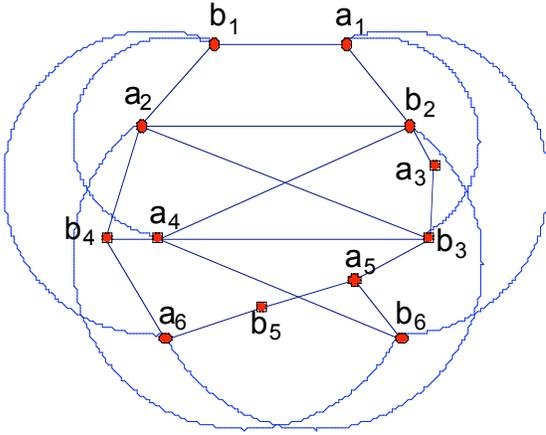
Figure 3. F'
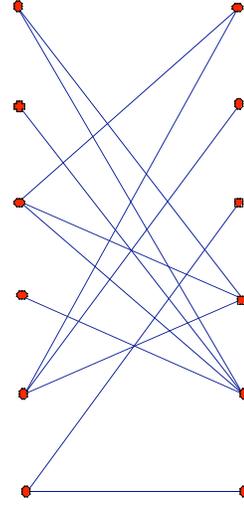
Figure 4. F'^{c}

Unlike the proof for $K_{6,6} \setminus 5e$, we do not make an exhaustive list of all ways to remove six edges from $K_{6,6}$: we may remove a vertex with three edges missing from the original graph to obtain a $K_{6,5} \setminus 3e$ subgraph, which we know to be IK from Corollary 2.5 of [CHPS]. Similarly, a graph having a vertex with four, five, or six edges missing from it is IK.

Here all remaining cases use H' to show the graph is IK except for case 10, which uses F'.

We considered partitions of 6 to find all ways to remove six edges from $K_{6,6}$. We found 16 cases to consider:

1. $K_{6,6}$-{$(a_1,b_4),(a_2,b_5),(a_4,b_1),(a_4,b_4),(a_5,b_2),(a_6,b_6)$};   {{2,1,1,1,1},{1,1,1,1,1,1}}
2. $K_{6,6}$-{$(a_1,b_4),(a_2,b_5),(a_3,b_5),(a_4,b_1),(a_5,b_4),(a_6,b_6)$};   {{2,2,1,1},{1,1,1,1,1,1}}
3. $K_{6,6}$-{$(a_1,b_5),(a_2,b_5),(a_4,b_1),(a_4,b_4),(a_5,b_3),(a_6,b_6)$};   {{2,1,1,1,1},{1,1,2,1,1}}
4. $K_{6,6}$-{$(a_1,b_4),(a_2,b_5),(a_4,b_1),(a_4,b_4),(a_5,b_2),(a_6,b_6)$};   {{2,1,1,1,1},{1,2,1,1,1}}
5. $K_{6,6}$-{$(a_1,b_4),(a_3,b_5),(a_4,b_1),(a_5,b_1),(a_5,b_4),(a_6,b_6)$};   {{2,2,1,1},{1,2,1,1,1}}
6. $K_{6,6}$-{$(a_1,b_4),(a_2,b_5),(a_3,b_5),(a_5,b_1),(a_5,b_4),(a_6,b_6)$};   {{2,1,2,1},{1,2,1,1,1}}
7. $K_{6,6}$-{$(a_1,b_4),(a_2,b_5),(a_3,b_5),(a_4,b_1),(a_5,b_1),(a_5,b_4)$};   {{2,2,2},{1,2,1,1,1}}
8. $K_{6,6}$-{$(a_2,b_5),(a_3,b_5),(a_4,b_1),(a_4,b_4),(a_5,b_1),(a_5,b_2)$};   {{2,2,1,1},{1,2,1,2}}
9. $K_{6,6}$-{$(a_3,b_1),(a_3,b_5),(a_5,b_2),(a_5,b_4),(a_6,b_3),(a_6,b_6)$};   {{2,2,2},{1,1,1,1,1,1}}
10. $K_{6,6}$-{$(a_1,b_4),(a_2,b_5),(a_3,b_5),(a_4,b_4),(a_5,b_1),(a_5,b_2)$};   {{2,1,1,2},{1,1,2,1,1}}
11. $K_{6,6}$-{$(a_1,b_4),(a_2,b_5),(a_4,b_1),(a_4,b_5),(a_5,b_2),(a_5,b_4)$};   {{2,1,2,1},{1,2,1,2}}
12. $K_{6,6}$-{$(a_1,b_4),(a_1,b_5),(a_2,b_5),(a_5,b_1),(a_5,b_4),(a_6,b_6)$};   {{2,2,1,1},{1,2,2,1}}
13. $K_{6,6}$-{$(a_1,b_4),(a_2,b_5),(a_4,b_1),(a_4,b_5),(a_5,b_1),(a_5,b_4)$};   {{2,2,2},{1,2,2,1}}
14. $K_{6,6}$-{$(a_1,b_5),(a_1,b_4),(a_4,b_4),(a_4,b_5),(a_5,b_1),(a_6,b_6)$};   {{2,2,1,1},{2,2,1,1}}
15. $K_{6,6}$-{$(a_1,b_5),(a_2,b_5),(a_4,b_1),(a_4,b_4),(a_5,b_1),(a_5,b_4)$};   {{2,2,2},{2,2,1,1}}
16. $K_{6,6}$-{$(a_1,b_4),(a_1,b_5),(a_4,b_1),(a_4,b_5),(a_5,b_1),(a_5,b_4)$};   {{2,2,2},{2,2,2}}

