USING LINK INVARIANTS TO DETERMINE SHAPES FOR LINKS

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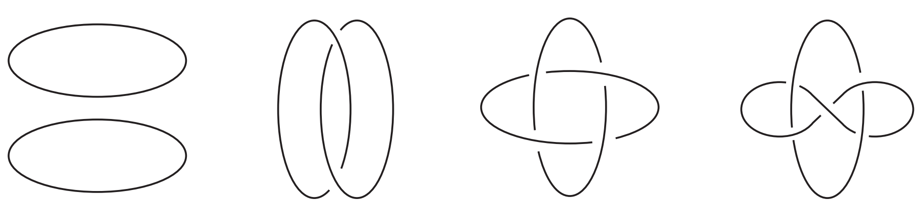
Abstract. In the tables of two component links up to nine crossing there are 92 prime links. These different links take a variety of forms and, inspired by a proof that Borromean circles are impossible, the questions are raised: Is there a possibility for the components of links to be geometric shapes? How can we determine if a link can be formed by a shape? Is there a link invariant we can use for this determination? These questions are answered with proofs along with a tabulation of the link invariants; Conway polynomial, linking number, and enhanced linking number, in the following report on "Using Link Invariants to Determine Shapes for Links".

1. An Introduction to Links

By definition, a **link** is a set of knotted loops all tangled together. Two links are **equivalent** if we can deform the one link to the other link without ever having any one of the loops intersect itself or any of the other loops in the process [1].We tabulate links by using **projections** that minimize the number of crossings. Some basic links are shown below.



unlink   Hopf link   Solomon's link   Whitehead link

Notice how each of these links has two loops, or components. Although links can have any finite number of components, we will focus

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on links of two and three components. A famous 3-link is the Borromean Rings (fig. 1). This link has the Brunnian property: if any one component is removed, the entire link falls apart.

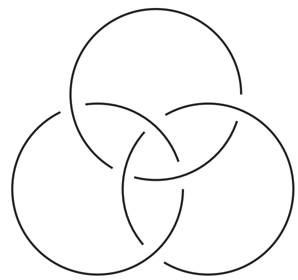


FIGURE 1. The Borromean Rings

The **linking number** measures how linked up two components are. We say that the linking number is a link invariant because it remains unchanged no matter what projection is representing the link. We can compute the linking number using the following algorithm[1]:

1. Orient the two components of the link. We do this by choosing a direction to travel around each component.
2. For each crossing between the two components determine a crossing sign of "+1" or "-1" (fig. 2). This is decided by rotating the understrand so that the arrows match. If the understrand is rotated clockwise, the crossing is a "+1". Similarly, if the understrand is rotated counterclockwise, the crossing is a "-1". *Note: Use only crossings between the two components, not self crossings of one component with itself.*
3. Sum the crossing signs and take the absolute value of this result. Then divide this number by 2. We now have the linking number!

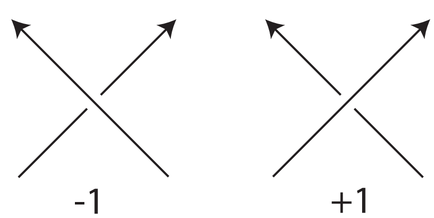
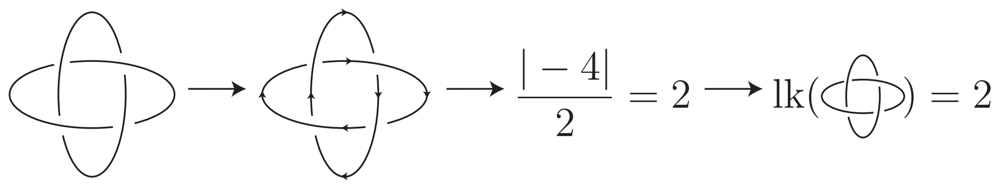


FIGURE 2. Crossing Signs



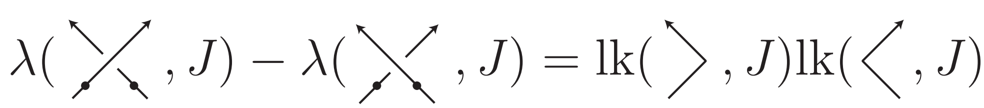
Solomon's link

Figure 3. Determining the Linking Number

2. Enhanced Linking Number - A New Invariant

Traditionally, knot theory has focused on the study of knot or link invariants. In most cases, a knot or link invariant is simply a function that assigns to each knot or link an integer. Recently there has been a shift in knot theory from studying knot invariants themselves to examining how the invariants change as the knot or link is changed.[1]

