INTRINSIC KNOTTING OF PARTITE GRAPHS

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Abstract. We introduce theorems that identify partite graphs that are intrinsically knotted (IK) since they contain subgraphs that are intrinsically knotted. We show that if we increase the number of vertices in each of the parts except one and delete an edge of an intrinsically knotted graph of two or more parts, the resulting graph is also IK. This is then extended to show that $K_{5,5}$ minus 3 edges is not intrinsically knotted when there exists a vertex of degree 2.

1. Introduction

A **graph** is a pair $G = (v, e)$, where v represents the set of vertices, and e represents the set of edges. An embedding is a representation of the graph in 3-space.

The **degree of a vertex** is the number of edges at that vertex. Two vertices are **adjacent**, or neighbors, if they have an edge connecting them. Two edges are adjacent if they have a vertex in common. A **cycle** is a path that ends and starts on the same vertex. A graph is **intrinsically knotted (IK)** if any embedding of the graph in 3-space contains a nontrivially knotted cycle.

A graph is **planar** if it can be drawn in a plane such that its edges do not intersect. Otherwise, it is non-planar. Planar graphs can have non-planar representations.

A graph is considered to be a **complete graph** if all vertices are connected to one another by one edge. We denote the complete graph on n vertices K_n. A partite graph is one in which the vertices are partitioned into disjoint sets called parts, such that each edge connects vertices of different sets. A complete partite graph has each vertex adjacent to all vertices in the other parts. A graph is **k-deficient** if it is k edges short of being a complete partite graph. A complete partite graph, with parts having $n_1, n_2, \ldots, n_m$ vertices is represented $K_{n_1,n_2,\ldots,n_m}$.

An **expansion** of a graph is formed by the addition of vertices and edges and by splitting vertices. Contracting edges and deleting any vertices and edges yields a **minor** of the graph. A graph which has a property that no proper minor of that graph has is **minor minimal** with respect to that property.

Conway and Gordon [CG] first noticed that every embedding of K_7 contains a knotted cycle. Likewise, they showed that every embedding of the K_6 graph contains at least one pair of nontrivially linked cycles. (This was also discovered, independently, by Sachs [S].)

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2. Partite Graphs with IK Subgraphs

Theorem 2.1. *A graph $K_{a,a+n}$ - $(a+n-3)$ edges is IK for $a \geq 5$, $n \geq 0$.*

Proof: For $n = 0$, $a = 5$: $K_{5,5+0}$ - $(5-3)$e $= K_{5,5}$ - 2e and $K_{5,5}$ - 2e is IK [CMOPRW].

We will assume $a + n \geq 6$. We now show $K_{5,5}$ - 2e is a subgraph of $K_{a,a+n}$ - $(a+n-3)$ edges. In order to obtain these subgraphs, we want to choose vertices on the right-hand side, i.e., the $a+n$ part, with the least number of edges deleted. There are 3 cases to examine: Case 1: exactly 3 vertices on the right-hand side of the graph ($a+n$ part) do not have edges deleted, Case 2: exactly 4 vertices on the RHS do not have edge deletions, and Case 3: 5 or more vertices do not have edge deletions. These cases are the ones that exist since we are removing $a+n-3$ edges and there are $a+n$ vertices on the RHS. So at least 3 vertices have no edges removed. In Case 2, $\exists$ one vertex with only one edge removed because there are exactly 4 vertices with no edges removed. So $a+n-3$ edges are removed from the remaining $a+n-4$ vertices. This means exactly one vertex has 2 edges removed and the others have 1 edge deleted.

Case 1: $\exists$ 3 vertices on the RHS that do not have edge deletions. Then all other vertices have one edge deletion each. Thus, by choosing the 3 and any 2 others, and using any 5 vertices on the LHS (the part with a vertices), we have a $K_{5,5}$ - 2e subgraph, and thus, the graph is IK.

Case 2: $\exists$ 4 vertices on the right with no edges removed and one with at most 1 edge removed. Therefore, by choosing these 5 vertices on the right, and using any 5 vertices on the LHS, we have a $K_{5,5}$ - e subgraph. $K_{5,5}$ - 2e is a subgraph since we are able to delete an additional edge from $K_{5,5}$ - e to obtain $K_{5,5}$ - 2e, and so the graph is IK.

