Torus Links and the Bracket Polynomial

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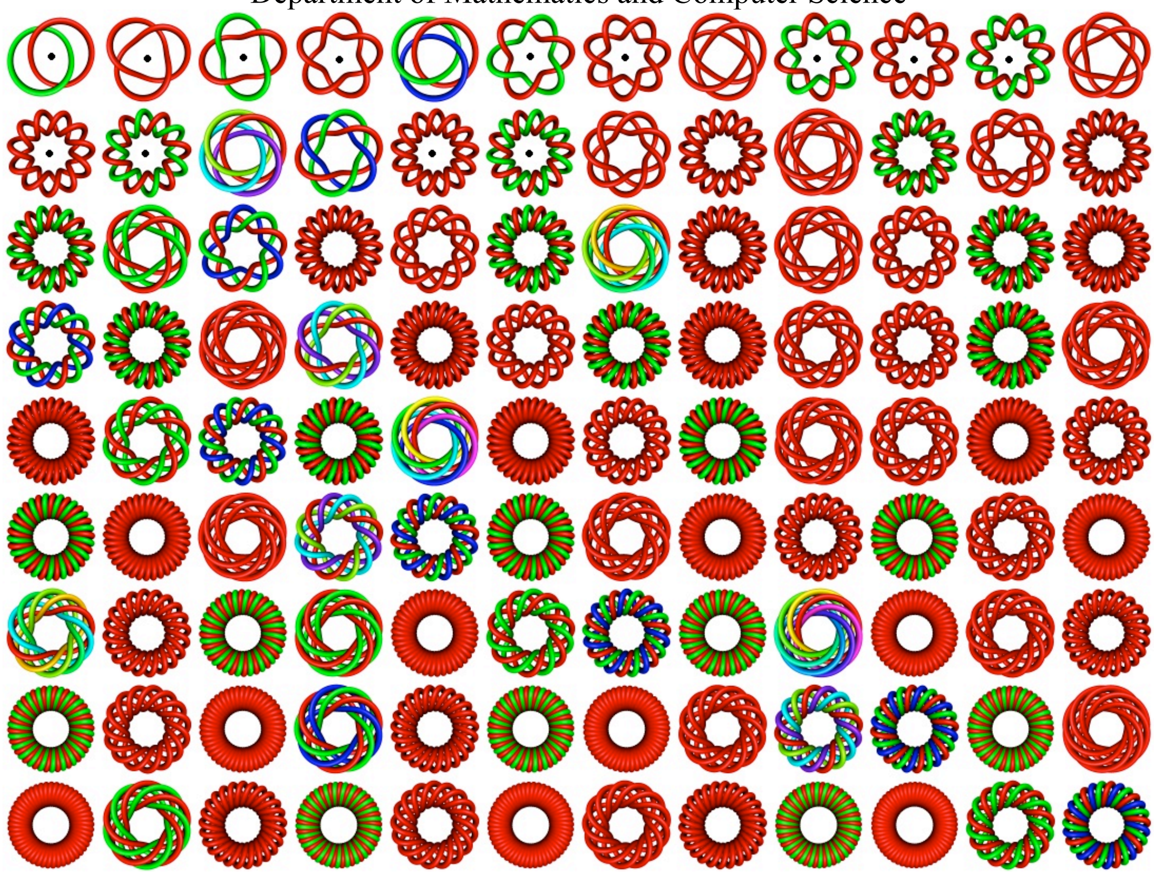

A picture of some torus knots and links. The first several (n,2) links have dots in their center. [1]

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Abstract

In this paper torus knots and links will be investigated. Links is a more generic term than knots, so a reference to links includes knots. First an overview of the study of links is given. Then a method of analysis called the bracket polynomial is introduced. A specific class called (n,2) torus links are selected for analysis. Finally, a recurrence relation is found for the bracket polynomial of the (n,2) links. A complete list of sources consulted is provided in the bibliography.

Chapter 1: Knots and Links

Knot theory belongs to a branch of mathematics called topology. Topology is a relatively new area of mathematics; it only began to develop in the nineteenth century. Topology is concerned with objects that can be continuously deformed from one object to another. The field is sometimes called rubber sheet geometry. An illustrative example is the fact that topologically a torus (a donut) is equivalent to a mug (see Figure 1). A mug can be compacted and then massaged around to resemble a torus. The hole rather than the geometry is what relates them.

Figure 1.1 Torus and Mug.

Knot theory is the subdivision of topology concerning entwined circles. A knot is a single knotted circle, while a link is composed of two or more entangled circles. The generic term link will refer to both knots and links; where a distinction is needed it will be provided. The main question in knot theory is: are two knots equivalent? If we can take one knot and deform it without any self-intersections into another knot, then the two knots are equivalent. Failure to transform one knot into another does not mean the two are not equivalent; perhaps a few more moves will solve the problem. Knot theory may appear an idle academic pursuit but it has practical applications.

1.1 The History of Knots and Links

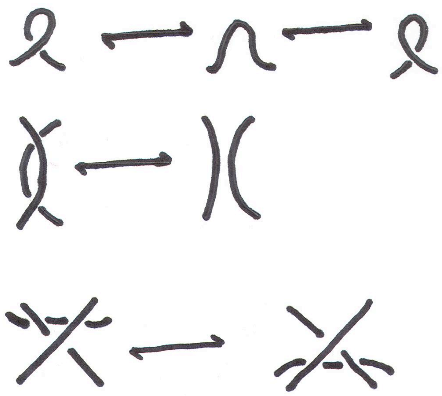
In 1833 knots were ushered into mainstream mathematics when Gauss began to use them to work on some problem related to electrodynamics. In the mid-1850s knots came to the attention of several Scottish mathematical physicists: Lord Kelvin, James Clerk Maxwell, and Peter Tait. They hypothesized that chemical elements were knotted vortexes of the ether. This group of very talented physicists had good intuition, however they did not prove their conjectures. Tait attempted to create a list of knots relating a knot to a particular element (O'Connor and Robertson).

Figure 1.2 Example of a knot (left) and a link (right)

Knotted vortex atoms fell out of favor because of other advances. First, Dimitri Mendeleyev created of the periodic table that explained the properties of the elements. Then the Michelson-Morley experiment in 1887 failed to detect the motion of the Earth with respect to the ether. Finally, Albert Einstein's Theory of Special Relativity rendered the idea of the ether as obsolete in 1905. The study of knots disappeared from physics.

