THE ALEXANDER POLYNOMIAL, COLORING, AND DETERMINANTS OF KNOTS

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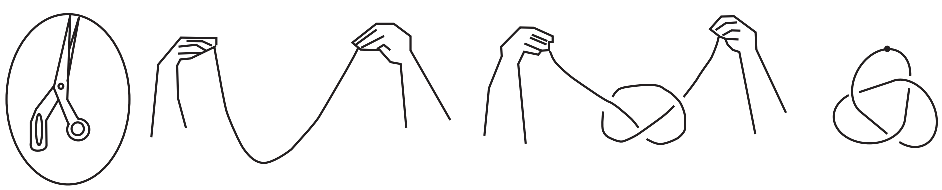
Abstract. We give an algorithm to calculate the Alexander polynomial $\Delta_K(t)$ of a knot K. We focus on the polynomials of pretzel knots and their connection to the concept of colorability and determinants of knots. In addition, we explore a different, easier approach (due to Conway) to computing the Alexander polynomial of knots and links. To conclude, we illustrate that it is possible to compute $\Delta_K(t)$ for non-alternating knots by the Conway polynomial approach.

1. Introduction

Take a piece of string. Tie a knot in it. Now glue the two ends of the string together to form a knotted loop. The result is a string that has no loose ends and that is truly knotted. Unless we use scissors, there is no way that we can untangle this string. (Figure from [A].)



A knot is just a loop of string, except that we think of the string as having no thickness, its cross-section being a single point. The knot is then a closed curve in space that does not intersect itself anywhere.

2. The Alexander Polynomial

The first polynomial associated with knots and links was due to J. Alexander in about 1928. This polynomial invariant was very good at distinguishing between knots and links, and mathematicians utilized

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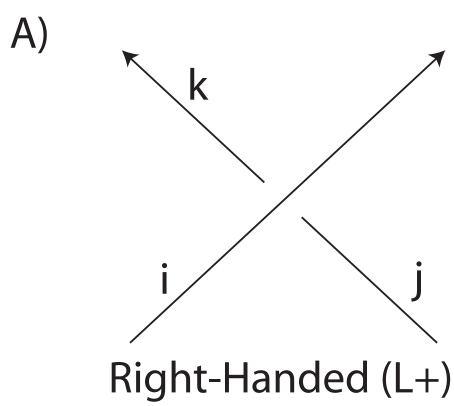
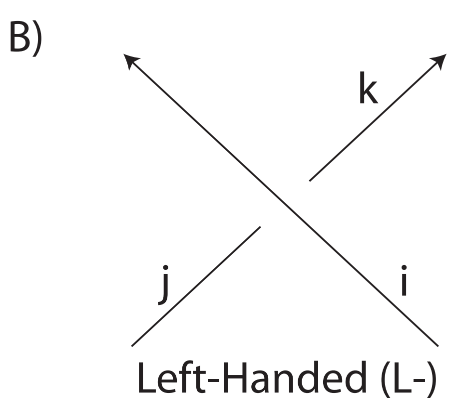
the Alexander polynomial to distinguish knots and links for the next 58 years.

An approach to finding the coloring number of a knot is via the Alexander polynomial, $\Delta_K(t)$, a polynomial in t. A knot can be n-colored iff n divides the determinant $|\Delta_K(-1)|$. Here is an algorithm, (taken from [L]) to compute the Alexander Polynomial $\Delta_K(t)$ of a knot,

1. Give your knot any orientation desired.
2. Number the arcs of the diagram and separately number the crossings.
3. Define an $(n \times n)$ matrix, where n = number of crossings (and arcs) according to the following procedure:
   (a) If the crossing x is right handed (see Figure A), with arc i passing over arcs j and k, enter a $1 - t$ in column i of row x, enter -1 in column j of that row, enter a t in column k of that row.