The graph $K_{6,6} \setminus \{(a_1,b_1),(a_2,b_2),(a_3,b_3),(a_4,b_4),(a_5,b_5),(a_6,b_6)\}$ was the only $K_{6,6} \setminus 6e$ graph for which this method was not successful. We have yet to determine whether this graph is IK or not.

We shall henceforth refer to the graph $K_{6,6} \setminus \{(a_1,b_1),(a_2,b_2),(a_3,b_3),(a_4,b_4),(a_5,b_5),(a_6,b_6)\}$ as G_{666}.

Theorem 4. *Any graph of the form $K_{7,7} \setminus 10e$ is IK.*

Proof: We proceed by deleting vertices and the edges connected to them from $K_{7,7} \setminus 10e$. If three or more deleted edges come from a single vertex, that vertex may be removed to obtain a $K_{7,6} \setminus me$ subgraph with $m \leq 7$, shown to be IK in Theorem 5, below. Now assume that each vertex has at most two edges removed. We can then remove a vertex from each side, removing four distinct edges, resulting in a $K_{6,6} \setminus 6e$ subgraph. We may assume this graph is G_{666} as otherwise the original graph has an IK subgraph and is therefore IK.

Below we present the three ways to remove 2 vertices from $K_{7,7} \setminus 10e$ and attain the graph G_{666}. Remove the circled vertices instead.
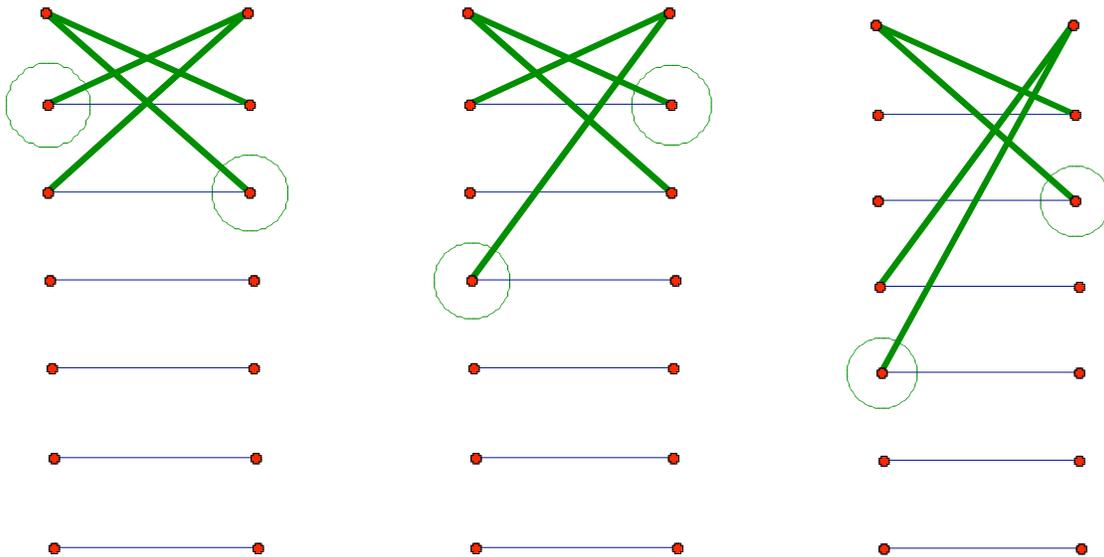
4. General Results