While scientists' interest in knots waned, mathematicians adopted the subject. An important idea that the physicists invented was the link projection. A link projection is the shadow that the three dimensional link would cast on a two dimensional surface. By indicating over and under crossings on a knot projection mathematicians could encoded a three dimensional object in a two dimensional diagram.

In 1926 Reidemeister formulated three moves that would not change the 'knottedness' of a knot (see Figure 1.3). The manipulations of the knots are intuitive when they are demonstrated using strings. The moves would allow for a complicated knot projection to be reduced to a more manageable projection. The Reidemeister moves will be referred to as R I, R II, and R III. Reidemesiter recognized that a knot did not change under planar isotopy. Planar isotopy deforms a knot projection by altering the loops while leaving the crossings alone.



I

II

III

Figure 1.3 The Reidemesiter moves.

In the late 1920s Alexander discovered the first polynomial knot invariant. The Alexander polynomial is a Laurent polynomial with positive and negative powers of the variable t. The Alexander polynomial is useful for telling knots apart. Polynomials continue to play an important role in knot theory. The next major advance was by John Conway in the 1970s. Conway developed a polynomial that could be used to easily obtain the Alexander polynomial. The Conway polynomial was computed using a skein relation. Skein relation is one of the most used tools in computing knot polynomials. A full explanation will be given when discussing polynomials in sections 2.2 and 2.3.

In the early 1980s V.F.R. Jones made a connection between the operator algebras that he was researching and knots. He created the Jones polynomial, a new knot invariant. It is stronger than the Alexander polynomial in that it is able to distinguish more knots. After Jones' discovery, many new of polynomial invariants were discovered. Many of these new polynomials are related or can be converted into the Jones polynomial.

Figure 1.4 The unknot

A simple polynomial is the bracket polynomial discovered by Louis Kauffman. He saw a connection between a physical model and knots. The bracket polynomial is based on a skein relation, which is why it is easy to calculate. This will be covered in greater detail in section 2.3. The Jones polynomial is related to the bracket polynomial.

1.2 Applications of Knots

Knots find application in both pure mathematics and actual physical structures. Often times the knots are able to offer new insight into an object that was previously not well understood. Other times knots suggests new structures.

Once a knot is understood, it can be used to explain properties of three manifolds encountered in topology, which would otherwise be impossible to visualize. Basing three manifolds on knots also would allow one to know if two three manifolds are the same. Knots are also interesting in their own right. Knot theory is also a popular branch of modern mathematics because it is so accessible to a general audience.

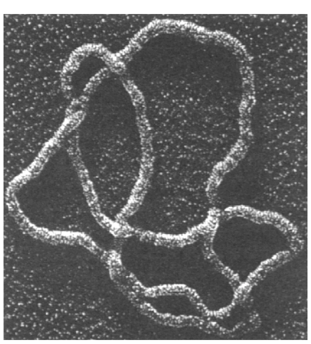


Figure 1.5 Knotted DNA. [2]

In biology knots occur in the ribbon like structure DNA. This makes it possible for a molecule of DNA to knot. Molecules of DNA are large; this allows them to have the necessary stretching and bending abilities to form knots. Using knot theory biologists are able to predict what more complex structures will look like. Researchers tested such predictions and found that they hold true. To experimentally verify the results they examine knotted DNA using scanning electron microscopes (see Figure 1.5).

Chemists are also interested in knotted molecules. The properties of a molecule that is unknotted and one that is knotted may be vastly different even if they are both composed of exactly the same atoms. When creating knotted atoms it is possible to make both a knot and its mirror image. If the knot and its mirror image are not equivalent under Reidemeister moves and planar isotopy they are said to be chiral. The major hurdle is that the creation of knotted molecules is difficult. The problem lies in being able to twist the bonds between atoms to form a knot. This requires any knotted molecule to be fairly large. The idea of knots as a model made of sticks (discussed below) is especially appropriate for chemical applications. Chemist also have to design the molecule and what elements go into it, as well as how to get the atoms to bond in an appropriate way. Such challenges fuel chemists' interest in knots.

Another major field that at first glance seem unrelated to knots is statistical mechanics. This is a branch of physics that models the behavior of a large number of particles. Often times a system is modeled with a lattice structure. Louis Kauffman and other knot theorists have found connections between certain models in statistical mechanics and knots. At this point in time statistical mechanics has provided insight into knot theory, but knot theory has not yet contributed to statistical mechanics. It is certain these two fields will remain closely related.

1.3 Classification of Knots

There are two major classes of knots: tame knots and wild knots. Wild knots are knots that are not piecewise linear. So a wild knot may have smaller and smaller knotting that shrinks down into a point. Wild knots have bizarre behaviors that make them difficult to analyze (see Figure 1.6). The study of knots focuses on tame knots. Tame knots are piecewise linear. It is easy to think of knots in terms of sticks. It is also possible to create minimal stick models using as few sticks as possible (see Figure 1.6). Finding the minimal stick model gives us a measure of the complexity of the knots.

Links are identified by the number of components that they have. Knots are a special subset of links, namely links with a single component. Another characteristic used to identify knots and links are the number of crossings of a particular projection. A few knots have special names (see Figure 1.7). Knot theorists look for a projection of the knot that has the minimal number of crossings. They do this because, in a sense, these projections depict the simplest representation of that knot. Knot theorists have many other techniques that can be deployed to understand knots.