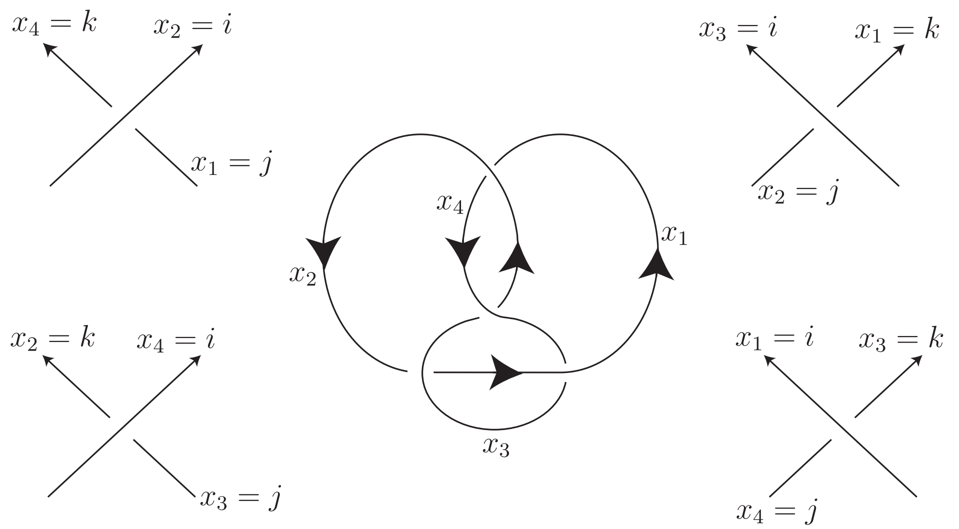


   (b) If the crossing is left handed (see Figure B), enter a $1 - t$ in column i of row x, enter a t in column j, enter a -1 in column k of the row, then all the remaining entries of x are 0.



Deleting any one of these columns and any one row yields an $(n-1) \times (n-1)$ matrix. The $(n-1) \times (n-1)$ matrix is the Alexander matrix of K. The determinant of the Alexander Matrix is called the Alexander polynomial of K.

Here is an example: the Alexander polynomial of the Figure 8 knot.



$$\begin{bmatrix} -1 & 1-t & 0 & t \\ 0 & t & -1 & 1-t \\ t & -1 & 1-t & 0 \\ 1-t & 0 & -1 & t \end{bmatrix} \Longrightarrow \begin{vmatrix} -1 & 1-t & 0 \\ 0 & t & -1 \\ t & -1 & 1-t \end{vmatrix} \Longrightarrow 1-2t+2t^2$$

$$|\Delta_K(-1)| = 5$$

3. Trees

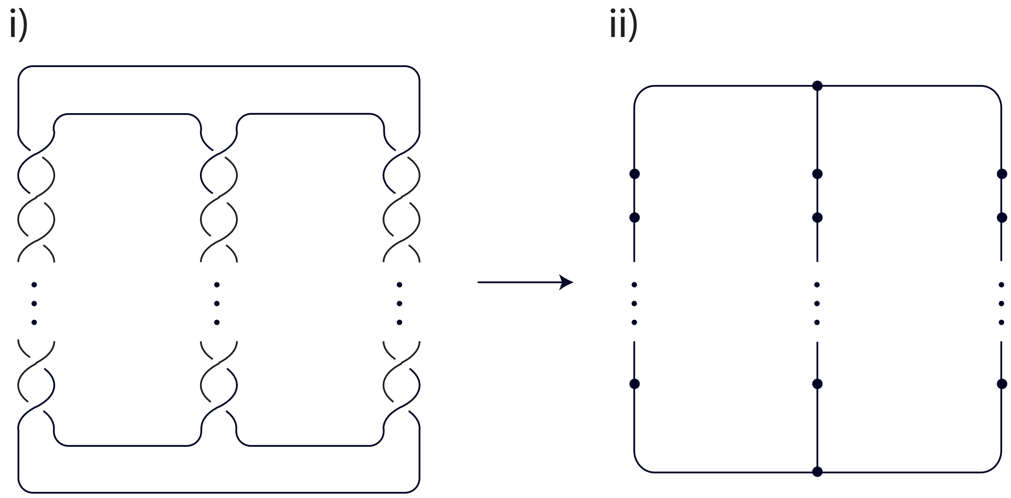
Another way of finding the determinant is using trees, which are special kinds of graphs. A graph consists of a set of points called vertices and a set of edges that connect them. A tree is a graph that is connected and has no closed cycles.

