Journal of Knot Theory and Its Ramifications

Fundamentally Different m-Colorings of Pretzel Knots

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ABSTRACT

We show that the number of fundamentally different m-colorings of a knot K depends only on the m-nullity of K, and develop a formula for the number of such colorings. We also determine the m-colorability and m-nullity of any (p, q, r) pretzel knot, and therefore determine the number of fundamentally different m-colorings for any (p, q, r) pretzel knot.

1. Introduction

Tricolorability is a rather coarse knot invariant (it only divides the category of knots into two types), but it has many useful applications; for example, tricolorability can be used to determine whether a link is non-splittable [1]. The set of integers m for which a knot is m-colorable (the "coloring number set") is a more powerful invariant (see Section 4.6 of [2]). Although m-colorability is defined for any integer $m > 2$, in this paper we will focus only on the cases where m is prime.

In Section 2 we give a quick overview of colorability of knots, and discuss the number of ways that a knot can be m-colored. The number of different m-colorings is a finer invariant than the m-colorability of a knot, and is used in [3] to distinguish knots with identical HOMFLYPT polynomials. In Section 3 we introduce the notion of "fundamentally different" colorings, that is, m-colorings that differ by more than just permutations of the colors, and develop a formula for the number of such colorings based only on the m-nullity of the knot.

In Section 4 we focus our attention on pretzel knots. In particular, we will show that the determinant of a (p, q, r) pretzel knot is $\Delta = pq + qr + pr$, and thus that a (p, q, r) pretzel knot is m-colorable only if m divides Δ. A simpler version of this fact was proved in [4] (without the use of determinants), where it is shown that a (p, q, r) pretzel knot fails to be m-colorable for any m if $\Delta = 1$. We will then determine the m-nullity of a (p, q, r) pretzel knot, and use the results from Section 3 to describe the number of fundamentally different m-colorings that a (p, q, r) pretzel knot admits. Specifically, we show that for any (p, q, r) pretzel knot K, if m divides Δ, then the m-nullity of K is either 2 or 3, depending on whether or not any of p, q, and r are relatively prime to m. This will mean that every m-colorable (p, q, r) pretzel knot must have either one or four fundamental m-colorings.

2. Definitions and Background

We begin by presenting some basic definitions and results concerning colorability of knots. (See also [4] and [5].)

2.1. *Colorability*

Definition 1. A knot K is *m-colorable* if each strand in a projection of K can be assigned a number (called a "color") from the set $\{0, 1, \cdots, m-1\}$ so that (i) at least two colors are used, and (ii) at each crossing, if x and y are the colors of the understrands and z is the color of the overstrand, then $x + y - 2z \equiv 0 \bmod m$.

In the case where $m = 3$, condition (i) is equivalent to requiring that the three strands meeting at each crossing are all the same color, or all different colors. In a general m-coloring, the conditions $x + y - 2z \equiv 0 \bmod m$ are more restrictive; having three strands all the same or all different at each crossing is not enough to guarantee that $x + y - 2z \equiv 0 \bmod m$. Throughout this paper, the only m-colorings we will consider are those where m is prime. Note that m-colorability is invariant under the Reidemeister moves, and therefore independent of the projection used.

Given a knot projection with a labeling of its crossings and strands, the associated *crossing matrix* C is the matrix whose (i, j) entry is 1 if strand j is an understrand at crossing i, and -2 if strand j is an overstrand at crossing i. A matrix obtained by removing one row and one column from a crossing matrix will be called a *minor crossing matrix*. Every minor crossing matrix of a knot K has the same determinant, which we denote $\det(K)$. The system of equations represented by the minor crossing matrix of a knot K has a solution mod m if and only if $\det(K) \equiv 0 \bmod m$. Therefore:

Theorem 1. Suppose m is prime. A knot K is m-colorable if and only if m divides $\det(K)$.

Each vector in the mod m nullspace of the crossing matrix C corresponds to an m-coloring of K. The *m-nullity* of a knot K is the dimension of the nullspace of the crossing matrix of K. (Note: Some authors use the mod m nullity of the *minor* crossing matrix rather than the mod m nullity of the full crossing matrix.) We can see immediately that if a knot has m-nullity 1, it is not m-colorable, because in this case, the only m-colorings are the trivial colorings, that is, those represented by multiples of the vector $(1, 1, \dots, 1, 1)$.

2.2. *Counting m-colorings*

The m-nullity of a knot K determines the number of m-colorings of K, regardless of projection or number of strands. Suppose K is a knot with m-nullity $n > 2$. Then the system of equations describing the m-colorability conditions at each crossing of K will have n free variables. This means that there is a set of n strands of K that can be assigned any color in $0, 1, 2, ..., m-1$ (as long as they are not all assigned the same color). We call this a set of *significant strands* for that projection of K. The colors for the remaining strands of the projection will be determined by the colors of the significant strands. Theorem 2 describes a formula for the number of m-colorings of a knot with m-nullity n. (This formula is presented without proof in [6].)

Theorem 2. A knot with m-nullity n has $m^n - m$ different m-colorings.

Proof. Let K be a knot with m-nullity n. This implies that K has n significant strands. Because each strand can be assigned any of m different colors, there exist m^n different ways to assign colors to the strands such that the condition $x + y - 2z \equiv 0 \mod m$ is satisfied. However, this number includes all the trivial m-colorings, that is, the m-colorings in which the entire knot is colored with only one color. Because an m-coloring must use at least two colors, we subtract the m trivial colorings, leaving us with $m^n - m$ different m-colorings. □

Example 1. For example, the trefoil knot has 3-nullity 2, and thus $3^2 - 3 = 6$ different 3-colorings. In this example, *any* two strands can be chosen as the significant strands. The six 3-colorings in Figure 1 correspond to the six different ways that we can assign colors from $\{0, 1, 2\}$ to the two significant strands (without both strands being the same color).