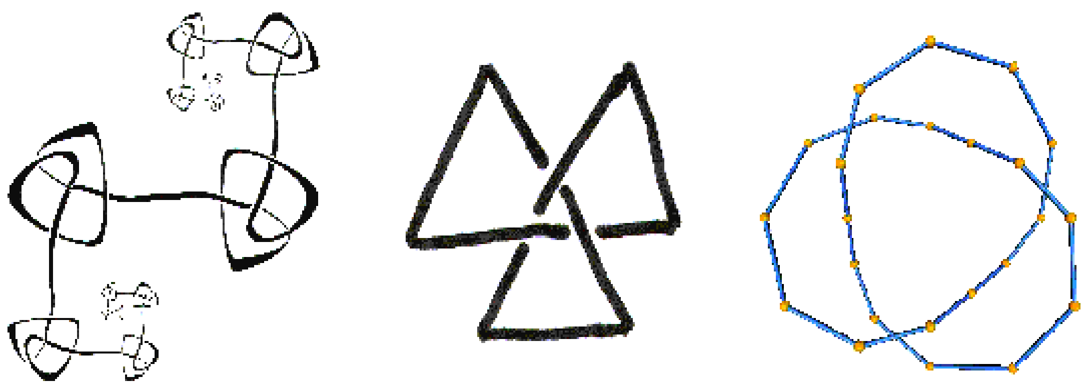


Figure 1.6 (L to R) A wild knot [3], A minimal stick model of the trefoil [4], A stick model of the trefoil [5]

In this paper links that can go around a torus are the links that will be analyzed. In Chapter 3 the rational for picking these links and some of their properties will be explained. In Figure 1.7 the first few torus links are shown. Immediately a pattern seems to emerge, a torus fits nicely within the link's loops.

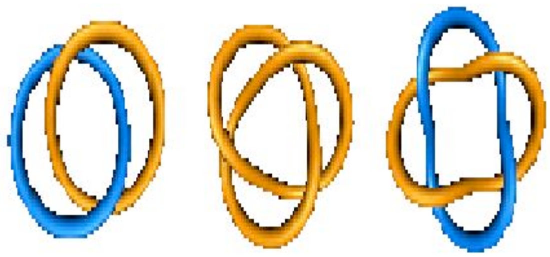


Figure 1.7 (L to R) The Hopf link, The Trefoil knot, and King Solomon's knot [5]

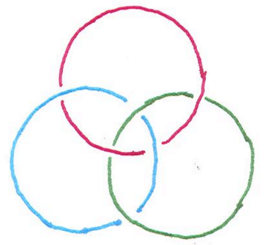


Figure 2.8 Borromean Rings

Chapter 2: Mathematics of Knots and Links

At first glance knots might seem to have little or no connection to mathematics. The connections to mathematics arise through algebraic topology along with other branches of topology. Concepts from algebraic topology lead to major advances in knot theory. The most successful mathematical constructs come in the form of polynomials derived from a link's projection.

One physical invariant that is useful is the number of components a link has. A knot has only one component, but when looking at links the number of components is somewhat important. A link with two components cannot become a link with three components. This scheme is useful for initially sorting a number of knots and links.

2.1 Knot Polynomials

Knot theorists have developed several polynomials for distinguishing one knot from another. Polynomials are much stronger than the physical features of a knot projection, and as of yet none are foolproof. Knot theorist will continue to search for better invariants.

Most polynomial invariants are obtained via skein relations. The skein relations are defined in terms of the knot projection. Specific skein relations are particular to a polynomial. Some polynomials are not invariant under the type I Reidemeister move, so they are referred to as framed

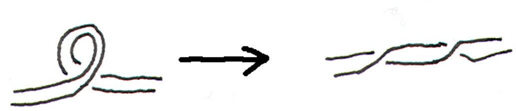


Figure 2.9 A type I move when viewing the knot as made of a ribbon.

polynomials. This means that instead of being thought of as strings they are better thought of as ribbons though the analysis remains very much the same. These polynomials are Laurent polynomials. They have one or more variables that can take on both positive and negative powers. Orientation gives the link an arrow associated with each of its components. The most discriminating polynomials often have two variables. In recent years, many polynomials have been discovered. Some are tedious to calculate while others are straightforward. Most of the polynomials are very similar.

The first polynomial developed was the Alexander polynomial. This polynomial was expressed in powers of t^{-}. This polynomial is often described in terms of the determinant of a matrix. Being the oldest polynomial, it is well understood in terms of algebraic topology. Work by John Conway in the late 1960s and early 1970s found an easier way to calculate the Alexander polynomial. The problem with the Alexander polynomial is that it is unable to distinguish between some knots.

An important part of identifying a knot was the use of skein relation in the 1960s, again work by John Conway. These are the major tools in many of the latest polynomials.

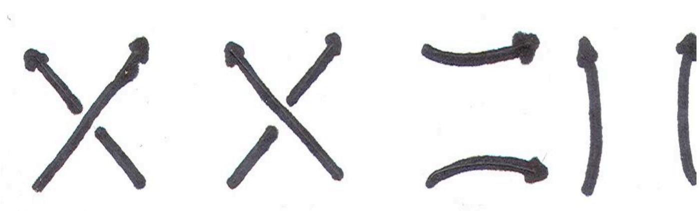


Figure 2.10 A skein relation. (L to R) The original crossing, a 'flip', the last two are smoothings.

The next polynomial was discovered by V.F.R. Jones in the early 1980s and generalized by a number of mathematicians. Adopting Lickorish and Millets designations for polynomials, the Jones polynomial is denoted by V(L) and the generalized Jones polynomials P(L). The polynomial P(L) is stronger than the original

Jones polynomial because it has two variables. The generalized polynomial can be converted into the Jones polynomial but an appropriate change of variables. Both of these polynomials are rather difficult to calculate.

Closely related to the Jones polynomial is the single variable bracket polynomial, <L>, of a link L. It is simple to calculate compared with the V(L) and P(L) polynomials. The bracket polynomial can be calculated directly from the knot projection. The span of <L> is the span (difference between highest and lowest powers in the polynomial) for any projection of a knot is invariant. The bracket polynomial is good at discovering whether mirror images of a knot are equivalent. One shortcoming is that the polynomial is not invariant under a type I move. One can correct for this and can obtain a new polynomial, X(L), which is an invariant. Once again the Jones polynomial, V(L), can be obtained from X(L) by an appropriate change of variables.

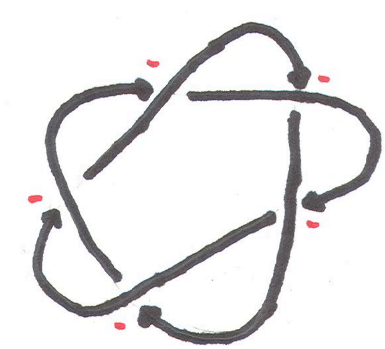


Figure 2.11 The writhe of this knot is five from looking at the over and undercrossings.

The calculation of the polynomial <L> of a link L is simply calculated from looking at the pictures and using the appropriate coefficients.

