p-Coloring Classes of Torus Knots

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Abstract

We develop a theorem for determining the p-colorability of any (m, n) torus knot. We also prove that any p-colorable (m, n) torus knot has exactly one p-coloring class. Finally, we show that every p-coloring of the braid projection of an (m, n) torus knot must use all of the p colors.

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1 Introduction

Our first result is a theorem specifically determining the p-colorability of any (m, n) torus knot. It has been previously shown that a $(m, m-1)$ torus knot is always p-colorable for p equal to m or $m-1$ depending on which is odd (see [5] and [10]). Another proven result is that a $(2, n)$ torus knot is always p-colorable for p equal to n and a $(3, n)$ torus knot is always 3-colorable if n is even [10]. A result similar to ours was also stated as a lemma without proof in [3].

Our second result shows that any p-colorable (m, n) torus knot has only one p-coloring class. A general result investigating colorings of torus knots by finite Alexander quandles appears in [2]. Our result is a special instance of this result; however, we present a proof using only elementary techniques. p-coloring classes have also previously been investigated in relationship to pretzel knots by [4].

Using our second result, we were also able to show a minor result concerning the distribution of colors in a p-coloring of a torus knot. We showed that any p-coloring of the braid representation of an (m, n) torus knot must use each of the p colors. Distribution of colors in p-colorings of knots has been

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previously investigated with the Kauffman-Harary Conjecture. This conjecture is concerned with the distribution of colors in a p-coloring of an alternating knot with prime determinant. Asaeda, Przytcki, and Sikora prove the Harary-Kauffman Conjecture is true for pretzel knots and Montesinos knots in [1].

2 Notation

For this paper we will define a *knot diagram* to be the projection of a knot from 3-space to the plane, where appropriate gaps are left at intersections to show which parts of the knot pass over other parts. We will also define a *strand* to be any connected component of a knot diagram.

Let $G_p(K)$ be the set of all p-colorings for a knot diagram K. Note that $G_p(K)$ is empty if K is not p-colorable. We wish to count the number of p-colorings in $G_p(K)$ that differ by more than just a permutation of the colors. To do this precisely, suppose $\mathcal{S}_K$ is the set of all strands of K. A *p-coloring* of a knot diagram K is a map $\gamma : \mathcal{S}_K \to \mathbb{Z}_p$ satisfying the condition that $2\gamma(s_j) - \gamma(s_i) - \gamma(s_k) = 0 \mod p$ for all $s_i, s_j, s_k \in \mathcal{S}_k$ at a crossing of K, where s_j is the overcrossing strand and s_i, s_k are the undercrossing strands. We also require that at least 2 of of the colors assigned to s_i, s_j, s_k be relatively prime. It is an easy exercise to see that the relation $\sim$ defined by $\gamma \sim \delta \iff \gamma = \rho \circ \delta$ for some permutation $\rho\colon \mathbb{Z}_p \to \mathbb{Z}_p$ for $\gamma, \delta \in G_p(K)$ is an equivalence relation on $G_p(K)$.

The *p-coloring class of* $\gamma \in G_p(K)$ is the set $\bar{\gamma} = \{\delta \in G_p | \delta \sim \gamma\}$. Two p-colorings are said to be *equivalent* if they are in the same p-coloring class, and *fundamentally different* if they are in different p-coloring classes. The set of p-coloring classes for a given knot K will be denoted by $C_p(K)$. The number of p-coloring classes for K will be denoted by $|C_p(K)|$. It happens that this definition of p-coloring classes corresponds directly to the $\mod p$ rank discussed in chapter 3 of [8].

Let $T_{m,n}$ represent the torus knot characterized by the number of times m that it circles around the meridian of the torus and the number of times n that it circles around the longitude of the torus. $T_{m,n}$ is a knot (rather than a 2 component link) if and only if m and n are relatively prime. A *braid* is a diagram of n strings which are attached to a horizontal bar at the top and the bottom. Each string in a braid can only intersect a horizontal plane exactly once. Connecting each of the strands on the top bar with the corresponding strands on the bottom bar will yield a knot, known as the *closure* of the braid. It is known that every knot is the closure of some braid (see chapter 3 of [8]). For example, the trefoil knot is the closure of the braid shown in Figure 1.