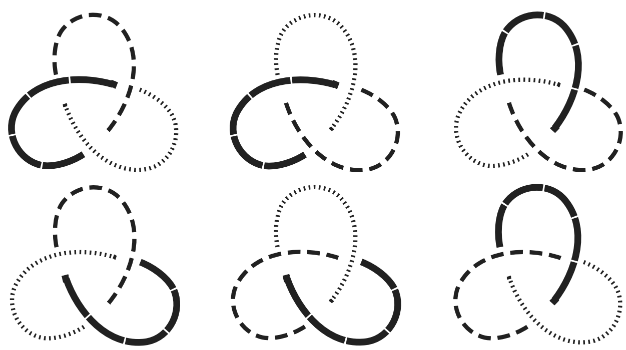

Figure 1: The six 3-colorings of the trefoil knot.

3. Fundamentally Different Colorings

We will now examine whether a knot can be colored in two or more ways that are not simply permutations of each other.

3.1. *Counting fundamentally different m-colorings*

The different 3-colorings of the trefoil knot in Figure 1 are all permutations of each other, and thus we say that the trefoil has only one *fundamental 3-coloring*. Some knots have more than one *fundamentally different m-coloring*.

Example 2. The 9_{35} knot has exactly four fundamentally different 3-colorings (this will be guaranteed by Theorem 1). Representatives of these four fundamentally different 3-colorings are shown in Figure 2.

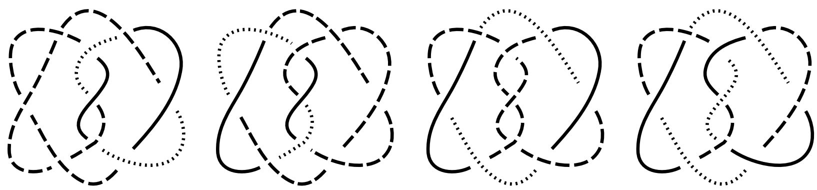

Figure 2: Four fundamentally different 3-colorings of the 9_{35} knot.

To find the number of fundamentally different m-colorings of a knot, we will separate the possible m-colorings into "types," and then count how many fundamentally different colorings are possible of each "type." A knot with m-nullity n will have n significant strands that can each be colored with any number $0, 1, 2, \ldots, m-1$ (as long as at least two colors are used). A choice of these m-colorings corresponds to a *coloring vector* $(c_1, c_2, \ldots, c_n)$. For example, a knot with 5-nullity 4 could have coloring vectors $(0, 2, 1, 4)$, $(1, 1, 1, 4)$, or $(3, 3, 0, 3)$. We will say that two coloring vectors (for m-nullity n) are of the same *type* if they are the same up to permutations of the order and choice of colors. For example, $(1, 1, 1, 4)$ and $(3, 3, 0, 3)$ are of the same type, but $(0, 2, 1, 4)$ and $(1, 1, 1, 4)$ are not.

We can classify a type of coloring vector using a *type vector* $\vec{d} = \langle d_1, d_2, \ldots, d_n\rangle$ that records the sizes of the sets of strands that are similarly colored. Each d_i in the type vector describes a number of strands that are assigned the same color. For example, the coloring vector $(3, 3, 0, 3)$ is made up of a set of 3 strands of the same color (3), and a set of 1 strand of a different color (0). Therefore we say $(3, 3, 0, 3)$ is of type $\langle 3, 1, 0, 0\rangle$. (We append zeroes so that the type vector has length $n = 4$.)

Definition 2. Suppose K is a knot with m-nullity n. A *type vector* for K is a vector $\vec{d} = \langle d_1, d_2, \ldots, d_n\rangle$ in $\mathbf{Z}_+^n$ satisfying the following three conditions:

(i) $d_{i+1} \le d_i < n$ for $1 \le i \le n-1$;

(ii) $\sum_{k=1}^{n} d_i = n$;

(iii) At least $n - m$ of the d_i are zero.

The set of all type vectors for m-nullity n is denoted $D_{n,m}$.

Part (i) of Definition 2 ensures that the entries d_i are decreasing (to avoid repetition), and omits the type of m-coloring where all n of the significant strands are assigned the same color (or the trivial coloring). We require that the entries d_i add to n in part (ii) since the sum of the d_i will be the number of significant strands. Part (iii) is only relevant if $n > m$. The number of non-zero entries in $\vec{d}$ represents the number of colors that are used. An m-coloring can use at most m colors, so if $n > m$ then at least $m - n$ of the d_i must be zero.

Our formula for the number of m-colorings of a knot with m-nullity n will also require the following definition.

Definition 3. The *k-count* of a type vector $\vec{d} = \langle d_1, d_2, \ldots, d_n \rangle$, denoted $s_k(\vec{d})$, is the number of entries of $\vec{d}$ which are equal to k: for $0 \leq k < n$,

$$s_k(\vec{d}) := \left| \{ d_i \mid d_i = k, 1 \leq i \leq n \} \right| .$$

Note that we will never need to consider $s_n(\vec{d})$, because any coloring whose type has $s_n(\vec{d}) \neq 0$ will be the trivial coloring. In this notation, part (iii) of Definition 2 says that $s_0 \leq m - n$. A few examples: The type vector $\langle 2, 1, 0 \rangle$ has $s_0 = 1$, $s_1 = 1$, and $s_2 = 1$. The type vector $\langle 1, 1, 1 \rangle$ has $s_0 = 0$, $s_1 = 3$, and $s_2 = 0$. A type vector with $s_1 = n$ corresponds to m-colorings where all the significant strands are colored differently.

We are finally in a position to describe our first main result, a formula for the number of fundamentally different m-colorings of a knot with m-nullity n. Note that every knot with m-nullity n will have the same list of type vectors, with the same k-counts. Therefore the formula in Main Theorem 1 depends only on the prime number m and the m-nullity n of the knot. The proof of Main Theorem 1 is left until Section 3.2.