2.2 Developing the Bracket Polynomial

The bracket polynomial is chosen because it is easy to calculate. This polynomial based on a simple skein relation, which takes a crossings and smoothing them (see Figure 2.5).

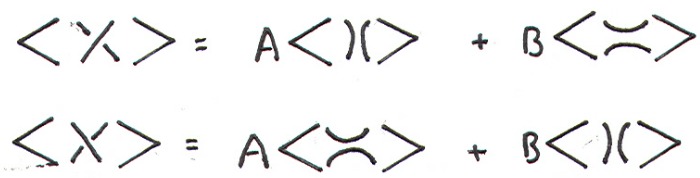


Figure 2.5 Rules for smoothings to obtain the bracket polynomial.

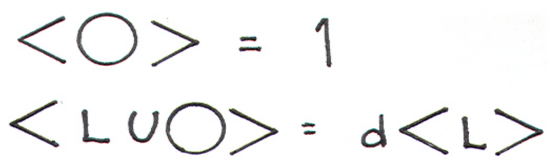


Figure 2.6 Therules for concerning unknots.

There are two ways to smooth a crossing depending on the whether it is a positive or negative crossing. 'Right handed' knots have positive crossings while 'left handed' knots have negative crossings. A smoothing removes one crossing and creates two possible projections of the knot. This process continues until the projection of a disjoint union of unknots. The other two rules are then applied (see Figure 2.6).

To make the bracket polynomial invariant under Reidemeister moves we select R II and show that the skein relation is an invariant. Following the proof in Figure 2.7 to be invariant AB = 1 and $A^2 + ABd + B^2 = 0$. So A = A and B = A^-

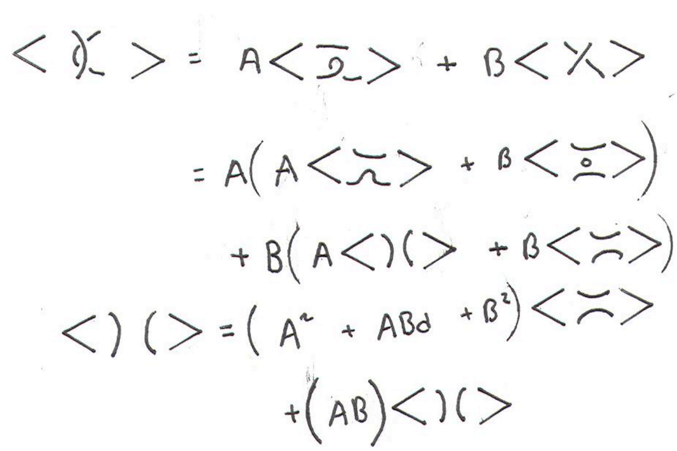


Figure 2.7 A pictorial proof that the bracket polynomial is invariant under R II move.

[1] (Kauffman p.206). Then from this relation we can conclude the $d = -A^4 - A^{-4}$. A similar proof can be constructed to show that these rules are invariant under the R III move (see Appendix D). In section 4.1 the bracket polynomial of torus knots will be calculated from these rules.

The bracket polynomial is not invariant under R I. So the bracket polynomial is a framed invariant. The span, the lowest power subtracted from the highest power of the polynomial, is invariant. To make it invariant it must be multiplied by the writhe of the link, w(L), which will be discussed in Section 3.3. The bracket polynomial can also be made invariant by multiplying $\langle L \rangle$ by $A^{-3w(L)}$. The X(L) polynomial is defined as $X(L) = A^{-3w(L)}\langle L \rangle$.

With a wide array of invariants to chose from it is difficult to pick out which one to use. In this case the bracket polynomial is selected. The reason is twofold; first it is easy to calculate and second, it is good at distinguishing mirror images. The next question is what kind of knots to study?

Chapter 3: Torus Knots and Links

Knots and links that can be entwined around a torus are an interesting class. They

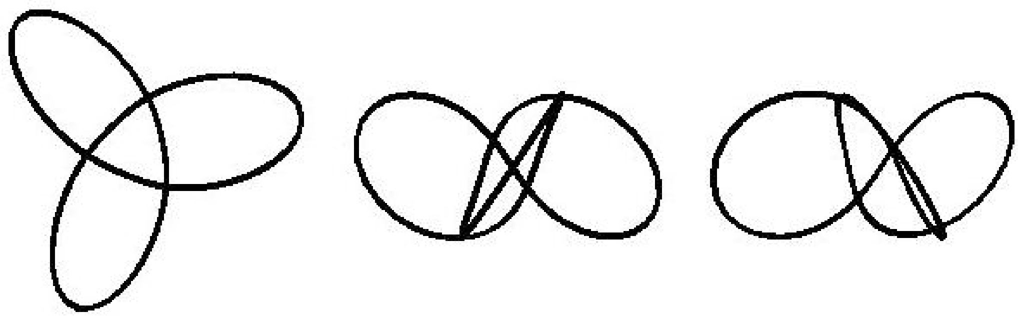


Figure 3.12 Several views of the trefoil. From above (left) and from the side (middle and right). [6]

are immediately intriguing because they seem to follow a pattern. The minimal crossing diagrams are symmetric. This pattern leads one to suspect there is some kind of structure underlying torus knots. Visualizing a torus knot such as the trefoil knot is quite simple (see Figure 1). It can be seen that a torus would fit through the trefoil's loops. Torus knots are a productive collection of knots to study.

3.1 Properties of Torus Links

Figure 3.2 Longitude on the left and the meridian on the right.

Torus knots are identified by the number of times the strand wraps around the torus meridianly and longitudinally. We speak of a (p, q) torus knot where p and q are relatively prime; when p and q are not relatively prime we obtain a link of two or more components. The convention established by Colin Adams (p. 108) will be used to identify torus knots (meridianal wraps, longitudinal wraps). A (3, 2) torus knot, the trefoil, is the same as the (2, 3) knot. It is possible to deform one trefoil into the other.

In this paper we will treat the (n, 2) torus knots (for $n \geq 2$). When n is odd the object is a knot, if n is even it is a two-component link. These knots wrap n times around the meridian and twice longitudinally. These knots were selected because they are the simplest of all torus knots (excluding the unknot). The minimal crossing diagrams are easy to construct thus it is easy to find the bracket polynomial. The fact that they wrap around twice longitudinally will simplify their analysis; we can use some concepts regarding braids to find properties of (n, 2) torus knots.