The torus knot $T_{m,n}$ can be drawn as the closure of the n-strand braid word $(\sigma_1\sigma_2 \cdots \sigma_{n-2}\sigma_{n-1})^m$. We will refer to the word $(\sigma_1\sigma_2 \cdots \sigma_{n-2}\sigma_{n-1})$ as the *base* for the braid word of $T_{m,n}$, and say that a *cycle* is a single completion of the base for the braid word of a knot $T_{m,n}$. For the duration of the paper, any braid representation of a knot $T_{m,n}$ is considered to have n strands and m cycles, where the 0^{th} strand is the strand on the far left of the braid representation.

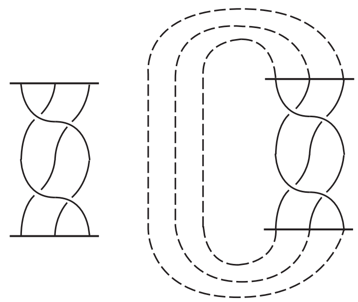

Figure 1: The braid whose closure is the trefoil knot.

3 p-Colorability of Torus Knots

Since $T_{m,n}$ is equivalent to $T_{n,m}$, the following theorem completely characterizes the p-colorability of torus knots.

Theorem 1. *Suppose $T_{m,n}$ is a torus knot and p is prime.*

i) If m and n are both odd, then $T_{m,n}$ is not p-colorable.

ii) If m is odd and n is even, then $T_{m,n}$ is p-colorable if and only if $p|m$.

Note that if $T_{m,n}$ is a torus knot, as opposed to a link, then m and n are relatively prime, and thus m and n cannot both be even. Results similar to those in Theorem 1 were stated without proof by Asami and Satoh in [3].

It is well known that if p is a prime number, then a knot K is p-colorable if and only if p divides $\det(K)$ (see chapter 3 of [8]). We will prove Theorem 1 by using the Alexander polynomial to show that $\det(T_{m,n})$ is given as in the following lemma.

Lemma 1. *Given any torus knot $T_{m,n}$, we have*

$$\det(T_{m,n}) = \begin{cases} 1, & \textit{if } m \textit{ and } n \textit{ are both odd} \\ m, & \textit{if } m \textit{ is odd and } n \textit{ is even} \end{cases}$$

Proof. Given a knot K, it is well known that $\det(K) = |\Delta_K(-1)|$, where $\Delta_K(t)$ is the Alexander polynomial of K (see [9]). It is also known that the Alexander polynomical for a knot $T_{m,n}$ has the following formula

$$\Delta_{T_{m,n}}(t) = \frac{(t^{mn}-1)(t-1)}{(t^m-1)(t^n-1)}. \tag{1}$$

Therefore we can directly calculate the Alexander polynomial and hence the determinant of $T_{m,n}$.

Case 1. Suppose m and n are both odd. Then, $\det(T_{m,n}) = \Delta_{T_{m,n}}(-1) = \frac{(-2)(-2)}{(-2)(-2)} = 1$.

Case 2. Suppose m is odd and n is even.

$$\begin{aligned}\det(T_{m,n}) &= \Delta_{T_{m,n}}(-1)\\ &= \frac{(mn+1)+mn-1}{(m+n)-m+n} \text{ (L'Hospital's rule)}\\ &= \frac{2mn}{2n}\\ &= m.\end{aligned}$$

□

4 Counting p-Coloring Classes of Torus Knots

Our second result shows that every p-colorable torus knot has only one p-coloring class.