Main Theorem 1. Suppose K is a knot with m-nullity n. Then K has $C(m, n)$ fundamentally different m-colorings, given by the formula:

$$C(m,n) = \sum_{\vec{d} \in D_{n,m}} \frac{n!}{(d_1!\, d_2! \cdots d_n!)\,(s_1!\, s_2! \cdots s_n!)}$$

Example 3. If a knot K has 5-nullity 4, then $D_{n,m}$ consists of four type vectors: $\langle 3, 1, 0, 0 \rangle$, $\langle 2, 2, 0, 0 \rangle$, $\langle 2, 1, 1, 0 \rangle$, and $\langle 1, 1, 1, 1 \rangle$. If we use these vectors (and the corresponding k-counts) in the formula in Main Theorem 1, we obtain the sum:

$$\frac{4!}{(3!\,1!\,0!\,0!)(1!\,0!\,1!)} + \frac{4!}{(2!\,2!\,0!\,0!)(0!\,2!\,0!)} + \frac{4!}{(2!\,1!\,1!\,0!)(2!\,1!\,0!)} + \frac{4!}{(1!\,1!\,1!\,1!)(4!\,0!\,0!)} = 14.$$

Since the sum is equal to 14, there exist 14 fundamentally different 5-colorings of a knot with 5-nullity 4. One knot with 5-nullity 4 is the composition of three Figure-8 (or 4_1) knots, shown in Figure 3. Using the four boldly marked strands as the set of significant strands, the $4_1\#4_1\#4_1$ knot can be colored 14 fundamentally different ways.

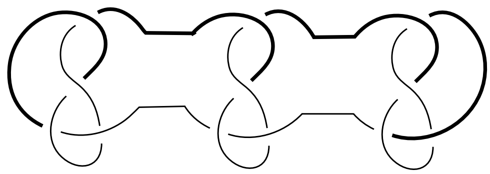

Figure 3: The $4_1\#4_1\#4_1$ knot has 14 fundamentally different 5-colorings.

In fact, as long as $m \geq 4$, any knot with m-nullity 4 will have 14 fundamentally different m-colorings. Table 1 lists the number of fundamentally different m-colorings of any knot with m-nullity n (provided that $m \geq n$). If $m < n$, then there are more significant strands than colors, which affects the number of possible m-colorings (see Table 2).

m-nullity	fundamental colorings
1	0
2	1
3	4
4	14
5	51
6	202

Table 1: The number of fundamentally different m-colorings for a given m-nullity n (the table holds for any m provided $m \geq n$).

3-nullity	fundamental colorings
1	0
2	1
3	4
4	13
5	40
6	121

Table 2: The number of fundamentally different 3-colorings for a 3-nullity n knot (note the tables differ only when $m < n$).

3.2. *Proof of Main Theorem 1*

Proof. Suppose K is a knot with m-nullity n. Then K has n significant strands, which we can label with (not necessarily distinct) colors $a_1, a_2, \ldots, a_n$. It is unimportant what specific color each strand is assigned; what matters is how many different ways we can assign colors to the strands so that a given type vector is satisfied. Before developing the general formula, we will focus on a few specific examples.

For example, the type vector $\langle 3, 2, 1, 0, 0, 0\rangle$ for a set of six significant strands represents a color assignment where three strands are one color, two more strands are another color, and the remaining strand is a third color. We will now count the number of arrangements of the strands a_i that satisfy this type vector. We must first choose three out of the six strands to be assigned the first color; there are $\binom{3}{6} = 20$ ways to do this. Similarly, there are $\binom{3}{2} = 3$ ways to choose two strands (from the remaining three) to be the second color. Finally, there is only one strand left to be assigned the remaining color ($\binom{1}{1} = 1$ choices). Therefore, there are $20 \cdot 3 \cdot 1 = 60$ strand arrangements satisfying the type vector. For example, two of these 60 arrangements are:

$$(a_1 = a_2 = a_3) \neq (a_4 = a_5) \neq (a_6),$$

$$(a_1 = a_4 = a_5) \neq (a_2 = a_6) \neq (a_3).$$

The counting method used above will overcount the number of arrangements if there are two or more equal entries in the type vector. For example, consider the type vector $\langle 2, 2, 1, 0, 0\rangle$ for a set of five significant strands, and the following two arrangements:

$$(a_1 = a_3) \neq (a_2 = a_5) \neq (a_4);$$

$$(a_2 = a_5) \neq (a_1 = a_3) \neq (a_4).$$

The counting method predicts that there will be $\binom{5}{2}\binom{3}{2}\binom{1}{1} = 30$ different arrangements for this type vector. However, this counts the two arrangements above differently, although they are obviously the same. Since there are two sets of the same size in the arrangements for this type vector, we must divide our result by 2; therefore there will be 15 different arrangements for the type vector $\langle 2, 2, 1, 0, 0\rangle$. If there are three sets of the same size, we must divide by $3 \cdot 2$; if there are s sets of the same size, we must divide by $s!$. We must divide for *every* multiple set, so if we had type vector $\langle 2, 2, 1, 1, 1, 0, 0\rangle$, we would have to divide our result by $2! \cdot 3!$. Notice that the k-counts from Definition 3 are exactly what we need to measure how many sets of the same size occur in any type vector.

Applying this counting method to a general type vector $\vec{d} = \langle d_1, d_2, \ldots, d_n\rangle$, we see that there are at most

$$\binom{n}{d_1}\binom{n-d_1}{d_2}\binom{n-d_1-d_2}{d_3}\cdots\binom{n-d_1-d_2-\cdots-d_{n-1}}{d_n} \tag{3.1}$$

different arrangements. The expression in (3.1) may overcount the number of arrangements, depending on the k-counts $s_k(\vec{d})$. To make an accurate count of the arrangements, we must divide the expression in (3.1) by the product of the factorials of the k-counts $s_k = s_k(\vec{d})$:

$$\frac{1}{s_1!\, s_2!\, s_3! \cdots s_{n-1}!}$$

(Note that if $s_i(\vec{d}) = 1$, then we divide by $1!$, which does not change the expresssion.)