The use of braids to understand torus knots was selected because it is easier to work with a braid than a knot. A torus knot redrawn in the form of a braid has all the same properties as a torus knot. In this analysis it will be assumed that they are indeed the same.

3.2 Drawing Torus Links

All (n, 2) torus knots are alternating knots. With a little imagination the torus can be deformed to have two long straight sections and U-like sections at the ends. Then all the meridian loops are pushed to one side so that only the two longitudes are left on the other straight section. This will be called the braid representation of the torus knots. Then the longitudinal loops can be cut and the torus knot becomes a braid. It is important to bear in mind that there is a cylinder running down the center of the braid. A (n, 2) torus knot has n crossings caused by the knot wrapping around the meridian. The two strands of the braid have only nontrivial crossings. There is no place in the projection where there would be a use for the second Reidemeister move. Therefore the crossings of such a braid must alternate.

Figure 3.13 A deformed (7,2) torus knot. The left hand side can be thought of as a braid.

Before going any further the concept of an orientation should be established. Putting an arrow on a strand and following it through the entire knot will give the desired orientation. On a link drawn as knotted around a torus putting a point in the center of the projection and going out and putting arrows both in the same direction will achieve the desired orientation. This method must be used consistently. This will be the standard orientation in this paper; other orientations are possible. When working with the braid representation arrows pointing in the same direction can be placed on any two parallel components to give the standard

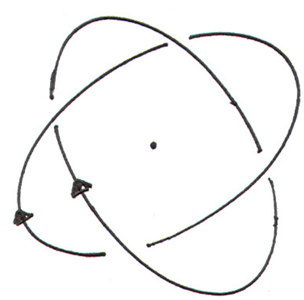


Figure 3.14 Orienting a link.

orientation. When working with just the braid it suffices to put directional arrows pointing down from the top of the braid.

3.3 Writhe of Torus Links

Another important concept to formally define is the writhe of a link. The writhe informally is a measure of the 'twistedness' of the link. In the case of a torus knot it also tells the 'handedness'. A positive writhe means a right-handed knot a negative writhe mean a left-handed knot. The writhe is determined by finding whether a crossing is a postive or negative crossing as determined by the orientation of the knot. A right hand rule is used to establish the sign of a crossing. Point the thumb along the overcrossing and curl the fingers. If the fingers curl in the direction of the undercrossing it is a positive crossing. When the fingers curl against the direction of the undercrossing it is a negative crossing. The sum of the positive and negative crossings is the writhe. The writhe of a link is indicated by w(L).

Figure 3.15 The writhe of the first few torus links. These are left-handed so add up the negative signs.

After examining several (n, 2) knots it seems that they all have a writhe ±n. At an intuitive level this means that all the crossings of a torus knot are the same. Combined with the fact that torus knots are alternating, the concept of the torus knot as a braid, and the orientation of the braid, an argument can be constructed to support this conjecture.

We can consider a torus link to be made up of 'unit braids' (my terminology) that are either left or right-handed (see Firgure 6). We can then connect unit braids together to create a torus link. The braids must be either all left-handed (negative) or right-handed (positive). If this was not the case the opposite braids would create a situation where an R II move would eliminate both crossings. So all crossings are of the same sign then the writhe is the sum of the number of crossings.

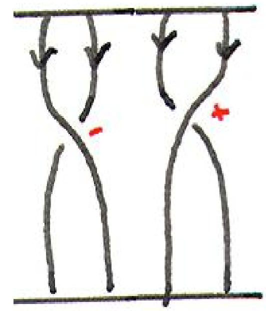


Figure 3. 16 Left and right-handed 'unit braids'.

The ability to calculate the writhe allows the construction of an invariant polynomial from the bracket polynomial. It is also interesting to note the relation that braids have with an (n, 2) torus knot. These facts are useful for extracting information from the projection of a torus knot.

Chapter 4: Computing the Bracket Polynomial of (n,2) Links

The (n, 2) torus knots are simpler to study than most knots,since they are easy to draw and visualize. They also exhibit some symmetry because they wrap around a torus. Intuitively a pattern should appear when a particular invariant is calculated for various torus knots. Once a pattern is found it can then be exploited to reach results that are normally unobtainable.

4.1 Computation of Bracket Polynomial

Calculations can be sped up in a number of ways. First it is helpful to begin with the simplest knots. Then as more complex knots are examined they reduce into a knot whose bracket polynomial has already been found.

Knowing the effect of the R I move allows for the quick calculation of the polynomial of a twisted unknot. This fact makes it easy to deal with links that can be simplified by R I moves. Figure 4.1 represents what would happen when an R I move is performed on a left-handed crossing. The proof for a right-handed crossing is similar.

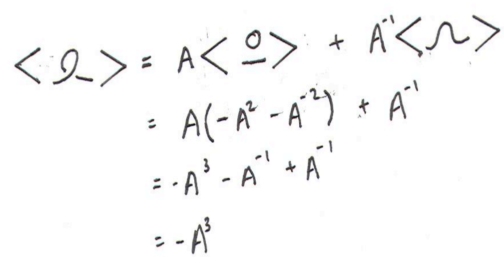


Figure 4.1 A quick method for calculating the contribution to bracket polynomial of a left-handed crossing when a type I move is preformed.

It becomes difficult to draw a projection of a torus knot with seven or more crossings. It is easier to make a line of twists and then connect the ends appropriately. This can be thought of as stretching the torus out and pushing the meridianal twists to one side. This is the connection between knots and braids as well as how (n, 2) knots wraps around the torus.

The bracket polynomial demonstrates that the two mirror images are distinct. In a sense mirror images are opposites of one another. If the braid portion of a right-handed knot is added to the left-handed knot the trivial braid is the result. If we know either the left or right-handed bracket polynomial substituting in A^{-1} for A the other's bracket polynomial will be obtained.