From a projection of an alternating knot or link, we create a corresponding planar graph in the following way.

1. Shade the diagram in a checkerboard pattern.
2. Put a vertex at the center of each shaded region.
3. Connect the vertices with edges that pass through a crossing.

Note: the determinant of an alternating knot is the number of maximal trees in its planar graph.

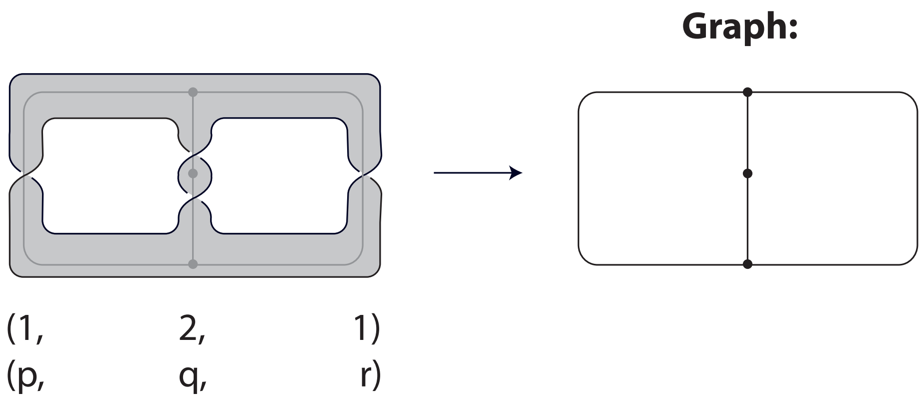
A knot in the form shown in the Figure i) below is called a (p, q, r) pretzel knot. The integers p, q, and r refer to the number of crossings in the three tangles at left, center and right. A (p, q, r) pretzel knot is alternating iff p, q, and r all have the same sign.



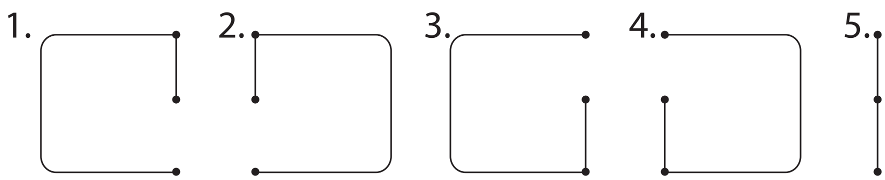
Theorem 1. *For alternating (p, q, r) pretzel knots, $pq + pr + qr$ gives the number of trees.*

Proof: A (p, q, r) pretzel is alternating if and only if p, q, and r all have the same sign. Lets assume they are all positive. (If they are all negative, the argument is similar.) So p, q, and r represent the number of crossings in the three parts of the pretzel. We are interested in creating a planar graph from which we can derive its trees. If we follow the procedure given above for finding trees, we can obtain the planar graph shown in Figure ii). This graph has p edges at left, q edges down the center, and r at right. Deleting any two edges, one each from the left and center of the graph will leave a tree as there will remain no closed cycles. There are pq ways to choose the two edges and pq trees. By following the same procedure, we can obtain pr trees by removing edges at left and at right and, finally, qr trees by removing edges at center and at right. Hence, $pq + pr + qr =$ total number of maximal trees for alternating pretzel knots. $\square$

Example: Another representation of the figure 8 knot is as the $(1, 2, 1)$ pretzel knot. Given this, can you find its trees?



$$pq + qr + qr = (1)(2) + (1)(1) + (2)(1) = 5 \text{ Trees (see below)}$$

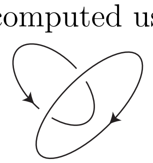


Since the number of maximal trees is five, that is also the determinant. This agrees with the calculation made using the Alexander polynomial $\Delta_K(t)$.