In this section we present some general results.

Theorem 5. $K_{6+n, 6} \setminus (2n + 5)e$ is IK where $n \geq 1$.

Proof: (by Induction on n)

For the base case, let n = 1. We will look at four subcases:

　　　Case 1: Consider the $K_{7,6} \backslash$ 7e graph where each of the 7 a-vertices has exactly one edge removed, and exactly one of the 6 b-vertices (call it b_6) has exactly 2 edges removed.

We can remove one vertex to show that this graph has a $K_{6,6} \backslash$ 6e minor, all of which have been show to be IK, except G_{666}. We can avoid this case by not removing one of the two a-vertices that is connected, in our complement graph, to b_6. Removing any of the other a-vertices will leave us with one of the $K_{6,6} \backslash$ 6e graphs which is known to be IK

　　　Case 2: Consider the $K_{7,6} \backslash$ 7e graphs where each of the 7 a-vertices has exactly one edge removed and at least 1 of the b-vertices has 3 or more edges removed. We can choose any a-vertex to remove, because in any case we will have a $K_{6,6} \backslash$ 6e minor that is known to be IK.

　　　Case 3: Consider the $K_{7,6} \backslash$ 7e graphs where each of the 7 a-vertices has exactly one edge removed and at least 2 b-vertices have at least 2 missing edges. We can remove any of the a-vertices, because in any case we will have a $K_{6,6} \backslash$ 6e minor that is known to be IK.

　　　Case 4: Consider the $K_{7,6} \backslash$ 7e graphs where one or more of the 7 a-vertices has more than one edge removed. By removing one of those vertices, we will be left with a $K_{6,6} \backslash$ me minor where m < 6, all of which are known to be IK.

As these four cases cover all possibilities, we've shown that all the $K_{7,6} \backslash$ 7e graphs are IK.

Now, for the inductive step, let n≥1 and assume all graphs of the form $K_{6+n,6} \backslash (2n + 5)$e are IK. Using Lemma 1 with a = 6 + n, b = 6, m = 2n + 5, and k = 2, we can show that every $K_{6+n+1,6} \backslash (2(n+1) + 5)$e graph G is also IK.

Thus, by induction $K_{6+n,6} \backslash (2n + 5)$e is IK for every n≥1.

Theorem 2 shows that the above result is also true when n = 0.

Corollary 1: A bipartite graph with exactly 6 vertices in one part and at least 6 vertices in the other part and E $\geq$ 4V–17 is IK.

Now we have shown that our conjecture holds for at least the case of the $K_{6+n,6}$ graphs.

Theorem 6. *$K_{7+n,7} \backslash (2n + 10)$e is IK where n $\geq$ 1.*

Proof: (by Induction on n)

For the base case, let n = 1. This gives us a $K_{8,7}$\12e graph. We know using Lemma 1 (with a = b = 7, m = 10, and k = 2) and Theorem 4 that $K_{8,7}$\12e is IK.

Now, for the inductive step, let n ≥ 1 and assume all graphs of the form $K_{7+n,7}$\(2n + 10)e are IK. Using Lemma 1 with a = 7 + n, b = 7, m = 2n + 10 and k = 2, we can show that every $K_{7+n+1,7}$\ (2(n + 1) + 10)e graph G is also IK.

Note that Theorem 4, which proves that $K_{7,7}$\10e is IK, is the case of this theorem where n = 0.