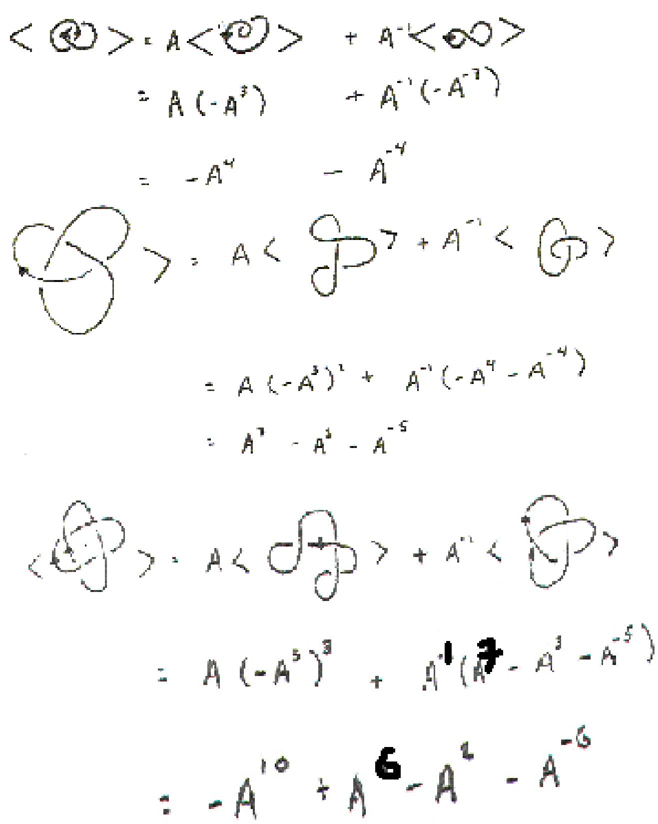


Figure 4.2 Calculating the bracket polynomial for the left handed Hopf Link, Trefoil, and (4,2) link.

Initially the rules depicted in Figure 2.5 are applied then the method shown in Figure 4.1 and previous knowledge is enough to obtain the polynomial of any torus link. The bracket polynomial for the Hopf link is symmetric and requires only one smoothing. It can then be made into the unknot by R I (see Figure 4.1). The left-handed trefoil is smoothed once. One smoothed projection is the Hopf link and the other a twisted unknot. The twisted unknot can be calculated using the R I (see Figure 4.1) and the bracket polynomial of the Hopf Link is complete. Combining these with the rules from section

2.2, the trefoil's polynomial is found. A left-handed (4,2) once smoothed gives a twisted unknot and a left trefoil. Using the methods depicted in Figure 4.1 and the knowledge of the trefoil the bracket polynomial of the (4,2) torus link is known

A few other observations are in order. In the bracket polynomial the difference between one polynomial's power and the next seems to be four in all cases except the terms that come from the polynomial of the Hopf link. The difference between the highest and lowest powers in the case of the Hopf link is eight. It stands to reason that this is what causes mirror images of the (n, 2) torus knots not to be symmetric. If that is the case it explains why the Hopf link is equivalent to its mirror image, but no others are. The polynomial of (n, 2) torus knots has n terms. Also when n is even all the powers in the polynomial are even. When n is odd the powers are all odd. The reason for this can be understood in terms of the recurrence relation.

4.2 Recurrence Relation

Why would we want to find a formula to calculate the bracket polynomial? When calculating the bracket polynomial for the simple knots up to about 7 crossings a pattern emerges. Beyond that point the pictures become difficult to draw. The knot's bracket polynomial reduces to a twisted unknot and the previous (n-1, 2) torus knot. This pattern seems like a 'ladder'; once one polynomial is known we can then construct the next one. The 'ladder' pattern readily suggests that a recurrence relation may be a good way to create a formula for the (n, 2) torus knots.

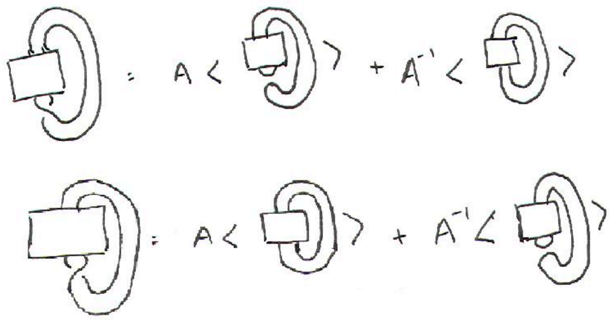


Figure 4.3 Smoothings of braids left-handed (top) and right-handed (bottom)

Figure 4.4 α smoothing (L) and β smoothing (R)

Creating a recurrence relation can be difficult. First a number of bracket polynomials need to be calculated by hand. As noted there is a pattern that emerges, but how exactly this would become a formula is still unknown. Drawing and computing some polynomials is a good place to start to see the pattern. Examine Figure 4.2 to see the pattern. The Hopf link is the easiest to compute of all (n,2) torus links. When finding the recurrence relation it is best to understand in terms of braids. In figure 4.3 the results of a smoothing on a braid are shown. The box represents the alternating twists that are unchanged by the smoothing. An α smoothing reduces the link by one crossing creating the (n-1,2) link. This forms a connection to the next lowest link that has been calculated. A β smoothing cuts the longitude of the torus off and connects the ends together. This creates a knotted tangle with a single strand. The connecting together of the ends necessarily reverses the orientation of the knot. The sign of all the crossings will change. It can then be noted that R I moves are capable reducing this smoothing to the unknot. So the polynomial of this is merely $-A^{\pm 3(n-1).}$ These two components added together gives us the bracket polynomial for the knot. The Hopf link requires slight modification of the use of braids; it is omitted because the bracket polynomial can be obtained quickly from the simple diagram.

Looking at the polynomials from the perspective of braids explains the even odd nature of the powers and the spacing of four between the powers of the polynomial. The Hopf link is anomalous in that it is equivalent to its mirror image. Also the Hopf link has a difference of eight rather than four between its powers. This propagates up into the polynomials for other links ensuring that they are not mirror images.

$$B_2 := -A^4 - \frac{1}{A^4}$$

$$B_3 := A^7 - A^3 - \frac{1}{A^5}$$

$$B_4 := -A^{10} + A^6 - A^2 - \frac{1}{A^6}$$

$$B_5 := A^{13} - A^9 + A^5 - A - \frac{1}{A^7}$$

$$B_6 := -A^{16} + A^{12} - A^8 + A^4 - \frac{1}{A^8} - 1$$

$$B_7 := A^{19} - A^{15} + A^{11} - A^7 + A^3 - \frac{1}{A} - \frac{1}{A^9}$$

$$B_8 := -A^{22} + A^{18} - A^{14} + A^{10} - A^6 + A^2 - \frac{1}{A^2} - \frac{1}{A^{10}}$$

$$B_9 := A^{25} - A^{21} + A^{17} - A^{13} + A^9 - A^5 + A - \frac{1}{A^3} - \frac{1}{A^{11}}$$