4. The Conway Polynomial

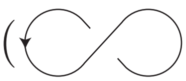
As we saw above, the very first polynomial for knots was the Alexander polynomial, invented back in 1928. It is a polynomial for oriented links, and we described it in terms of a variable t. At the time of its invention, it was defined in terms of relatively abstract mathematical concepts. It wasn't until 1969, that John Conway found a way to calculate the Alexander polynomial of a link using a so-called skein relation, which is an equation that relates the polynomial of a link to the polynomial of links obtained by changing the crossings in a projection of the original link. Conway showed that the Alexander polynomial $\Delta_K(t)$ can be computed using just three rules.

Rule 1:

$\Delta($  $) = 0$

This rule says that the polynomial of the unlink is equal to 0.

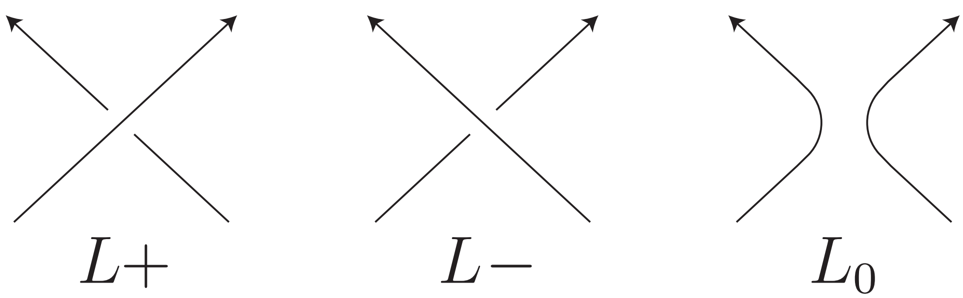
Rule 2:

$\Delta($  $) = 1$

This rule is the usual one, namely that the trivial knot has a polynomial equal to 1. This holds true for any projection of the trivial knot, not just the usual one.
Rule 3:

$$\Delta(L_+) = \Delta(L_-) - z\Delta(L_0)$$

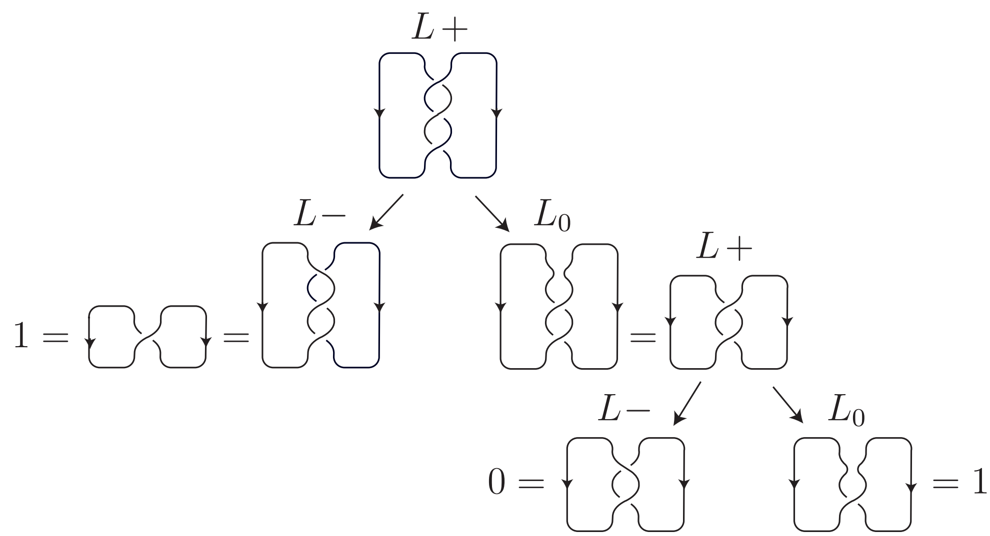
This is the skein relation. We take three projections of links L_+, L_-, and L_0 such that they are identical except in the region depicted below.