Theorem 2. *If p is prime and $T_{m,n}$ is any p-colorable torus knot, then we have $|C_p(T_{m,n})| = 1$.*

Theorem 2 is a special case of a result found by Asami and Kuga in [2]. They prove that if a knot $T_{m,n}$ can be p-colored using a finite Alexander quandle, it has a total of p^2 trivial and non-trivial colorings. If $T_{m,n}$ cannot be colored by such a quandle, then it has only the p trivial colorings. It is important to note that Asami and Kuga only consider the total number of all p-colorings without distinguishing between equivalent colorings, while we consider equivalence classes of p-colorings, or p-coloring classes.

To prove Theorem 2 it suffices to show that if $T_{m,n}$ is p-colorable, then every p-coloring of $T_{m,n}$ is equivalent. To this end, in Section 4.1 we will exhibit a specific p-coloring of $T_{m,n}$ called the *main p-coloring*. Then in Section 4.2 we will prove Theorem 2 by showing that all p-colorings of $T_{m,n}$ are equivalent to the main p-coloring.

4.1 A main p-coloring for every torus knot

Given a torus knot $T_{m,n}$, consider two p-colorings $\alpha, \beta \in G_p(T_{m,n})$. Let α_i^j and β_i^j be the colors of the i^{th} strand of any braid knot K after j cycles for the p-colorings α and β, respectively. Note that the p-colorings α and β are in the same coloring class in $C(T_{m,n})$ if and only if

$$\alpha_i^j = \alpha_l^k \iff \beta_i^j = \beta_l^k \tag{2}$$

for all $i, l \in \{0, 1, \dots, n-1\}$ and $j, k \in \{0, 1, \dots, m\}$.

Given a p-colored braid representation of a knot $T_{m,n}$, we will say that the j^{th} *color array* of $T_{m,n}$ is the element of $(\mathbb{Z}_p)^n$ whose i^{th} component is the color of the i^{th} strand of the braid representation of $T_{m,n}$ after j cycles. For a knot

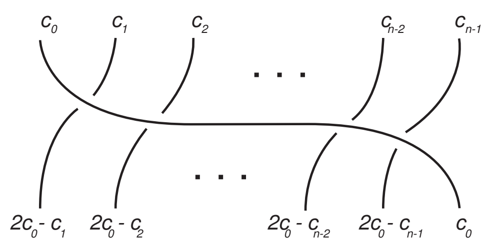

Figure 2: The action of ϕ on the j^{th} color array of $T_{m,n}$.

$T_{m,n}$ to be p-colorable it is a necessary and sufficient condition that the initial color array of its braid representation be exactly the same as its final color array.

Let $(c_0, c_1, \cdots, c_{n-1})$ be the j^{th} color array for the braid representation of some knot $T_{m,n}$, and consider the map $\phi : (\mathbb{Z}_p)^n \to (\mathbb{Z}_p)^n$ defined by

$$\phi(c_0, c_1, \cdots, c_{n-1}) = (2c_0 - c_1, 2c_0 - c_2, \cdots, 2c_0 - c_{n-1}, c_0). \tag{3}$$

Notice that ϕ is the map that, given the j^{th} color array of a knot $T_{m,n}$, returns the $(j+1)^{\text{st}}$ color array according the rules of p-colorability, as seen in Figure 2. Define the map ϕ^j to be the composition of j copies of ϕ, or in other words:

$$\phi^j(\text{ initial color array of } T_{m,n}) = j^{\text{th}} \text{ color array of } T_{m,n}.$$

A p-coloring of any knot $T_{m,n}$ is entirely determined by its initial color array in a braid representation for $T_{m,n}$. Furthermore, for a knot $T_{m,n}$ to be p-colorable, it is a necessary and sufficient condition that we have $\phi^m = \text{id}$ when applied to the initial color array of the braid representation.