To find the total possible number $C(n, m)$ of fundamentally different m-colorings, we must add together the number of arrangements corresponding to each type vector. This process results in the following formula:

$$C(n,m) = \sum_{\vec{d} \in D_{n,m}} \left(\prod_{1 \le i \le n,\, d_i \neq 0} \binom{n - \sum_{j=1}^{i} d_{j-1}}{d_i} \cdot \prod_{1 \le k < n} \frac{1}{s_k!} \right). \tag{3.2}$$

It now only remains to simplify the equation in (3.2) and show it is equivlant to the formula in Main Theorem 1. We begin by considering the part of (3.2) that comes from expression 3.1. It is easy to show that:

$$\prod_{1\leq i\leq n\, d_i\neq 0} \binom{n-\sum_{j=1}^{i} d_{j-1}}{d_i} = \frac{n!}{(d_1!d_2!\cdots d_n!)(n-d_1-d_2-\cdots-d_n)!}.$$

Since $\sum_{k=1}^{n} d_i = n$, the second factor in the denominator of the expression above is $(n-n)! = 0! = 1$. Thus we have:

$$\prod_{1\leq i\leq n\, d_i\neq 0} \binom{n-\sum_{j=1}^{i} d_{j-1}}{d_i} = \frac{n!}{d_1!d_2!d_3!\cdots d_n!}.$$

Therefore, the equation in (3.2) can be written as:

$$C(n,m) = \sum_{\vec{d}\in D_{n,m}} \frac{n!}{(d_1!d_2!d_3!\cdots d_n!)(s_1!s_2!s_3!\cdots s_n)}.$$

□

Given a type vector $\vec{d}$, the corresponding term in the sum from Main Theorem 1 counts the number of possible strand arrangements that satisfy the type vector $\vec{d}$. To count the total number of m-colorings, including those that differ only by permutations, we must multiply each of the terms in the sum from Main Theorem 1 by $\frac{m!}{(m-(n-s_0))!}$, where $s_0 = s_0(\vec{d})$ is the 0-count for the type vector $\vec{d}$. This means that the total number of m-colorings of a knot with m-nullity n is given by the expression:

$$\sum_{\vec{d}\in D_{n,m}} \left(\frac{m!}{(m-(n-s_0))!} \cdot \frac{n!}{(d_1!d_2!d_3!\cdots d_n!)(s_1!s_2!s_3!\cdots s_n)} \right). \tag{3.3}$$

It is interesting to note that this complicated expression is in fact equal to $m^n - m$, the formula from Theorem 2.

4. Pretzel Knots

The remainder of this paper will be dedicated to applying our m-colorability results, including the formula from Main Theorem 1, to pretzel knots.

4.1. *Basic properties of pretzel knots*

A *twist* is a part of a knot comprised of two strands twisted together one or more times (see Figure 4). The top right strand is called the *northeast* strand, the bottom right is called the *southeast* strand, and so on. A twist is *positive* if its northeast strand is an overstrand; otherwise it is *negative*. An twist is *odd* if it has an odd number of crossings; otherwise it is *even*.

A *pretzel knot* is a knot composed of three twists joined together such that each northeast strand connects to the adjacent northwest strand, and each southeast

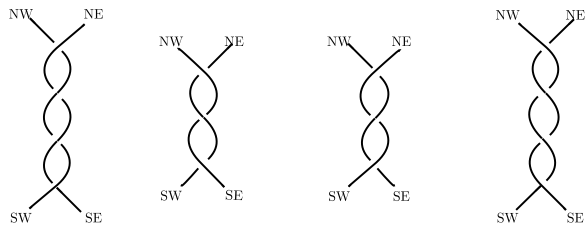

Figure 4: Four twists: a positive even twist, a positive odd twist, a negative odd twist, and a negative even twist, respectively.

strand connects to the adjacent southwest strand (we consider the first and last twists to be adjacent). In this paper we will focus solely on three-twist pretzel knots. We name a pretzel knot based on the number of crossings in each twist. For example, the three twists in Figure 5 have 5, 4, and 7 crossings respectively, and each twist is positive, so the knot is named $(5, 4, 7)$.

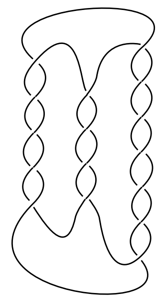

Figure 5: A $(5, 4, 7)$ pretzel knot.

The following three theorems describe basic properties of pretzel knots. Their proofs are elementary and we do not include them here. The results in this paper apply to both pretzel knots and pretzel links (as described in Theorem 3), and from this point forward we will refer to both pretzel knots and pretzel links simply as pretzel knots. Theorems 4 and 5 show that, for example, the $(5, 4, 7)$, $(4, 7, 5)$, and $(-7, -4, -5)$ pretzel knots all represent the same knot. In addition, these theorems show that to prove a theorem for all (p, q, r) pretzel knots, it suffices to prove two special cases: the case where p, q, and r are all positive, and the case where exactly one of p, q, and r is negative.

Theorem 3. A (p, q, r) pretzel is a knot if at most one twist is even, a 2-component link if two twists are even, and a 3-component link if all twists are even.

Theorem 4. Permuting the twists in a (p, q, r) pretzel knot changes the projection but does not change the knot.

Theorem 5. Suppose p, q, and r are integers. Then the (p, q, r) pretzel knot and the $(-p, -q, -r)$ pretzel knot are different projections of the same knot.

4.2. *Labelings and crossing matrices*

If K is a (p, q, r) pretzel knot where p, q, and r are all positive integers, then we will label the crossings of K as shown in Figure 6, and the strands of K as shown in Figure 7.

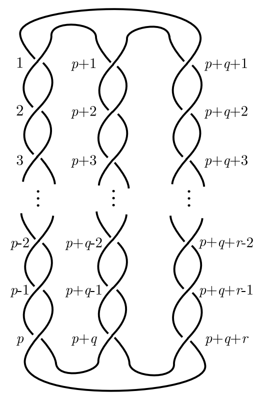

Figure 6: Crossing labeling for a positive-twist pretzel knot.

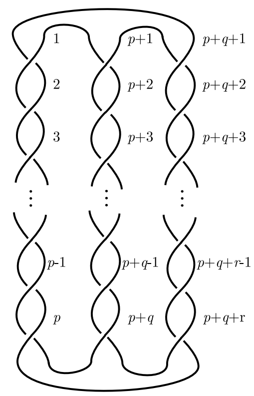

Figure 7: Strand labeling for a positive-twist pretzel knot.