Figure 4.5 The bracket polynomials up to the (9,2) torus links. [6]

The recurrence relation is easy to implement. As a base case the Hopf link's polynomial is used. Once we have this initial polynomial we can then obtain all the above polynomials using the recurrence relation. For right-handed knots it is:

$$\langle(n,2)\rangle = A\langle(n-1,2)\rangle + A^{-1}(-A^{-3})^{n-1}$$

For left-handed knots it is:

$$A^{-1}\langle(n-1,2)\rangle + A(-A^{3})^{n-1}$$

Appendix A shows the implementation of the recurrence relation. Appendix B provides the bracket polynomial up to the (20,2) torus link.

Conclusion

Upon closer examination it is discovered that (n,2) torus links have a number of unique features. The fact that a recurrence relation can be found suggests that there may be even more close relations between torus links. The discovery that these links can be thought of as braids and that their writhe is equal to the number of the minimal crossing diagram are interesting discoveries that help in calculating the bracket polynomial. The bracket polynomial is nice in that it can tell the difference between mirror images. Even though the bracket polynomial is only an invariant for framed links it can be used to obtain other polynomials. This fact makes it a useful starting point for the analysis of torus knots. Perhaps there is a more general recurrence relation among (n,m) torus links (where m≥2). Using braids to understand (n,m) torus links would be advantageous because these torus links are hard to draw. It is easy to create a braid for a link with n components. Finding the braid of other knots would be more difficult. Cutting the knot into pieces each a strand in the braid. From this the crossing relationship can be discovered. This would present a greater challenge than finding the braids for (n,2) is a reasonable next step.

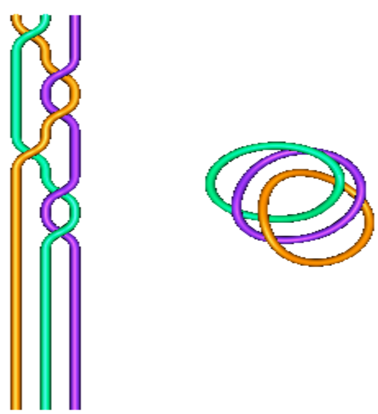


Figure C.17 The braid that forms the Borromean Rings [5].

Appendix A: Maple Code

This Maple worksheet calculates the bracket polynomial for torus knots of the form (N,2) where N is a natural number greater that 3. The polynomial is found using a recurrence relation found in the first for loop. At the bottom thier is a second loop that outputs the span of the bracket polynomial. Note that in some places where the information that would be output in redundant the output is suppressed using the : symbol. Press the !!! button to execute the entire worksheet at once.
This sheet was created using Maple 8 by Paul Corbitt.

```
> restart:
```

N controls the number of bracket polynomials to be generated from 2 up to N

```
> N:=20;
```

$$N := 20$$

Hand decribes the handedness of the knots. Hand >=0 is a right handed knot/link, hand <0 is a left handed knot/link.

```
> Hand:=-2;
```

$$Hand := -2$$

The variable k is takes the handedness of the knot for use in the calculation.

```
> k:=-sign(Hand):
```

<L> is the array from 2 up to N that will hold the bracket polynomial.

```
> <L>:=array[2..N]:
```

<L>[2] is the base step for generating all higher (i,2) knots/links.

```
> <L>[2]:=-A^(4)-A^(-4):
```

The for loop goes through hand calculates either right or left handed bracket polynomials from 3 up to selected N.

```
> for i from 3 to N do
>   <L>[i]:=sort(expand(A^(-k)*<L>[i-1]+A^(k)*(-A^(3*k))^(i-
1)));
> end do:
```

Span[2..N] intializes the array holds the span of the bracket polynomial.

```
> Span:=array[2..N]:
```

Span[i] holds the span of the bracket polynomial B[i].

```
> for i from 2 to N do
>     Span[i]:=degree(B[i])-ldegree(B[i]);
> end do:
```

Appendix B: Bracket Polynomial to N = 20

$$\langle L\rangle_2 := -A^4 - \frac{1}{A^4}$$

$$\langle L\rangle_3 := A^7 - A^3 - \frac{1}{A^5}$$

$$\langle L\rangle_4 := -A^{10} + A^6 - A^2 - \frac{1}{A^6}$$

$$\langle L\rangle_5 := A^{13} - A^9 + A^5 - A - \frac{1}{A^7}$$

$$\langle L\rangle_6 := -A^{16} + A^{12} - A^8 + A^4 - \frac{1}{A^8} - 1$$

$$\langle L\rangle_7 := A^{19} - A^{15} + A^{11} - A^7 + A^3 - \frac{1}{A} - \frac{1}{A^9}$$

$$\langle L\rangle_8 := -A^{22} + A^{18} - A^{14} + A^{10} - A^6 + A^2 - \frac{1}{A^2} - \frac{1}{A^{10}}$$

$$\langle L\rangle_9 := A^{25} - A^{21} + A^{17} - A^{13} + A^9 - A^5 + A - \frac{1}{A^3} - \frac{1}{A^{11}}$$

$$\langle L\rangle_{10} := -A^{28} + A^{24} - A^{20} + A^{16} - A^{12} + A^8 - A^4 - \frac{1}{A^4} - \frac{1}{A^{12}} + 1$$

$$\langle L\rangle_{11} := A^{31} - A^{27} + A^{23} - A^{19} + A^{15} - A^{11} + A^7 - A^3 + \frac{1}{A} - \frac{1}{A^5} - \frac{1}{A^{13}}$$

$$\langle L\rangle_{12} := -A^{34} + A^{30} - A^{26} + A^{22} - A^{18} + A^{14} - A^{10} + A^6 - A^2 + \frac{1}{A^2} - \frac{1}{A^6} - \frac{1}{A^{14}}$$

$$\langle L\rangle_{13} := A^{37} - A^{33} + A^{29} - A^{25} + A^{21} - A^{17} + A^{13} - A^9 + A^5 - A + \frac{1}{A^3} - \frac{1}{A^7} - \frac{1}{A^{15}}$$