We now consider a second n-tuple that can be defined from the braid representation of a p-colored knot $T_{m,n}$. The *color variance* between any two adjacent strands in the projection colored with c_i and c_j respectively is $c_j - c_i \bmod p$. (We consider the far left and far right strands to be adjacent.) Given a p-colored braid representation of a knot $T_{m,n}$, the j^{th} *variance vector of* $T_{m,n}$ is the element of $(\mathbb{Z}_p)^n$ whose i^{th} component is the color variance between the $(i-1)^{\text{th}}$ and i^{th} strands after j cycles. The 0^{th} variance vector of $T_{m,n}$ is referred to as the *initial variance vector*. A *constant variance vector* is a variance vector $V = (v_0, v_1, \cdots, v_{n-1})$ where $v_0 = v_1 = \cdots = v_{n-1}$. If $(c_0, c_1, \cdots, c_{n-2}, c_{n-1})$ is the j^{th} color array for some knot $T_{m,n}$, then the j^{th} variance vector for $T_{m,n}$ is

$$(v_0, v_1, \cdots, v_{n-2}, v_{n-1}) = (c_1 - c_0, c_2 - c_1, \cdots, c_{n-1} - c_{n-2}, c_0 - c_{n-1}). \tag{4}$$

Let $\psi \colon (\mathbb{Z}_p)^n \to (\mathbb{Z}_p)^n$ denote the map that takes as input the j^{th} variance vector of a knot $T_{m,n}$ and returns the $(j+1)^{\text{st}}$ variance vector. By Equations (3) and (4) we have:

$$\begin{aligned}
\psi(v_0, v_1, \cdots, v_{n-2}, v_{n-1}) &= ((2c_0 - c_2) - (2c_0 - c_1), (2c_0 - c_3) - (2c_0 - c_2), \\
&\qquad \cdots, c_0 - (2c_0 - c_{n-1}), (2c_0 - c_1) - c_0 \\
&= (c_1 - c_2, c_2 - c_3, \cdots, c_{n-1} - c_0, c_0 - c_1) \\
&= (-v_1, -v_2, \cdots, -v_{n-1}, -v_0).
\end{aligned}$$

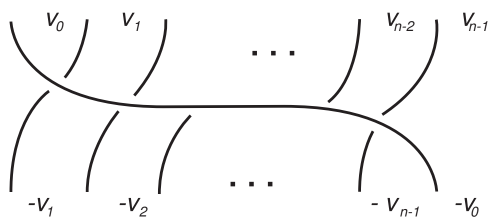

Figure 3: The action of ψ on the j^{th} variance vector of $T_{m,n}$.

Figure 3 shows how Equation (5) looks as we move from the j^{th} to the $(j+1)^{\text{th}}$ variance vector. In this figure, the color variance v_i between the $(i-1)^{\text{st}}$ and i^{th} strands is shown between those two strands.

We define $\psi^j : (\mathbb{Z}_p)^n \to (\mathbb{Z}_p)^n$ to be the composition of j copies of ψ. Since for every application of ψ to $(v_0, v_1, \cdots, v_{n-1})$, all entries get multiplied by -1 and move over one position to the left while wrapping around, we have that $\psi^j(v_0, v_1, \cdots, v_{n-1})$ acts such that for all $i \in \{0, 1, \cdots, n-1\}$

$$v_i \longrightarrow (-1)^j v_{i+j \bmod n}. \tag{5}$$

In a more general form this means that

$$\psi^j(v_0, v_1, \cdots, v_{n-1}) = \begin{cases} (v_j, v_{j+1}, \cdots, v_0, v_1, \cdots, v_{j-1}), & \text{if } j \text{ is even} \\ (-v_j, -v_{j+1}, \cdots, -v_0, -v_1, \cdots, -v_{j-1}), & \text{if } j \text{ is odd,} \end{cases}$$

where all subscripts are taken modulo n.