Using the projection and labeling in Figures 6 and 7, the general crossing matrix for a pretzel knot where all three twists are positive is shown in Figure 8. Each unlabeled entry in the matrix is a 0, which have been removed for ease in reading. Note that the crossing matrix, and all of its diagonal blocks, are square.

$$
\begin{array}{l}
1 \\ 2 \\ \vdots \\ p-1 \\ p \\ p+1 \\ p+2 \\ \vdots \\ p+q-1 \\ p+q \\ p+q+1 \\ p+q+2 \\ \vdots \\ p+q+r-1 \\ p+q+r
\end{array}
\left(\begin{array}{ccccc|ccccc|ccccc}
-2 & 1 & & & & & & & & & 1 & & & & \\
1 & -2 & 1 & & & & & & & & & & & & \\
 & & \ddots & & & & & & & & & & & & \\
 & & 1 & -2 & 1 & & & & & & & & & & \\
 & & & 1 & -2 & & & & & 1 & & & & & \\
\hline
1 & & & & & -2 & 1 & & & & & & & & \\
 & & & & & 1 & -2 & 1 & & & & & & & \\
 & & & & & & & \ddots & & & & & & & \\
 & & & & & & & 1 & -2 & 1 & & & & & \\
 & & & & & & & & 1 & -2 & & & & & 1 \\
\hline
 & & & & & 1 & & & & & -2 & 1 & & & \\
 & & & & & & & & & & 1 & -2 & 1 & & \\
 & & & & & & & & & & & & \ddots & & \\
 & & & & & & & & & & & & 1 & -2 & 1 \\
 & & & & 1 & & & & & & & & & 1 & -2
\end{array}\right)
$$

Figure 8: General crossing matrix of a (p, q, r) pretzel knot when p, q, and r are all positive.

If any of the twists in a pretzel knot are negative, we will require a different labeling. We choose this labeling so that the connecting strands are labeled the same way as in the all-positive pretzel labeling in Figures 6 and 7. For example, a $(p, q, -r)$ pretzel knot (where p, q, and r are positive) is labeled as in Figures 9 and 10, and has the general crossing matrix as shown in Figure 11.

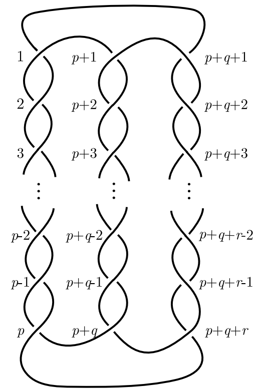

Figure 9: Crossing labeling for a$(p, q, -r)$ pretzel knot.

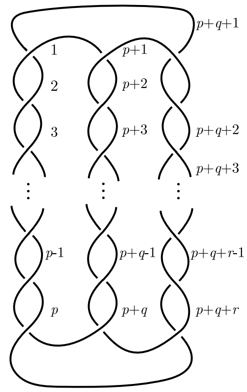

Figure 10: Strand labeling for a $(p, q, -r)$ pretzel knot.

$$
\begin{array}{l}
1\\ 2\\ 3\\ \vdots\\ p-2\\ p-1\\ p\\ p+1\\ p+2\\ p+3\\ \vdots\\ p+q-2\\ p+q-1\\ p+q\\ p+q+1\\ p+q+2\\ p+q+3\\ \vdots\\ p+q+r-2\\ p+q+r-1\\ p+q+r
\end{array}
\left(\begin{array}{ccccccc|ccccccc|ccccccc}
-2 & 1 & & & & & & & & & & & & & 1 & & & & & & \\
1 & -2 & 1 & & & & & & & & & & & & & & & & & & \\
 & 1 & -2 & 1 & & & & & & & & & & & & & & & & & \\
 & & & \ddots & & & & & & & & & & & & & & & & & \\
 & & & 1 & -2 & 1 & & & & & & & & & & & & & & & \\
 & & & & 1 & -2 & 1 & & & & & & & & & & & & & & \\
 & & & & & 1 & -2 & & & & & & & 1 & & & & & & & \\
\hline
1 & & & & & & & -2 & 1 & & & & & & & & & & & & \\
 & & & & & & & 1 & -2 & 1 & & & & & & & & & & & \\
 & & & & & & & & 1 & -2 & 1 & & & & & & & & & & \\
 & & & & & & & & & & \ddots & & & & & & & & & & \\
 & & & & & & & & & & 1 & -2 & 1 & & & & & & & & \\
 & & & & & & & & & & & 1 & -2 & 1 & & & & & & & \\
 & & & & & & & & & & & & 1 & -2 & & & & & & & 1 \\
\hline
 & & & & & & & -2 & & & & & & & 1 & 1 & & & & & \\
 & & & & & & & 1 & & & & & & & & -2 & 1 & & & & \\
 & & & & & & & & & & & & & & & 1 & -2 & 1 & & & \\
 & & & & & & & & & & & & & & & & & \ddots & & & \\
 & & & & & & & & & & & & & & & & & 1 & -2 & 1 & \\
 & & & & & & 1 & & & & & & & & & & & & 1 & -2 & \\
 & & & & & & -2 & & & & & & & & & & & & & 1 & 1
\end{array}\right)
$$

Figure 11: General crossing matrix of a $(p, q, -r)$ pretzel knot.

4.3. *m-colorability of pretzel knots*

In this section we will use the determinant to determine the m-colorability of a general pretzel knot.

Theorem 6. Suppose p, q, and r are any integers. A (p, q, r) pretzel knot is m-colorable if and only if m divides $pq + pr + qr$.

The proof of Theorem 6 involves using row operations to write the general minor crossing matrix of a pretzel knot as an upper-triangular matrix so that we can easily find the determinant.

Proof. It suffices to prove Theorem 6 in two cases: when all three twists are positive, and when two twists are positive and one is negative (see Theorems 4 and 5). We will show that in these two cases, the determinant of a (p, q, r) pretzel knot is $\Delta = pq + pr + qr$. By Theorem 1, this implies that the knot will be m-colorable if and only if m divides Δ.