In 2003, C. Livingston introduced a new invariant called the enhanced linking number [6] in an article called "Enhanced Linking Numbers", although the idea behind this new invariant was first seen in an article by Livingston and Paul Kirk called "Vassiliev invariants of two component links and the Casson-Walker invariant". The enhanced linking number is based on a **skein relation**. This relation was contributed in 1969 by John Conway as a way to calculate the Alexander polynomial. This is an equation that relates the polynomial of a link to the polynomial of links obtained by changing the crossings in a projection of the original link [1]. The notion of the enhanced linking number, λ, is based on a crossing change formula for λ as follows:



The advantage to using this idea of a crossing change formula for enhanced linking numbers is that to apply λ to both computational and theoretical problems in link theory, one only needs to know the crossing change formula satisfied by λ – its precise value need not be known.[6]

In order to understand this notation, some explanations followed by an example would be helpful. The first notation to introduce is the idea of left-handed crossings and right-handed crossings (fig. 4). Notice that in the left-handed crossing, in order for the arrows to align, the

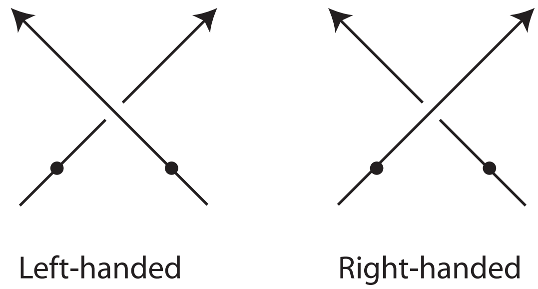


Figure 4. Left- and right-handed crossings

understrand must be rotated counterclockwise. Similarly in the right-handed crossing, the understrand must be rotated clockwise. Also note the single dot toward the bottom of each arrow. This represents the idea that the crossing is a "self-crossing" on one component.

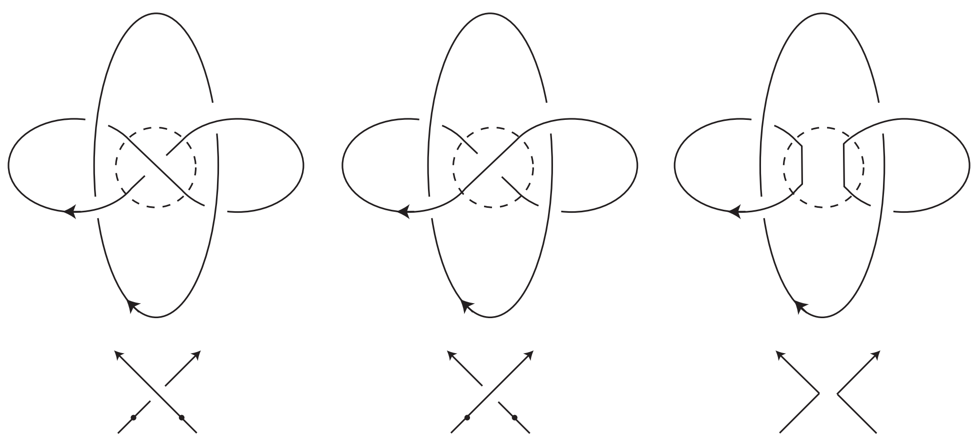
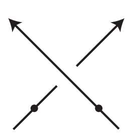
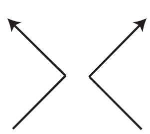


Figure 5. Notation Explanation (Whitehead Link)

The process of replacing



, with



is sometimes called smoothing the link at the crossing. With the notation explained, an illustration of the crossing change formula for λ is helpful.

Notice, the second diagram (see fig. 6) in this difference is the unlink. Hence, since the value of λ on the unlink and the Whitehead link are different, the Whitehead link can not be deformed into the unlink.