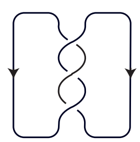
These rules are enough to ensure that the Alexander polynomial is an invariant for knots and links [A]. Given a particular projection, we could choose a crossing such that it is one of the crossings that we would like to change in order to turn the projection into a trivial projection. This process of repeatedly choosing a crossing, and then applying the skein relation to obtain two simpler links, yields a tree of links called the **resolving tree**. Note that the Conway polynomial is a polynomial in z, where $z = (t^{1/2} - t^{-1/2})$.

Example: Compute the Conway polynomial of the trefoil knot, using the rules above starting with a resolving tree.

We start by letting the original projection correspond to L_+, so using the skein relation equation, $\Delta(L_+) = \Delta(L_-) - z\Delta(L_0)$ the resolving tree is below.



So the trefoil Conway polynomial is

$\Delta($  $) = 1 - z(0 - z) = 1 - z(-z) = 1 + z^2$

5. The Alexander polynomial for Non-alternating Knots

We want to investigate whether or not the skein relation allows us to calculate the Alexander polynomial for non-alternating pretzel knots. It was always clear that the application of the skein relation to a crossing would eventually lead to a set of trivial links, for which we could calculate the polynomials. In the case of non-alternating pretzel knots, it is less clear, but is also possible.

We began by creating a table of Conway polynomials (see below) for $(2, n)$ torus knots and links. When the number of crossings n is odd, we get a knot, while an even number of crossings gives a link of two components. For knots, the choice of orientation will not change the answer. For links orientation is very important. We will limit ourselves to torus knots up to 5 crossings.