Case 1. Let p, q, and r be positive integers, and let K be a (p, q, r) pretzel knot with labelings as in Figures 6 and 7 and crosing matrix C as in Figure 8. To find $\det(K)$ we will reduce the matrix C so that only zeroes appear below the lower diagonal (we will choose the minor of C obtained by removing the first row and last column, but will not remove this row and column until the end of the proof).

We begin with the following sequences of row operations:

$$\begin{array}{rcl} R_{p+1} & \longrightarrow & R_{p+1} - \sum_{i=2}^{p}(i-1)R_i. \\ R_{p+q+1} & \longrightarrow & R_{p+q+1} - \sum_{i=p+2}^{p+q}(i-1)R_i. \end{array}$$

The row operations above change the matrix C into the matrix shown in Figure 12.

$$\begin{array}{r|ccccc|ccccc|ccccc|}
1 & -2 & 1 & & & & & & & & & 1 & & & & \\
2 & 1 & -2 & 1 & & & & & & & & & & & & \\
\vdots & & & \ddots & & & & & & & & & & & & \\
p-1 & & & 1 & -2 & 1 & & & & & & & & & & \\
p & & & & 1 & -2 & & & & & 1 & & & & & \\ \hline
p+1 & & & & & p & -2 & 1 & \cdots & \cdots & -(p-1) & & & & & \\
p+2 & & & & & & 1 & -2 & 1 & & & & & & & \\
\vdots & & & & & & & & \ddots & & & & & & & \\
p+q-1 & & & & & & & & 1 & -2 & 1 & & & & & \\
p+q & & & & & & & & & 1 & -2 & & & & & 1 \\ \hline
p+q+1 & & & & & & & & & & q & -2 & 1 & \cdots & \cdots & -(q-1) \\
p+q+2 & & & & & & & & & & & 1 & -2 & 1 & & \\
\vdots & & & & & & & & & & & & & \ddots & & \\
p+q+r-1 & & & & & & & & & & & & & 1 & -2 & 1 \\
p+q+r & & & & & 1 & & & & & & & & & 1 & -2
\end{array}$$

Figure 12: Partially reduced crossing matrix of a (p, q, r) pretzel knot.

The final row that must be reduced is R_{p+q+r}. We begin by switching this row with row R_{p+1} (because there is a p above the 1 in column C_p):

$$R_{p+q+r} \longleftrightarrow R_{p+1}.$$

Note that switching these rows will change the sign of the determimnant. The new row R_{p+q+r} can be partially reduced with the sequence of row operations:

$$R_{p+q+r} \longrightarrow R_{p+q+r} - \sum_{i=p+1}^{p+q}(i-1)R_i.$$

At this point, the last $r + 1$ rows of the matrix are as shown in Figure 13.

$$
\begin{array}{r|c|cc|ccccc|}
p+q & & 1 & -2 & & & & & 1 \\
\hline
p+q+1 & & & q & -2 & 1 & \cdots & \cdots & -(q-1) \\
p+q+2 & & & & 1 & -2 & 1 & & \\
\vdots & & & & & & \ddots & & \\
p+q+r-1 & & & & & & 1 & -2 & 1 \\
p+q+r & & & -p & -2 & 1 & \cdots & -p & 1+2p
\end{array}
$$

Figure 13: Final $r+1$ rows of the partially reduced crossing matrix of a (p,q,r) pretzel knot.

Since we now have a q and a $-p$ in the same column, we will have to use some creative operations to continue reducing:

$$
\begin{array}{rcl}
R_{p+q+1} & \longrightarrow & R_{p+q+1} + R_{p+q+r}, \\
R_{p+q+r} & \longrightarrow & q \cdot R_{p+q+r}, \\
R_{p+q+r} & \longrightarrow & R_{p+q+r} + p \cdot R_{p+q+1}.
\end{array}
$$

Note that multiplying by q will change the determinant by a factor of q. The final row is now:

$$
\left(\begin{array}{ccc|ccc|cccccccc} 0 & \cdots & 0 & 0 & \cdots & 0 & -2(q+p) & p+q & 0 & \cdots & 0 & -pq & p+q+pq \end{array} \right).
$$

The remainder of the row reduces fairly easily using the following sequence of row operations:

$$
R_{p+q+r} \longrightarrow R_{p+q+r} + \sum_{i=p+q+2}^{p+q+r-1} (i-p-q)(q+p)R_i.
$$

We have now reduced the crossing matrix C to the reduced matrix R shown in Figure 14.

$$
\begin{array}{r|ccccc|ccccc|ccccc|}
1 & -2 & 1 & & & & & & & & & 1 & & & & \\
2 & 1 & -2 & 1 & & & & & & & & & & & & \\
\vdots & & & \ddots & & & & & & & & & & & & \\
p-1 & & & 1 & -2 & 1 & & & & & & & & & & \\
p & & & & 1 & -2 & & & & & 1 & & & & & \\
\hline
p+1 & & & & & 1 & & & & & & & & & 1 & -2 \\
p+2 & & & & & & 1 & -2 & 1 & & & & & & & \\
\vdots & & & & & & & & \ddots & & & & & & & \\
p+q-1 & & & & & & & & 1 & -2 & 1 & & & & & \\
p+q & & & & & & & & & 1 & -2 & & & & & 1 \\
\hline
p+q+1 & & & & & & & & & & q & -2 & 1 & \cdots & \cdots & -(q-1) \\
p+q+2 & & & & & & & & & & & 1 & -2 & 1 & & \\
\vdots & & & & & & & & & & & & & \ddots & & \\
p+q+r-1 & & & & & & & & & & & & & 1 & -2 & 1 \\
p+q+r & & & & & & & & & & & & & & \Delta & \Delta
\end{array}
$$

Figure 14: Reduced crossing matrix R where $\Delta = pq + pr + qr$

The minor of R obtained by removing the first row and last column is an upper-triangular matrix with determinant $-q \cdot \Delta$. Therefore, the minor crossing matrix of K has determinant $\Delta = pq + pr + qr$ (since our row operations changed the determinant by a factor of q and switched the sign).