Given a p-colorable torus knot $T_{m,n}$, we can assume without loss of generality that m is even and n is odd. We will now exhibit a specific p-coloring of $T_{m,n}$ for every prime p that divides n, called the *main p-coloring* of $T_{m,n}$. Since p divides n there exists $r \in \mathbb{Z}^+$ such that $n = rp$. Consider the braid representation of $T_{m,n}$ with n strands and m cycles whose initial, or 0^{th}, color array M is the n-tuple whose entries are the series $0, 1, \cdots, p-2, p-1$, repeated r times:

$$M = (0, 1, \cdots, p-2, p-1, 0, 1, \cdots, p-1, \cdots, p-2, p-1). \tag{6}$$

Using Equation (3) we can deduce that the 1^{st} color array of $T_{m,n}$ is

$$\begin{aligned} \phi(M) &= \phi(0, 1, \cdots, p-2, p-1, 0, 1, \cdots, p-1, \cdots, p-2, p-1) \\ &= (2(0)-1, 2(0)-2, \cdots, 2(0)-(p-1), 2(0)-0, \\ &\qquad 2(0)-1, 2(0)-2, \cdots, 2(0)-0, \cdots, 2(0)-(p-1), 0) \\ &= (-1, -2, \cdots, -(p-1), 0, -1, -2, \cdots, 0, \cdots, -(p-1), 0) \\ &= (p-1, p-2, \cdots, 1, 0, p-1, p-2, \cdots, 0, \cdots, 1, 0). \end{aligned}$$

Note that the 2^{nd} color array of $T_{m,n}$ is

$$\begin{aligned}
&\phi^2(M)\\
&\quad = \phi(\phi(M))\\
&\quad = \phi(p-1, p-2, \cdots, 1, 0, p-1, p-2, \cdots, 0, \cdots, 1, 0)\\
&\quad = (2(p-1)-(p-2), 2(p-1)-(p-3), \cdots, 2(p-1)-0,\\
&\qquad\qquad 2(p-1)-(p-1), 2(p-1)-(p-2), 2(p-1)-(p-3), \cdots,\\
&\qquad\qquad 2(p-1)-(p-1), \cdots, 2(p-1)-0, (p-1))\\
&\quad = (p, p+1, \cdots, 2p-2, 2p-1, p, p+1, \cdots, 2p-1, \cdots, 2p-2, p-1)\\
&\quad = (0, 1, \cdots, p-2, p-1, 0, 1, \cdots, p-1, \cdots, p-2, p-1)\\
&\quad = M.
\end{aligned}$$

4.2 Every p-coloring of a torus knot is equivalent to the main p-coloring

To prove Theorem 2 it suffices to show that if $T_{m,n}$ is p-colorable, then every p-coloring of $T_{m,n}$ is equivalent to the main p-coloring of $T_{m,n}$. Notice that the initial variance vector for the main p-coloring of a knot $T_{m,n}$ as defined in Equation (6) is the constant variance vector $(1, 1, \cdots, 1)$. Our first step in proving Theorem 2 will be to prove the following lemma.

Lemma 2. *Suppose $T_{m,n}$ is a p-colorable torus knot where p is prime, and consider the n-strand braid representation of $T_{m,n}$. If the initial color array of the p-coloring has a constant variance vector, then that initial color array induces a p-coloring that is equivalent to the main p-coloring of $T_{m,n}$.*

Proof. Let C be an initial color array of $T_{m,n}$ with constant variance vector $(v, v, \ldots, v)$. The for some $a \in \mathbb{Z}_p$ we can write C as

$$C = (a + 0v, a + v, \cdots, a + (n-2)v, a + (n-1)v). \tag{7}$$

The first color array of $T_{m,n}$ with this p-coloring is

$$\begin{aligned}
\phi(C) &= \phi(a, a+v, \cdots, a+(n-2)v, a+(n-1)v)\\
&= (2a-(a+v), 2a-(a+2v), \cdots, 2a-(a+(n-1)v), a)\\
&= (a-v, a-2v, \cdots, a-(n-1)v, a)\\
&= (a+(n-1)v, a+(n-2)v, \cdots, a+v, a).
\end{aligned} \tag{8}$$

Notice that ϕ reverses the order of the entries in C. Applying ϕ again, we see that the second color array is

$$\begin{aligned}
&\phi^2(C)\\
&\qquad = \phi(a-v, a-2v, \cdots, a-(n-1)v, a)\\
&\qquad = (2(a-v)-(a-2v), 2(a-v)-(a-3v), \cdots, 2(a-v)-a, a-v)\\
&\qquad = (a, a+v, \cdots, a+(n-3)v, a-2v, a-v) notag && (9)\\
&\qquad = (a, a+v, \cdots, a+(n-3)v, a+(n-2)v, a+(n-1)v)\\
&\qquad = C. && (10)
\end{aligned}$$

Thus we can see that $\phi^2 = \mathrm{id}$ when applied to color arrays with constant variance.