Case 3: $\exists$ 5 or more vertices on the right with no edges removed. Thus we are able to use 5 of these vertices that have no edges removed and any 5 vertices on the LHS to create a $K_{5,5}$ subgraph. So $K_{5,5}$ - 2e is a subgraph since we are able to delete two additional edges from $K_{5,5}$ to get $K_{5,5}$ - 2e, and thus, the graph is IK. □

We can use the theorem to improve the sufficient condition for IK of [CMOPRW]: $E(G) \geq 5V(G) - 14$.

Corollary 2.2. *A bipartite graph with 5 vertices in one part and $E(G) \geq 4V(G) - 17$ is IK.*

Proof: If $a = 5$, $K_{5,n+5}$ - $(5+n-3)$e $= K_{5,n+5}$ - $(n+2)$e has $5n+25-(n+2) = 4n+23 = 4(n+10)-17$ edges and $5+(n+5) = n+10$ vertices. Therefore, if $E(G) \geq 4V(G) - 17$, the graph has a $K_{5,n+5}$ -$(n+2)$e minor and is IK. □

Corollary 2.3. *A bipartite graph with 6 vertices in one part and $E(G) \geq 5V(G) - 27$ is IK.*

Proof: If $a = 6$, $K_{6,n+6}$ - $(6+n-3)$e $= K_{6,n+6}$ - $(n+3)$e has $6n+36-(n+3) = 5n+33 = 5(n+12)-27$ edges and $6+(n+6) = n+12$ vertices. Therefore, if $E(G) \geq 5V(G) - 27$, the graph has a $K_{6,n+6}$ -$(n+3)$e minor and is IK. □

Lemma 2.4. *Let G be an intrinsically knotted partite graph of two or more parts. If the number of vertices in each of the parts except one is increased by one and an edge is deleted, the graph G' so formed is IK.*

Proof: We show that adding a vertex to each of the partitions except one allows us to avoid the edge deletion by removing a vertex from all but one part and recovering G as a subgraph. We will look at two cases where the edge deletion can occur: 1.) an edge is deleted between a part with a new vertex and a part that stays the same and 2.) an edge is deleted between parts that both have a new vertex. These are the only two cases that occur because we cannot have edges existing between vertices of the same part in a partite graph.

Case 1: We are able to avoid the deleted edge by not using the extra vertex affected by the edge deletion. Therefore, we are able to recover G as a subgraph.

Case 2: Similarly to Case 1, we can avoid the edge deletion because of the addition of a vertex to each part. We can choose to not include the two vertices of the deleted edge since we now have two new vertices that are unaffected by the edge deletion. Again, we are able to recover the IK subgraph.

Therefore, in both cases, we are able to recover the IK minor and have thus shown that G' is also IK. □

Let $K_{a,(b)^n}$ denote a complete partite graph with one part of a vertices and n parts of b vertices.

Theorem 2.5. *A graph of the form $K_{a,(a+1)^n}$ - e has $K_{a,(a)^n}$ as a subgraph*

Proof: In $K_{a,(a+1)^n}$ - e, the edge removed must be taken from either parts with a and $a+1$ vertices (1) or parts with $a+1$ and $a+1$ vertices (2).

Case 1: The edge is removed between parts with a and $a+1$ vertices. By choosing the a vertices with no edge removed in the $a+1$ part in question and a vertices in the other parts, we construct $K_{a,(a)^n}$.

Case 2: The edge is between $a+1$ and $a+1$. Again we can choose a vertices from those two parts and every other part to form $K_{a,(a)^n}$ as a subgraph.

Therefore $K_{a,(a+1)^n}$ - e has a $K_{a,(a)^n}$ as a subgraph. □

Corollary 2.6. *If $K_{a,(a+1)^n}$ - e is not IK, then $K_{a,(a)^n}$ is not IK.*

Corollary 2.7. *If $K_{a,(a)^n}$ is IK, then $K_{a,(a+1)^n}$ - e is IK.*

Theorem 2.8. *A graph of the form $K_{a,(a+n)^k}$ -(n+a-2)e is intrinsically knotted for $a \geq 3$, $n \geq 0$, and $k \geq 2$.*

Proof by Induction: When $n = 0$ and $a \geq 3$, $k \geq 2$: $K_{a,(a)^n}$ - $(a-2)$e always has a $K_{3,3,3}$ - e subgraph which is intrinsically knotted [CMOPRW]. Therefore $K_{a,(a+n)^k}$ -$(n+a-2)$e is intrinsically knotted for $n = 0$.