Case 2: Suppose p, q, and r are positive integers, and K is a $(p,q,-r)$ is a pretzel knot, with labelings as in Figures 9 and 10 and crossing matrix D as in Figure 11. The initial row operations for reducing crossing matrix D are similar to the row operations for reducing crossing matrix C in Case 1, and are as follows:

$$\begin{array}{rcl}
R_{p+1} & \longrightarrow & R_{p+1} - \sum_{i=2}^{p}(i-1)R_i, \\
R_{p+q+1} & \longleftrightarrow & R_{p+q+2}, \\
R_{p+q+2} & \longrightarrow & R_{p+q+2} + 2 \cdot R_{p+q+1}, \\
R_{p+q+1} & \longrightarrow & R_{p+q+1} - \sum_{i=p+2}^{p+q}(i-p-1)R_i, \\
R_{p+q+r} & \longrightarrow & R_{p+q+r} + 2 \cdot R_{p+q+r-1}, \\
R_{p+q+r-1} & \longleftrightarrow & R_{p+1}, \\
R_{p+q+r-1} & \longrightarrow & R_{p+q+r-1} + -p \cdot R_{p+1} \sum_{i=p+2}^{p+q}(i-p)R_i. \\
R_{p+q+r-1} & \longrightarrow & R_{p+q+r-1} + R_{p+q+1}, \\
R_{p+q+r-1} & \longrightarrow & q \cdot R_{p+q+r-1}, \\
R_{p+q+r-1} & \longrightarrow & R_{p+q+r-1} + p \cdot R_{p+q+1}, \\
R_{p+q+r-1} & \longrightarrow & R_{p+q+r-1} + \sum_{i=p+q+3}^{p+q+r-2}(i-p-q-1)(p+q)R_i, \\
R_{p+q+r-1} & \longleftrightarrow & R_{p+q+r}, \\
R_{p+q+r} & \longrightarrow & 2 \cdot R_{p+q+r}, \\
R_{p+q+r} & \longrightarrow & R_{p+q+r} + (-pr - qr - pq + 2p + 2q) \cdot R_{p+q+r-q}.
\end{array}$$

Note that we multiplied rows by q and 2, which will change the determinant. The fully reduced matrix S is shown in Figure 15.

$$\begin{array}{r|ccccccc|ccccccc|ccccccc|}
1 & -2 & 1 & & & & & & & & & & & & & 1 & & & & & & \\
2 & 1 & -2 & 1 & & & & & & & & & & & & & & & & & & \\
3 & & 1 & -2 & 1 & & & & & & & & & & & & & & & & & \\
\vdots & & & & \ddots & & & & & & & & & & & & & & & & & \\
p-2 & & & & 1 & -2 & 1 & & & & & & & & & & & & & & & \\
p-1 & & & & & 1 & -2 & 1 & & & & & & & & & & & & & & \\
p & & & & & & 1 & -2 & & & & & & & 1 & & & & & & & \\
\hline
p+1 & & & & & & & 1 & & & & & & & & & & & & & 1 & -2 \\
p+2 & & & & & & & & 1 & -2 & 1 & & & & & & & & & & & \\
p+3 & & & & & & & & & 1 & -2 & 1 & & & & & & & & & & \\
\vdots & & & & & & & & & & & & \ddots & & & & & & & & & \\
p+q-2 & & & & & & & & & & & 1 & -2 & 1 & & & & & & & & \\
p+q-1 & & & & & & & & & & & & 1 & -2 & 1 & & & & & & & \\
p+q & & & & & & & & & & & & & 1 & -2 & & & & & & & 1 \\
\hline
p+q+1 & & & & & & & & & & & & & & q & 0 & -2 & 1 & & \cdots & & -(q-1) \\
p+q+2 & & & & & & & & & & & & & & & 1 & -3 & 2 & & & & \\
p+q+3 & & & & & & & & & & & & & & & & 1 & -2 & 1 & & & \\
\vdots & & & & & & & & & & & & & & & & & & \ddots & & & \\
p+q+r-2 & & & & & & & & & & & & & & & & & & 1 & -2 & 1 & \\
p+q+r-1 & & & & & & & & & & & & & & & & & & & 2 & -3 & 1 \\
p+q+r & & & & & & & & & & & & & & & & & & & & -\Delta & \Delta
\end{array}$$

Figure 15: Fully reduced crossing matrix S of $(p,q,-r)$ pretzel knot where $\Delta = pq + pr + qr$.

The minor of the reduced matrix S obtained by removing the first row and last column is an upper-triangular matrix with determinant $2 \cdot q \cdot \Delta$. Our row operations changed the determinant by a factor of q and then by a factor of 2 (there is no sign change since we performed an even number of row switches). Therefore, the minor crossing matrix of K has determinant $\Delta = pq + pr + qr$. □

We should also note that it is possible for a pretzel knot to have determinant ± 1, in which case the m-nullity will be 1 for any m. (For example, the $(2, -3, -5)$ pretzel knot has determinant -1.) In this case, the knot is not m-colorable for any m. This can only happen if there are both positive and negative strands in the pretzel knot.

4.4. *Fundamentally different m-colorings of pretzel knots*

We now bring together the results from Sections 3.1 and 4.3 to determine the number of fundamentally different m-colorings of any pretzel knot.

Main Theorem 2. Suppose K is a (p, q, r) pretzel knot, and m is a prime number that divides $\det(K) = pq + pr + qr$.

(i) If at least one of p, q, and r is relatively prime to m, then K has only one fundamental m-coloring.

(ii) If none of p, q, and r are relatively prime to m, then K has exactly four fundamentally different m-colorings.

Proof. By Main Theorem 1, it suffices to show that in part (a) of Main Theorem 2, the m-nullity of K is 2, and in part (b), the m-nullity of K is 3.