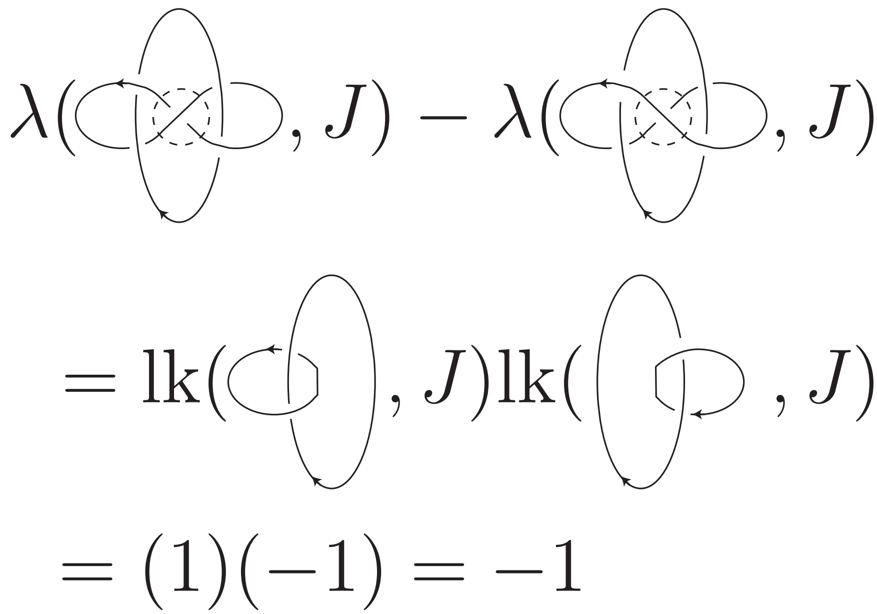


FIGURE 6. Crossing change formula for λ

2.1. Defining λ using the Conway Polynomial.

The definition of λ depends on the use of knot polynomials, in particular, the Conway polynomial. The definition of λ for two-component links is defined in terms of the Conway polynomial, so in order to understand the definition, we discuss basic properties of the Conway polynomial.

The individual coefficients of the Conway polynomial are link invariants in their own right. These coefficients, viewed as invariants $c_i(L)$ $(i \geq 0)$ for a link L, are defined by the expansion:

$$C(L)(z) = \sum_i c_i(L) z^i$$

Notice that smoothing a crossing changes the number of components of a link by one and that multiplication by z switches odd and even polynomials. It follows from induction that a link with an even number of components has an odd Conway polynomial and vice versa.

In order to tabulate the link invariants, the linking number and the enhanced linking number, we will use the following theorem and definition from [6].

Theorem 1 (Theorem 8.4 of [6]). *For a two-component link L, $c_1(L) = lk(L)$.*

Definition 2 (Definition 8.6 of [6]). *For a two-component link $L = (K, J)$, define $\lambda(L) = c_3(L) - c_1(L)(c_2(K) + c_2(J))$*

3. Guide to Tabulation of Links and Link Invariants

The table of link invariants at the end of this paper shows the Conway polynomial, linking number, and enhanced linking number for all prime links up to nine crossings. It also shows these invariants for the mirror images of these links, which is indicated by an asterisk. Those links without mirror images are determined to be **amphicheiral**, i.e., that the mirror image of the link can be deformed to equal the link. This table uses Conway polynomials and oriented projections of links from Doll and Hostes, "A Tabulation of Oriented Links" [7].

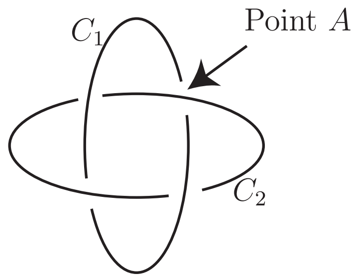
4. Using Shapes for Two Component Links

In 1991, Lindstrom and Zetterstrom proved that Borromean Circles are impossible [2]. This raises the questions, "Are circles possible for different links?"; "What shapes are possible for different links?" and "Is there an invariant we can use to determine which shapes are possible for each link?"

Understanding circles for links with two components is not difficult. We know that circles in three space project into a plane in three ways[3]; as a line, a circle, or an ellipse. This means that two linked circles project into a combination of a pair of these three possibilities. Notice that the maximum intersections (in the plane) of two unique lines is one point; the maximum intersections of a line and a circle, a line and an ellipse, or two circles is two; and the maximum intersections of a circle and an ellipse or two ellipses is four. This means two linked circles can be projected into two space with a maximum of four crossings. In the tables, there are only three links with four or fewer crossings; the unlink, the Hopf link, and Solomon's link. In addition to these prime links found in the tables, there are a few non-prime links with four or fewer crossings. Since we know that these non-prime links must contain at least one component that is a knot, and the unknot is the only knot that can be contained in a plane ([4] Corollary 4A), these non-prime links of four or fewer crossings can not be formed with circles or any other planar geometric shape. Looking at the unlink and Hopf link it is apparent that circles can be used. Solomon's link, on the other hand, is a different story.

Theorem 3. *Solomon's link can not be made with two circles.*

Proof: (By contradiction) Suppose we have circles C_1 and C_2 forming Solomon's link. Since C_1 is a circle, it lies in a plane. We now have



C_2 above the plane starting at point A and following C_2 in a clockwise manner C_2 goes below the plane, then above the plane again, then below the plane again, and back to point A above the plane. This means that circle C_2 meets the plane at least 4 times, which is impossible since the circle does not belong to that plane. Therefore Solomon's link can not be made with two circles. □

Notice that the basis of this proof is the idea that C_2 can not go above, then below, then above, then below the plane. This gives us the idea of alternating crossings. We will say that C_2 has four alternating crossings with C_1. The argument above also applies to any planar shape with the property that it crosses no plane beside its own 4 times. We will call such a shape a convex figure. For example, convex polygons, ellipses, and circles are all convex figures since they can intersect a (distinct) plane in at most 2 points.