Since p is prime, there exists $t \in \mathbb{Z}_p$ such that $a = tv \bmod p$. Thus, $C = (tv, tv+v, tv+2v, \cdots, tv+(n-1)v) = (tv, (t+1)v, (t+2)v, \cdots, (t+(n-1))v)$. We know if $v \in \mathbb{Z}_p/\{0\}$ and p is prime, then $\langle v \rangle = \mathbb{Z}_p$. Therefore the first p entries of C are distinct elements of $\mathbb{Z}_p$. Moreover, since p divides n, C is comprised of this exact pattern of all of $\mathbb{Z}_p$ repeated precisely $\frac{n}{p}$ times. Therefore,

$$C = (c_0, c_1, \cdots, c_{p-2}, c_{p-1}, c_0, c_1, \cdots, c_{p-1}, \cdots, c_{p-2}, c_{p-1}) \tag{11}$$

where $c_0, c_1, \cdots, c_{p-1}$ repeats $\frac{n}{p}$ times. Since $c_0, c_1, \cdots, c_{p-1}$ are distinct, we know that for $i \in \{0, 1, \cdots, p-1\}$ we have

$$c_i = c_j \iff i = j. \tag{12}$$

Using the notation in Equation (12), and the results in Equations (8) and (10), we see that the j^{th} color array for the p-coloring induced by the initial color array C is

$$\phi^j(C) = \begin{cases} (c_0, c_1, \cdots, c_{p-2}, c_{p-1}, c_0, c_1, \cdots, c_{p-1}, \cdots, c_{p-2}, c_{p-1}), & \text{if } j \text{ is even}\\ (c_{p-1}, c_{p-2}, \cdots, c_1, c_0, c_{p-1}, c_{p-2}, \cdots, c_0, \cdots, c_1, c_0), & \text{if } j \text{ is odd.} \end{cases}$$

On the other hand, from Section 4.1 we know that the j^{th} color array for the main p-coloring induced by the initial color array M from Equation (6) is

$$\phi^j(M) = \begin{cases} (0, 1, \cdots, p-2, p-1, 0, 1, \cdots, p-1, \cdots, p-2, p-1), & \text{if } j \text{ is even}\\ (p-1, p-2, \cdots, 1, 0, p-1, p-2, \cdots, 0, \cdots, 1, 0), & \text{if } j \text{ is odd.} \end{cases}$$

We wish to prove that C and M induce equivalent p-colorings in terms of the condition in Equation (2). Let $\pi_i : (\mathbb{Z}_p)^n \to (\mathbb{Z}_p)$ be the i^{th} projection map, and let $\phi_i^j = \pi_i \circ \phi^j : (\mathbb{Z}_p)^n \to (\mathbb{Z}_p)$. Notice that ϕ_i^j takes in an initial color array and returns the color of the i^{th} strand of the j^{th} color array of the induced p-coloring. From the expression for $\phi^j(C)$ above, we see that for $j \in \{0, 1, \cdots, m\}$ and $i \in \{0, 1, \cdots, n-1\}$ we have

$$\phi_i^j(C) = \begin{cases} c_i \bmod p, & \text{if } j \text{ is even}\\ c_{p-1-i} \bmod p, & \text{if } j \text{ is odd.} \end{cases} \tag{13}$$