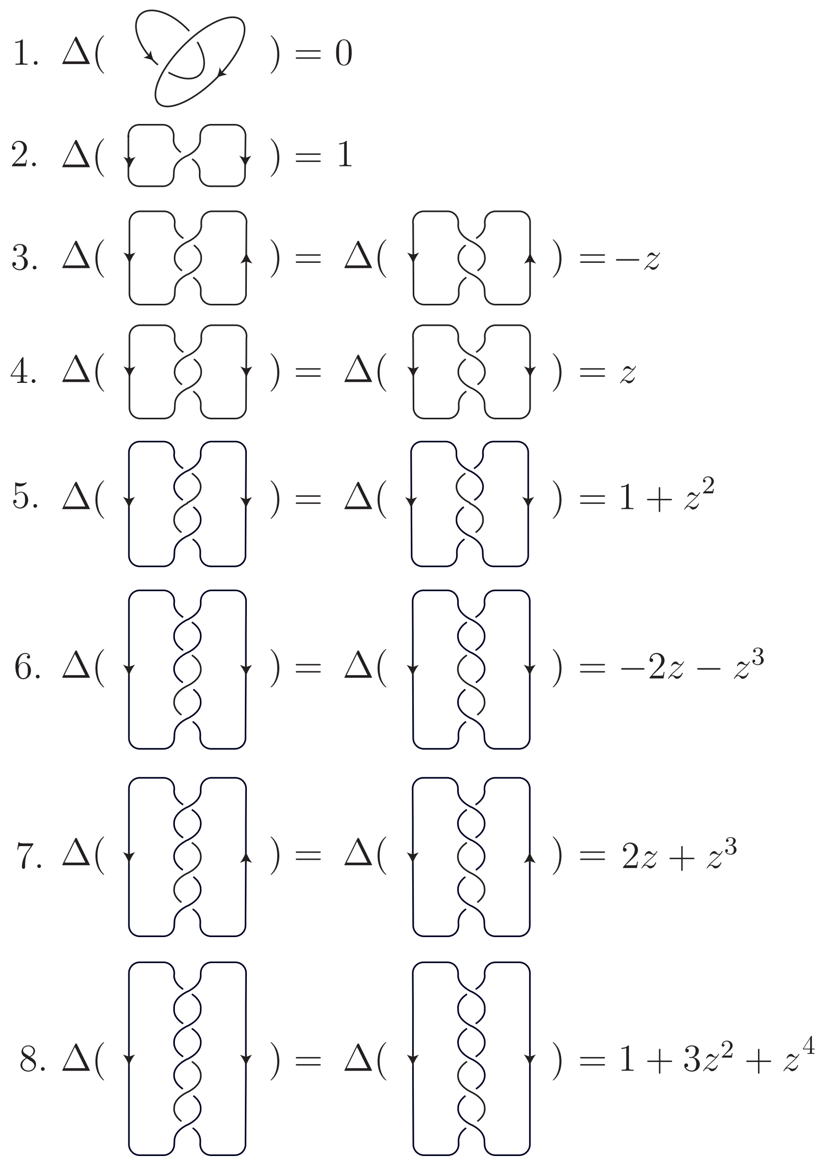


FIGURE 1. Table of Conway polynomials

Here is a corollary to compute non-alternating pretzel knots;

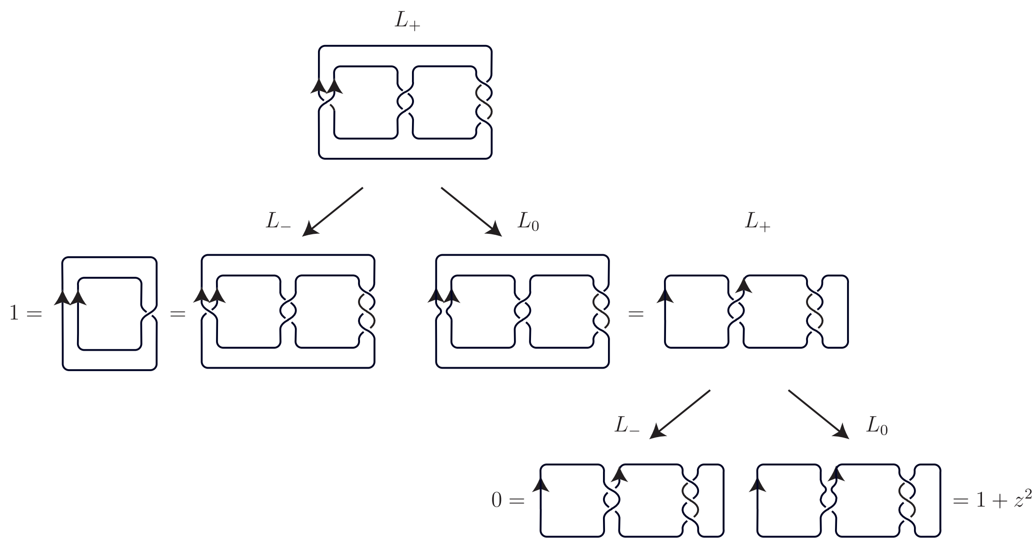
Corollary 2. $\Delta(p, q, r) = \Delta(p - 2, q, r) - z\Delta(p - 1, q, r)$

Example: Compute the Alexander polynomial of the $(1, 2, 3)$ pretzel knot.

Observe,

$$\begin{aligned}\Delta(1,2,3) &= \Delta(1-2,2,3) - z\Delta(1-1,2,3)\\ &= \Delta(-1,2,3) - z\Delta(0,2,3)\end{aligned}$$

Using a resolving tree and skein relation,



$$\begin{aligned}\Delta(1,2,3) &= 1 - z(-z - z^3)\\ &= 1 + z^2 + z^4 \text{ (Conway polynomial)}\\ &= 1 + (t^{1/2} - t^{-1/2})^2 + (t^{1/2} - t^{1/2})^4\\ &= t^2 - 3t - 3t^{-1} + t^{-2} + 1\\ &=\cdot\ t^2(t^2 - 3t - 3t^{-1} + t^{-2} + 1\\ &= t^4 - 3t^3 + 3t^2 - 3t + 1 \text{ (Alexander polynomial)}\end{aligned}$$

References

[A] Adams, Colin. "The Knot book," W.H. Freeman and Company, New York, 1994

[L] Livingston, Charles,"Knot Theory," Mathematical Association of America, Washington DC, 1993.

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