Lemma 4. *If every projection of a link has 4 alternating crossings, then it is impossible to form that link with convex figures.*

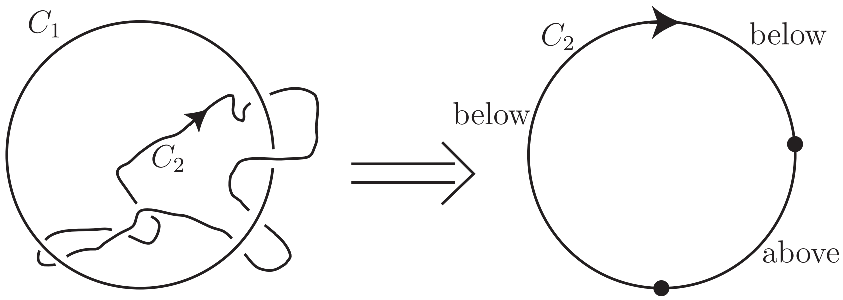
A this point it would be convenient to be able to distinguish whether or not it is possible to form certain links with convex figures using a link invariant, and in fact, we can.

Theorem 5. *If a link has two components whose linking number is greater than or equal to 2, then convex figures can not be used to form the link.*

Proof: By the lemma above, we need to show that if two components C_1, C_2 have a linking number, $\mathrm{lk}(C_1 \cup C_2)$, of 2 or greater, then every projection of the link has 4 alternating crossings. We are going to prove this using contradiction.

Assume the $\mathrm{lk}(C_1 \cup C_2) \geq 2$ and the link has a projection with less than 4 alternating crossings. Since the link's projection has less than 4 alternating crossings, it must have either 2 or 0 alternating crossings. An odd number of alternate crossings can not occur:

Assume the projection has 3 alternating crossings, C_1 is a fixed circle, and C_2 is an intertwining component producing 3 alternating crossings.(i.e., C_2 is split into 3 pieces by C_1's plane.)



Notice that if you start at the arrow on C_2 , we get the three alternating crossings, below-above-below. But the component, C_2, has to start either above or below and finish back at the same result, either above or below. This means C_2, only has 2 alternating pieces, below-above. In the case of 0 alternate crossings, C_2 is completely above or below C_1 and the linking number is zero.

Assume then that the projection has 2 alternating crossings. In this case, C_2 is above then below. Schematically, we will represent C_2 as being above C_1 in the right semicircle and below C_1 at the left. There are four cases to consider depending on whether the top and bottom points, A and B, are inside or outside C_1 .

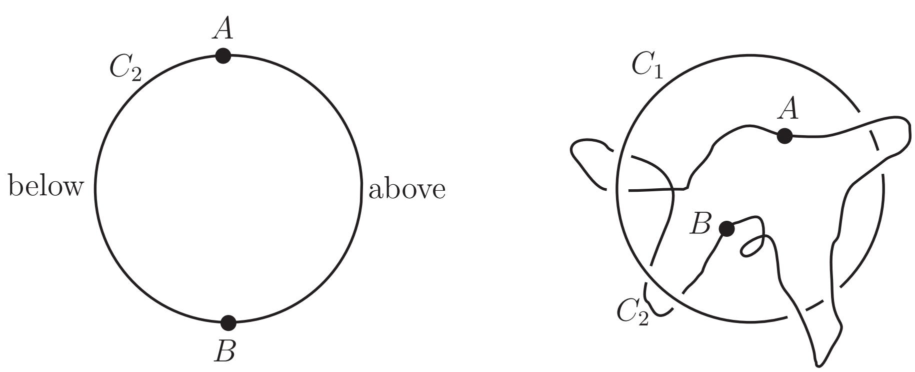


Figure 7. Two alternating crossings

If both are inside, then traveling from A to B on right semicircle of C_2, one crosses over C_1 an even number of times, half going out of C_1 , half coming back in. Outward and inward crossings have different signs, so the sum of these crossings is zero. Similarly, the sum of the crossing signs on the left semicircle of C_2 is 0. So, the linking number is 0. If A and B are both outside, again the linking number is 0. And if one of A or B is inside the circle and the other is out, then the linking number is 1.