Assume that $K_{a,(a+n)^k}$-$(n+a-2)$e is intrinsically knotted, with $a \geq 3$, $n \geq 0$, $k \geq 2$. If n is increased by 1 (i.e., $n \Rightarrow n+1$) then the number of edges removed is also increased by 1. The new edge that is removed is either removed between the parts with a and $a+(n+1)$ or between parts with $a+(n+1)$ and $a+(n+1)$ vertices.

Case 1: The edge is removed between parts with a and $a+(n+1)$. We still have $a+n$ choices of vertices from which we can choose our minor and avoid including the edge that has been removed. Therefore we will have a $K_{a,(a+n)^k}$ - $(n+a-2)$e subgraph which is intrinsically knotted (by induction).

Case 2: The edge is removed between two $a+(n+1)$ parts. Again there are still $a+n$ choices of vertices from which we can choose our minor and still avoid including that edge. Therefore we will have a $K_{a,(a+n)^k}$ - $(n+a-2)$e subgraph which is intrinsically knotted (by induction).

Therefore when $K_{a,(a+n)^k}$ - $(n+a-2)$e is intrinsically knotted, $K_{a,(a+(n+1))^k}$ - $((n+1)+a-2)$e is intrinsically knotted. By induction, the theorem is proved. $\square$

3. Not Intrinsically Knotted Graph

Theorem 2.1 is based on showing graphs of the form $K_{a,a+n}$ - $(a+n-3)$ edges have a $K_{5,5}$ - 2e IK minor. Here we show that this method cannot be improved because, in general, $K_{5,5}$ - 3e is not IK. Following [CMOPRW], we denote the vertices in $K_{l,m,n}$ by $\{a_1, a_2, \ldots, a_l\}$, $\{b_1, b_2, \ldots, b_m\}$, and $\{c_1, c_2, \ldots, c_n\}$. Thus, $K_{5,5}$ - $\{a_5b_1, a_4b_1, a_3b_1\}$ denotes $K_{5,5}$ with 3 edges removed, all of them incident to the same vertex b_1 in the second part.

Theorem 3.1. $K_{5,5}$ - $\{a_5b_1, a_4b_1, a_3b_1\}$ *is not IK.*

Proof: It is known that $K_{4,4,1}$ - $\{a_5c, a_4c\}$ is not IK [CMOPRW]. This implies that $K_{4,4,1}$ - $\{a_5c, a_4c, a_3c\}$ is also not IK. In a knotless embedding of $K_{4,4,1}$, add a vertex d on the edge a_2c. The graph is now bipartite as we will show. In adding d, the vertex c is no longer connected by an edge to any vertex in A. c is connected to $\{b_2, b_3, b_4, b_5, d\}$. Therefore we can consider c to be a vertex in A. d is connected to $\{a_2, c\}$. Therefore we can consider d to be a vertex in B. Since $\deg(c) = 5$, $\deg(d) = 2$, $\deg(a_3) = \deg(a_4) = \deg(a_5) = 4$ and $\deg(a_2) = \deg(b_2) = \deg(b_3) = \deg(b_4) = \deg(b_5) = 5$, we have a bipartite graph on 10 vertices. This graph is precisely $K_{5,5}$ - 3e with a vertex d of degree two. Therefore $K_{5,5}$ - $\{a_5b_1, a_4b_1, a_3b_1\}$ has at least one knotless embedding. $\square$

References

[CMOPRW] J. Campbell, T.W. Mattman, R. Ottman, J. Pyzer, M. Rodrigues, and S. Williams, *Intrinsic Knotting and Linking of Almost Complete Graphs*, (preprint) math.GT/0701422 (2007).

[CG] J.H. Conway and C.McA. Gordon, *Knots and Links in Spatial Graphs*, Journal of Graph Theory, **Vol 7**, (1983), 445-453.

[S] H. Sachs, 'On Spatial Representations of Finite Graphs', in: A. Hajnal, L. Lovasz, V.T. Sos (Eds.), Colloq. Math. Soc. Janos Bolyai, Vol 37, North-Holland, Amsterdam, (1984) 649-662.