$$\langle L\rangle_{14} := -A^{40} + A^{36} - A^{32} + A^{28} - A^{24} + A^{20} - A^{16} + A^{12} - A^8 + A^4 + \frac{1}{A^4} - \frac{1}{A^8} - \frac{1}{A^{16}} - 1$$

$$\langle L\rangle_{15} := A^{43} - A^{39} + A^{35} - A^{31} + A^{27} - A^{23} + A^{19} - A^{15} + A^{11} - A^7 + A^3 - \frac{1}{A} + \frac{1}{A^5} - \frac{1}{A^9} - \frac{1}{A^{17}}$$

$$\langle L\rangle_{16} := -A^{46} + A^{42} - A^{38} + A^{34} - A^{30} + A^{26} - A^{22} + A^{18} - A^{14} + A^{10} - A^{6} + A^{2} - \frac{1}{A^{2}} + \frac{1}{A^{6}} - \frac{1}{A^{10}} - \frac{1}{A^{18}}$$

$$\langle L\rangle_{17} := A^{49} - A^{45} + A^{41} - A^{37} + A^{33} - A^{29} + A^{25} - A^{21} + A^{17} - A^{13} + A^{9} - A^{5} + A - \frac{1}{A^{3}} + \frac{1}{A^{7}} - \frac{1}{A^{11}} - \frac{1}{A^{19}}$$

$$\langle L\rangle_{18} := -A^{52} + A^{48} - A^{44} + A^{40} - A^{36} + A^{32} - A^{28} + A^{24} - A^{20} + A^{16} - A^{12} + A^{8} - A^{4} - \frac{1}{A^{4}} + \frac{1}{A^{8}} - \frac{1}{A^{12}} - \frac{1}{A^{20}} + 1$$

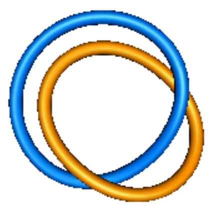
$$\langle L\rangle_{19} := A^{55} - A^{51} + A^{47} - A^{43} + A^{39} - A^{35} + A^{31} - A^{27} + A^{23} - A^{19} + A^{15} - A^{11} + A^{7} - A^{3} + \frac{1}{A} - \frac{1}{A^{5}} + \frac{1}{A^{9}} - \frac{1}{A^{13}} - \frac{1}{A^{21}}$$

$$\langle L\rangle_{20} := -A^{58} + A^{54} - A^{50} + A^{46} - A^{42} + A^{38} - A^{34} + A^{30} - A^{26} + A^{22} - A^{18} + A^{14} - A^{10} + A^{6} - A^{2} + \frac{1}{A^{2}} - \frac{1}{A^{6}} + \frac{1}{A^{10}} - \frac{1}{A^{14}} - \frac{1}{A^{22}}$$

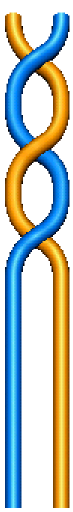
Appendix C: Braids and (n,2) Torus Links

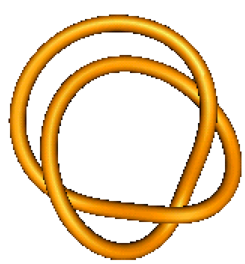
An easy way to imagine torus links is the idea of a braid. This discussion will focus on (n, 2) torus knots. First create a braid with 2 strands then alternate over and under crossings. Then connect the right top to the right bottom and the left top to the left bottom. A (n, 2) torus knot is created. The resulting diagrams provide insight into the (n,2) torus knot. It can be noted that the n twists represent the number of times wrapped around the meridian. The two strands that turn the braid into a knot represent the link wrapping twice longitudinally around the torus. The following diagrams (using [5]) are examples of the simplest (n,2) torus knots.

The braid: The torus knot:



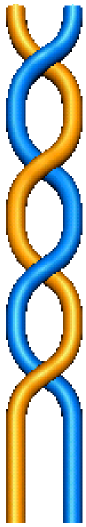
The Hopf Link [5]



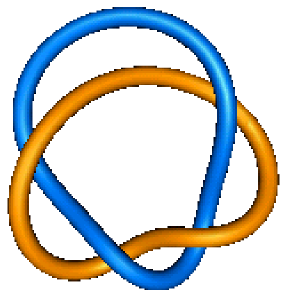


Trefoil Knot [5]

The braid:



The torus knot:

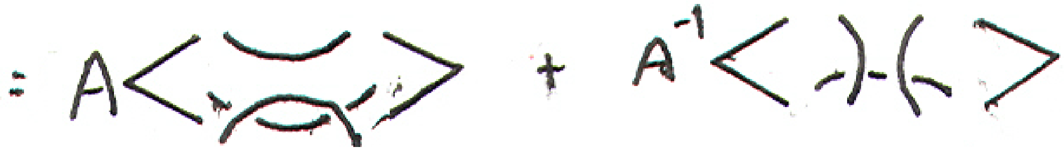


King Solomon's 'Knot' [5]

Appendix D: Proof of Invariance of the Kauffman Bracket under R III

First the crossing of the two straight lines is smoothed.

Then the diagram on the right is modified using the second Reidemeister move. A planar isotopy is then applied to the second diagram.



The digrams can be ‘unsmoothed’ and the proof is complete.

Bibliography

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O'Connor ,JJ and EF Robertson. Topology and Scottish Mathematical Physics. Jan 2001 <http://www-gap.dcs.st-and.ac.uk/~history/HistTopics/Knots_and_physics.html>

Sossinsky, Alexi. Knots Mathematics With a Twist. Trans. Giselle Weiss. Cambridge: Harvard UP, 2002.

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[1] 'Torus zoo' from:
http://www.cs.ubc.ca/nest/imager/contributions/scharein/knot-theory/torus-zoo-xing.jpg

[2] Knotted DNA from:
math.rice.edu/~amheap/ interests.html

[3] A wild knot from:
http://www.cs.ubc.ca/nest/imager/contributions/scharein/knot-theory/wild.pdf

[4] Picture adapted from The Knot Book on page 28.

[5] From and created using KnotPlot software and can be found at:
http://www.cs.ubc.ca/nest/imager/contributions/scharein/KnotPlot.html

[6] Created using Maple 8 software.

All other figures were drawn by the author.