Notice that in all cases, the linking number is either 0 or 1. But this is a contradiction. Therefore, if a link has $\mathrm{lk}(C_1 \cup C_2) \geq 2$, then the link can not be made with convex figures. $\square$

Corollary 6. *Links* 4^2_1, 6^2_1, 6^2_2, 6^2_3, 7^2_5, 7^2_7, 8^2_1, 8^2_2, 8^2_3, 8^2_4, 8^2_5, 8^2_6, 8^2_9, 8^2_{11}, 8^2_{14}, 8^2_{16}, 9^2_1, 9^2_2, 9^2_6, 9^2_7, 9^2_{14}, 9^2_{16}, 9^2_{17}, 9^2_{20}, 9^2_{22}, 9^2_{23}, 9^2_{24}, 9^2_{26}, 9^2_{28}, 9^2_{29}, 9^2_{30}, 9^2_{38}, 9^2_{40}, 9^2_{43}, 9^2_{45}, 9^2_{48}, 9^2_{49}, 9^2_{51}, 9^2_{53}, 9^2_{57}, 9^2_{58}, 9^2_{59}, 9^2_{60}, 9^2_{61} *can not be formed by convex figures.*

Using Rolfsen's Corollary, links 7^2_4, 7^2_8, 8^2_{10}, 8^2_{12}, 8^2_{15}, 9^2_{13}, 9^2_{15}, 9^2_{18}, 9^2_{19}, 9^2_{21}, 9^2_{25}, 9^2_{27}, 9^2_{31}, 9^2_{33}, 9^2_{36}, 9^2_{39}, 9^2_{44}, 9^2_{46}, 9^2_{47}, 9^2_{50}, 9^2_{52}, 9^2_{55}, 9^2_{56} also can not be formed by planar figures. Thus we've eliminated all but 23 links.

5. Using Shapes for Three Component Links

As previously mentioned, in 1991, Lindstrom and Zetterstrom proved that Borromean circles are impossible[2]. This leads to the idea of using different shapes for the Borromean rings.

Theorem 7. *Equilateral Borromean Triangles are possible.*

Proof: Consider ΔABC in a three dimensional coordinate system with vertices $A(0,0,0)$, $B(5,0,0)$, and $C(2.5,4.33,0)$; ΔDEF with vertices $D(1.25,-.72,.5)$, $E(3.75,3.61,.5)$, and $F(2.4,1.44,-3.83)$; and ΔGHI with vertices $G(2.5,1.44,1)$, $H(2.5,4.33,-2.54)$, and $I(5,0,-2.54)$. Note: We are using approximate values for simpler calculations. We want to show that line segments AB and BC intersect the plane DEF within ΔDEF, line segments DE and EF intersect the plane GHI within ΔGHI, and line segments GH and GI intersect the plane ABC within ΔABC. This will show that no pair of triangles is linked, but the three are linked (the definition of Borromean rings).

The intersection points are as follows:
segment $AB \cap \Delta DEF = (1.87,0,0)$
segment $BC \cap \Delta DEF = (3.44,2.7,0)$
segment $DE \cap \Delta GHI = (2.62,1.65,0.5)$
segment $EF \cap \Delta GHI = (2.99,1.65,0.5)$
segment $GH \cap \Delta ABC = (2.5,2.26,0)$
segment $GI \cap \Delta ABC = (3.21,1.03,0)$

We can now verify that the points of intersection are contained within the triangles by projecting each triangle into a plane and showing the points lie within the triangles. We will project ΔABC and ΔGHI into the xy plane and ΔDEF into the yz plane.