Part (i): Suppose K is a (p, q, r) pretzel knot where p, q, and r are positive and at least one twist is relatively prime to m (see Example 4). Without loss of generality, we can assume that q is relatively prime to m (see Theorem 4). To find the m-nullity of K we will reduce its crossing matrix modulo m. We have already reduced this matrix over the integers in Case 1 of the proof of Theorem 6; see Figure 14. Since m is relatively prime to q, the row operation where we multiplied by q is still valid modulo m. Therefore, the matrix in Figure 14 is the mod m reduced crossing matrix for K. Since m divides the determinant, this matrix clearly has m-nullity 2. The proof where q is relatively prime to m and K is a $(p, q, -r)$ pretzel knot is similar.

Part (ii): Suppose K is a (p, q, r) pretzel knot where p, q, and r are positive integers and no twist is relatively prime to m (see Example 5). When reducing the crossing matrix for K in the proof of Theorem 6, we multiplied a row by q. Since m divides q (and in fact, also divides p and r), this operation is not permitted when reducing $\bmod m$. The row operations before this step (up to Figure 13) are legal $\bmod m$, and result in the matrix shown in Figure 16 (after taking mod m).

Notice that in Figure 16, the rows R_{p+q+1} and R_{p+q+r} are the same. The following operation is enough to finish reducing the matrix:

$$R_{p+q+r} \longrightarrow R_{p+q+r} + (-R_{p+q+1}).$$

This leaves one row of zeroes, so this matrix clearly has m-nullity 3. The proof in the case where m is not relatively prime to any twist, and K is a $(p, q, -r)$ pretzel knot is similar. □

$$
\begin{array}{r|ccccc|ccccc|ccccc|}
1 & -2 & 1 & & & & & & & & & 1 & & & & \\
2 & 1 & -2 & 1 & & & & & & & & & & & & \\
\vdots & & & \ddots & & & & & & & & & & & & \\
p-1 & & & 1 & -2 & 1 & & & & & & & & & & \\
p & & & & 1 & -2 & & & & & 1 & & & & & \\
\cline{2-16}
p+1 & & & & & 1 & & & \cdots & & & & & & 1 & -2 \\
p+2 & & & & & & 1 & -2 & 1 & & & & & & & \\
\vdots & & & & & & & & \ddots & & & & & & & \\
p+q-1 & & & & & & & & 1 & -2 & 1 & & & & & \\
p+q & & & & & & & & & 1 & -2 & & & & & 1 \\
\cline{2-16}
p+q+1 & & & & & & & & & & 0 & -2 & 1 & \cdots & \cdots & 1 \\
p+q+2 & & & & & & & & & & & 1 & -2 & 1 & & \\
\vdots & & & & & & & & & & & & & \ddots & & \\
p+q+r-1 & & & & & & & & & & & & & 1 & -2 & 1 \\
p+q+r & & & & & & & & & & 0 & -2 & 1 & \cdots & 0 & 1
\end{array}
$$

Figure 16: Partially reduced crossing matrix of a (p, q, r) pretzel knot where no twist is relatively prime to the $\bmod m$.

Example 4. The $(3, 15, 5)$ pretzel knot has determinant $135 = 3^3 \cdot 5$, and thus is m-colorable (for prime m) if and only if $m = 3$ or $m = 5$. Because $m = 3$ is relatively prime to at least one of the twists, the $(3, 15, 5)$ pretzel knot has 3-nullity 2, and therefore only one fundamental 3-coloring (shown in Figure 17). Similarly, $m = 5$ is relatively prime to at least one twist, the $(3, 15, 5)$ pretzel knot has 5-nullity 2, and only one fundamental 5-coloring (shown in Figure 18).

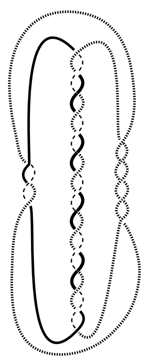

Figure 17: The 3-coloring of the $(3, 15, 5)$ pretzel knot.

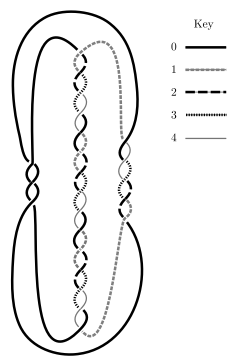

Figure 18: The 5-coloring of the $(3, 15, 5)$ pretzel knot.

Example 5. The pretzel knot $(3, 6, 9)$ has determinant $99 = 3^2 \cdot 11$, and thus is thus both 3-colorable and 11-colorable. Because 3 divides all three twists, the knot has 3-nullity 3, and therefore four fundamentally different 3-colorings (shown in Figure 19). On the other hand, since 11 is relatively prime to at least one twist (in fact, to all of the twists), the 11-nullity is 2. Therefore, the $(3, 6, 9)$ pretzel knot only one fundamental 11-coloring (shown in Figure 20).

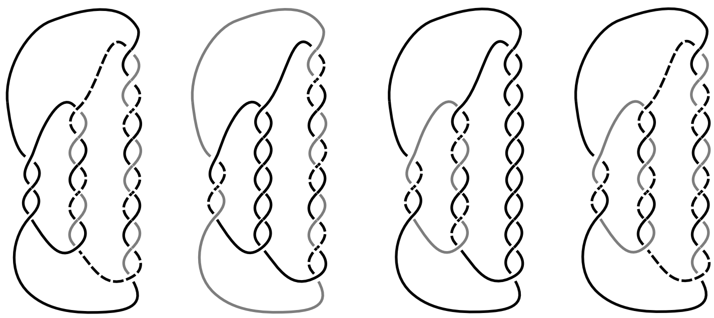

Figure 19: Four fundamentally different 3-colorings of the pretzel knot $(3, 6, 9)$.

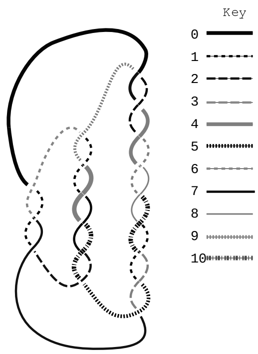

Figure 20: The $(3, 6, 9)$ pretzel knot has one fundamental 11-coloring.

Acknowledgements

This work is the result of an NSF-funded Research Experiences for Undergraduates project at James Madison University (NSF grant number NSF-DMS 0243845), and was written by the first and second authors under the direction of the third.

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