Similarly, from the expression for $\phi^j(M)$ above, we see that

$$\phi_i^j(M) = \begin{cases} i \bmod p, & \text{if } j \text{ is even} \\ p-1-i \bmod p, & \text{if } j \text{ is odd.} \end{cases} \tag{14}$$

It is now easy to see from Equations (12), (13), and (14) that we have

$$\phi_i^j(M) = \phi_l^k(M) \iff \phi_i^j(C) = \phi_l^k(C)$$

for $j, k \in \{0, 1, \cdots, m\}$ and $i, l \in \{0, 1, \cdots, n-1\}$. Therefore by Equation (2) we know that C and M induce equivalent p-colorings on $T_{m,n}$. $\square$

We will now use Lemma 2 to prove Theorem 2. The key will be to show that there cannot be a p-coloring of $T_{m,n}$ that does not have a constant variance vector.

Proof. Suppose $T_{m,n}$ is a p-colorable torus knot, and consider the n-strand braid representation of $T_{m,n}$. By Theorem 1 we can assume without loss of generality that n is odd and m is even, and that p divides n. Seeking a contradiction, assume that $|C_p(T_{m,n})| > 1$. Specifically, assume there is a p-coloring $\gamma \in G_p(T_{m,n})$ that is fundamentally different from the main p-coloring of $T_{m,n}$. Let $G = (g_0, g_1, \cdots, g_{n-2}, g_{n-1})$ be the initial color array for γ, and let $V = (v_0, v_1, \cdots, v_{n-2}, v_{n-1}) = (g_1 - g_0, g_2 - g_1, \cdots, g_{n-1} - g_{n-2}, g_0 - g_{n-1})$ be the initial variance vector for γ.

Using ψ as defined in Equation (5) we can apply ψ^q to V to get the q^{th} variance vector of γ (where all subscripts are taken mod n):

$$\psi^q(V) = \begin{cases} (v_q, v_{q+1}, \cdots, v_0, v_1, \cdots, v_{q-1}), & \text{if } q \text{ is even} \\ (-v_q, -v_{q+1}, \cdots, -v_0, -v_1, \cdots, -v_{q-1}), & \text{if } q \text{ is odd.} \end{cases}$$

Let r be the smallest positive integer for which V partitions into s repeating sections of length r. Note that $1 \le r \le n$ and that $n = rs$. Since by hypothesis γ is not equivalent to the main p-coloring of $T_{m,n}$, Lemma 2 tells us that the variance vector V is not constant. Therefore we have $r > 1$.

Since m is even, we know from Equation (3) that

$$\phi^m(V) = (v_m, v_{m+1}, \cdots, v_0, v_1, \cdots, v_{m-1}).$$

In other words, ϕ^m turns the initial variance vector V into the vector where all entries have been shifted to the left m positions and wrapped around. Because m is the number of cycles in our braid representation, and γ is a p-coloring of $T_{m,n}$, we must have $\psi^m(V) = V$. In order for this to occur, ψ^m must shift V over by some multiple of r, the length of a repeating section. Therefore we must have $m = kr$ for some $k \in \mathbb{Z}^+$.

We have now shown that $n = rs$, $m = kr$, and $r > 1$; these facts imply that $\gcd(m, n) \ge r > 1$. But this contradicts our assumption that $T_{m,n}$ is a knot and not a link. Therefore, there cannot exist a p-coloring γ that is fundamentally different from the main coloring, and hence we must have $|C_p(T_{m,n})| = 1$. $\square$

Because the main p-coloring of a torus knot $T_{m,n}$ uses all p colors, and every p-coloring that is equivalent to the main coloring must also use all p colors, we have the following immediate corollary to Theorem 2.

Corollary 3. *Every p-coloring of a braid projection of knot $T_{m,n}$ must use all p colors.*

5 Questions for future research

1. What is the significance of the color distribution in Corollary 3 and what other knots have a similar color distribution?
2. Do variance vectors have useful applications to other types of knots, especially those whose braid word is some power of a base word?
3. What other types of p-colorable knots have only one p-coloring class?

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