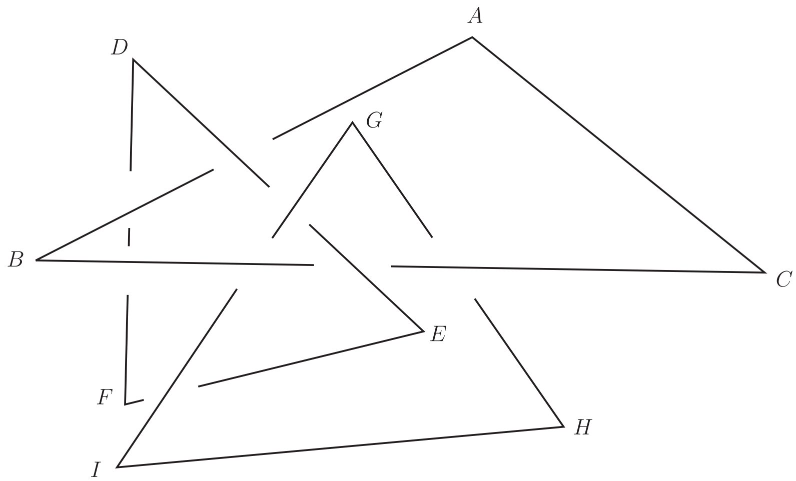
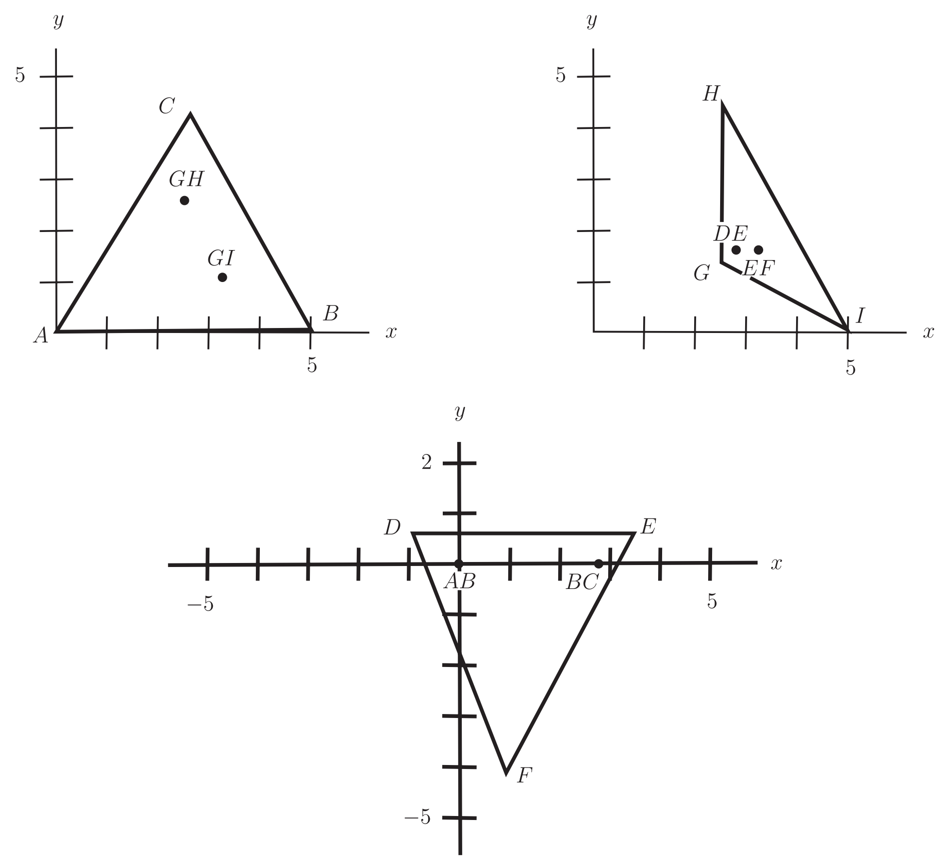


Figure 8. Borromean Triangles



Therefore, Equilateral Borromean Triangles are possible. □

Now, looking at constructing 6 crossing three links with circles, we know that link 6^3_2 is impossible (Borromean rings) and that link 6^3_3 is possible, as is easily verified with a physical model.

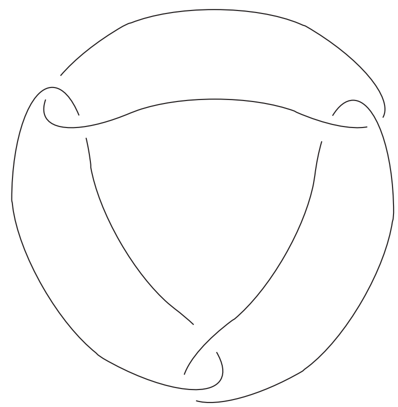


FIGURE 9. Link 6^3_3

But what about link 6^3_1?

Theorem 8. *The link 6^3_1 can not be constructed with circles.*

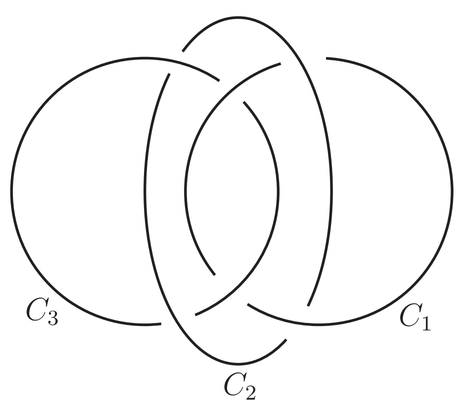


FIGURE 10. Link 6^3_1

Proof: (contradiction) Assume link 6^3_1 can be constructed with circles. Rotate C_3 lifting its top out of the page and dropping its bottom, so that it will meet C_1 in two points in its new position C_3'. Observe that C_3' will not meet C_2 since we twist it away from C_2. There are now two cases: either C_1 and C_3' are in the same plane or they are not. We now basically follow the same proof Lindstrom and Zetterstrom used to prove Borromean circles are impossible[2].