Although this Theorem does not allow us to prove our conjecture in this case, it does improve the previous best known bound (for graphs in general) [CMOPRW] of E ≥ 5V-14

Corollary 2: A bipartite graph with exactly 7 vertices in one part and at least 7 vertices in the other part and E ≥ 5V − 31 is IK.

Theorem 7. $K_{8+n,8}$\ (2n + 15)e is IK where n≥1.

Proof: (by Induction on n)

For the base case, let n = 1. This gives us a $K_{9,8}$\17e graph. We know from Theorem 6 that $K_{9,7}$\14e is IK. Applying Lemma 1 with a = 7, b = 9, m = 14, and k = 3, we can get a $K_{9,7}$\14e subgraph, which is IK, so our graph is IK.

Now, for the inductive step, let n ≥ 1 and assume all graphs of the form $K_{8+n,8}$\(2n + 15)e are IK. Using Lemma 1 with a = 8 + n, b = 8, m = 2n + 15, and k = 2, we can show that every $K_{8+n+1,8}$\ (2(n + 1) + 15)e graph G is also IK.

As far as we know, Theorem 7 does <u>not</u> hold when n = 0. We do, however, know that $K_{8,8}$\14e is IK (use Lemma 1 and that $K_{8,7}$\12e is IK by Theorem 6).

Theorem 8. *Any graph of the form $K_{a,a}$\ (6a -34)e is IK where a≥9.*

Proof: (by Induction on a)

For the base case, let a = 9.

        Consider a $K_{9,9}$\20e graph. We have shown, through Theorem 7, that $K_{9,8}$\17e is IK. By Lemma 1, with k = 3, if $K_{9,8}$\17e is IK, then $K_{9,9}$\ 20e is also IK.

        Now, for the inductive step, let a≥9 and assume all graphs of the form $K_{a,a}$\(6a -34)e are IK. We will show that every $K_{a+1,a+1}$\ (6(a + 1) - 34)e graph G is IK.

We have assumed that all graphs of the form $K_{a,a} \setminus (6a - 34)e$ are IK, where a is the number of vertices in each part. Applying Lemma 1 (with k = 3) shows that every $K_{a+1,a} \setminus (6a - 31)$ graph is IK. Applying the lemma one more time, with m = 6a-31 and k = 3, gives us that all $K_{a+1,a+1} \setminus (6a - 28)$ graphs are IK. Since 6a − 28 is 6(a+1) − 34, this means that every $K_{a+1,a+1} \setminus (6(a+1) - 34)e$ graph G is IK.

Thus, by induction $K_{a,a} \setminus (6a -34)e$ is IK for all a ≥ 9.

Theorem 9. *Any graph of the form $K_{a+n,a} \setminus (3n + 6a-34)e$ is IK where a ≥ 9 and n ≥ 0.*

Proof: (by Induction on n)

For the base case, let n = 0. This gives us a $K_{a,a} \setminus (6a-34)e$ graph. This is IK by Theorem 8.

Now, for the inductive step, assume that $K_{a+n,a} \setminus (3n + 6a-34)e$ is IK. Now, look at $K_{a+n+1,a} \setminus (3(n + 1) + 6a-34)e$. This can be a $K_{a+n+1,a} \setminus (3n + 6a-31)e$. Using Lemma 1, $K_{a+n+1,a} \setminus (3n + 6a-31)e$ is IK provided (3n + 6a-31) is more than twice (a + n + 1). This is true when a = 9: 3n + 23 is larger than n + 10. Each time a increases by one, 3n + 6a-31 will increase by 6 while the 2(a + n + 1) will only increase by 2. So, by induction, the inequality for Lemma 1 holds and $K_{a+n+1,a} \setminus (3n+6a-31)e$ is IK.

b/a	6	7	8	9	10	11	12	13	14
6	6*	7	9	11	13	15	17	19	21
7		10	12	14	16	18	20	23	25
8			14	17	19	21	23	26	29
9				20	23	26	29	32	35
10					26	29	32	35	38
11						32	35	38	41
12							38	41	44
13								44	47
14									50

TABLE 1. $K_{a,b}$/me known to be IK

Certain entries of Table 1 are improvements on the general results of Theorems 5 to 9. Those entries were achieved by applying Lemma 1.

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