(Case 1) If they are in the same plane, we follow C_2 in a clockwise direction, starting at "12 o'clock". We are first above the plane, then below, then above, then below, then back to the start. This means that C_2 meets the plane at least 4 times, which is impossible.

(Case 2) If C_1 and C_3' do not lie in a plane, then we want to show they will lie in a sphere. Let P, Q be the points in which C_1 and C_3' meet. Let R be the midpoint of segment PQ. Let S_1 and S_2 be the centers of the circles C_1 and C_3', respectively. The plane containing R, S_1, and S_2 is orthogonal to the line PQ. Let l_i be the normal to the plane of C_i through S_i $(i = 1, 2)$. The line l_i is orthogonal to the line PQ and contains the point S_i in the plane. This implies that l_i lies in the plane. Since l_1 and l_2 are not parallel lines they will meet in a point T. The distance between T and $C_1 \cap C_3'$ is a constant r. Therefore, C_1 and C_3' belong to a sphere with center T and radius r. We now have C_2 going in and out of this sphere 4 times, which is impossible. Therefore the link 6_1^3 is impossible to form with circles. $\square$

In an article by Cromwell, Beltrami, and Rampichini [5], they give ten interlaced patterns that can be formed by three circles. Figure 11 shows these ten geometrically distinct patterns that can be derived by choosing the crossings in different ways. In this case, the patterns are considered up to symmetry: 3-fold rotation, reflection, and reflection in the plane of the pattern [5]. This last symmetry operation means that the sense of all the crossings is switched.

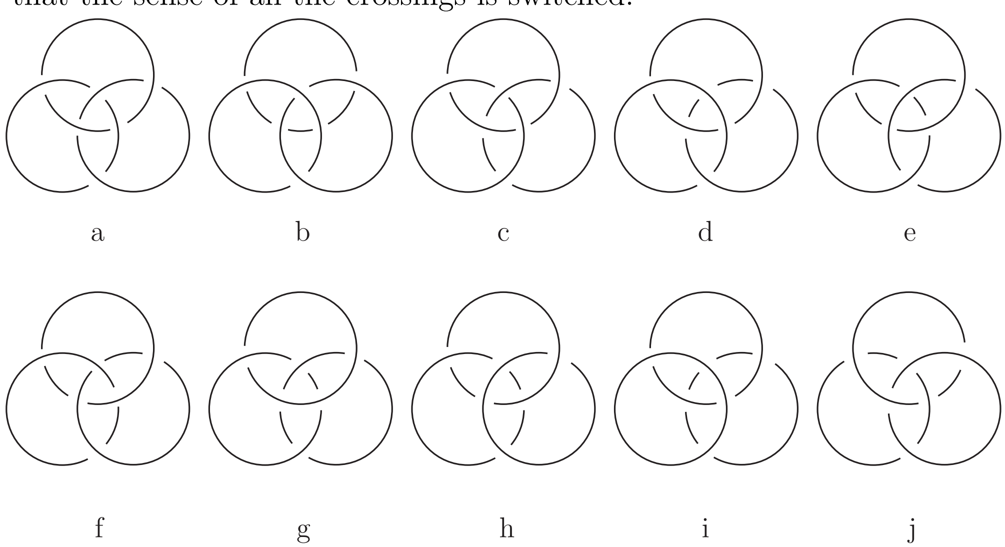


FIGURE 11. Links derived from Borromean Rings

Considered from a topological viewpoint, these ten patterns have only five distinct link types: the Borromean rings (a), the $(3, 3)$-torus link (e and f), a three component chain (g, h, and j), and Hopf link with a split component (b, c, and d), and the 3-component trivial link (i) [5]. Looking at these patterns we know that a three component chain, a Hopf link with a split component, and a 3-component trivial link can be formed with circles. We also know that the Borromean rings can not be formed with circles. As for the $(3,3)$ torus links, pattern (e) is link 6^3_3 so we know that it can be formed with circles. A physical model of pattern (f) can be made and it too can be formed with circles.

6. Table of Link Invariants of Two-Component Links

L	Conway	$lk(L)$	$\lambda(L)$	L	Conway	$lk(L)$	$\lambda(L)$
0^2_1	0	0	0	2^2_1	z	1	0
4^2_1	$2z$	2	0	4^2_{1*}	$2z+z^3$	2	1
5^2_1	z^3	0	1	6^2_1	$3z+4z^3+z^5$	3	4
6^2_{1*}	$3z$	3	0	6^2_2	$3z+2z^3$	3	2
6^2_3	$2z+2z^3$	2	2	6^2_{3*}	$-2z+z^3$	-2	1
7^2_1	$-z+2z^3+z^5$	-1	2	7^2_{1*}	$z+2z^3$	1	2
7^2_2	$z-2z^3$	1	-2	7^2_{2*}	$z+2z^3+z^5$	1	2
7^2_3	$-2z^3$	0	-2	7^2_4	$2z^3+z^5$	0	2
7^2_5	$2z+3z^3$	2	1	7^2_{5*}	$2z+2z^3+z^5$	2	0
7^2_6	$-z^3-z^5$	0	1	7^2_7	$2z+z^3$	2	1
7^2_{7*}	$2z+4z^3+z^5$	2	2	7^2_8	z^3	0	1
8^2_1	$4z+10z^3+6z^5+z^7$	4	10	8^2_{1*}	$4z$	4	0
8^2_2	$4z+7z^3+2z^5$	4	7	8^2_{2*}	$4z+3z^3$	4	3
8^2_3	$3z+6z^3+2z^5$	3	6	8^2_{3*}	$-3z+2z^3$	-3	2
8^2_4	$4z+4z^3$	4	4	8^2_{4*}	$4z+6z^3+2z^5$	4	6
8^2_5	$3z+4z^3$	3	4	8^2_{5*}	$-3z+z^5$	-3	0
8^2_6	$2z+3z^3$	2	3	8^2_{6*}	$-2z+2z^3$	-2	2
8^2_7	$-z+z^5$	-1	0	8^2_8	$z+z^5$	1	0
8^2_9	$2z-3z^3$	2	-1	8^2_{9*}	$2z-z^5$	2	2
8^2_{10}	$-z^5$	0	0	8^2_{11}	$2z+5z^3+2z^5$	2	3
8^2_{11*}	$-2z+z^5$	-2	2	8^2_{12}	$-z^5$	0	0
8^2_{13}	$-z^3+z^5$	0	-1	8^2_{14}	$2z+5z^3$	2	7
8^2_{14*}	$2z+z^5$	2	-2	8^2_{15}	z^3	0	1
8^2_{16}	$2z-z^3$	2	1	8^2_{16*}	$2z-2z^3-z^5$	2	0
9^2_1	$-2z+2z^3+4z^5+z^7$	-2	2	9^2_{1*}	$2z+3z^3$	2	3
9^2_2	$2z+4z^3+4z^5+z^7$	2	4	9^2_{2*}	$-2z+3z^3$	-2	3
9^2_3	$z-4z^3-2z^5$	1	-4	9^2_{3*}	$-z-4z^3$	-1	-4
9^2_4	$-5z^3-2z^5$	0	-5	9^2_5	$4z^3+4z^5+z^7$	0	4

L	Conway	$lk(L)$	$\lambda(L)$	L	Conway	$lk(L)$	$\lambda(L)$
9^2_{5*}	$-4z^3$	0	-4	9^2_6	$2z-3z^3-2z^5$	2	-3
9^2_{6*}	$2z+4z^3+2z^5$	2	4	9^2_7	$-2z+2z^3+2z^5$	-2	2
9^2_{7*}	$2z+3z^3+2z^5$	2	3	9^2_8	$-z-4z^3-2z^5$	-1	-4
9^2_{8*}	$z-4z^3$	1	-4	9^2_9	$-3z^3-2z^5$	0	-3
9^2_{10}	$-3z^3$	0	-3	9^2_{11}	$z-2z^3-2z^5$	1	-2
9^2_{11*}	$-z-2z^3+z^5$	-1	-2	9^2_{12}	$z+2z^3+2z^5$	1	2
9^2_{12*}	$-z+2z^3-z^5$	-1	2	9^2_{13}	$3z^3+4z^5+z^7$	0	3
9^2_{14}	$2z+8z^3+3z^5$	2	10	9^2_{14*}	$2z+5z^3+4z^5+z^7$	2	7
9^2_{15}	$3z^3+2z^5$	0	3	9^2_{16}	$2z+6z^3$	2	8
9^2_{16*}	$2z+3z^3+2z^5$	2	5	9^2_{17}	$2z+5z^3$	2	7
9^2_{17*}	$2z+4z^3+2z^5$	2	6	9^2_{18}	$2z^3+2z^5$	0	2
9^2_{19}	$z-3z^3-4z^5-z^7$	1	-4	9^2_{19*}	$z+5z^3+2z^5$	1	4
9^2_{20}	$3z+5z^3+4z^5+z^7$	3	2	9^2_{20*}	$3z+5z^3$	3	2
9^2_{21}	$z-3z^3-2z^5$	1	-4	9^2_{21*}	$z+5z^3+4z^5+z^7$	1	4
9^2_{22}	$3z+3z^3+2z^5$	3	0	9^2_{22*}	$3z+7z^3+3z^5$	3	4
9^2_{23}	$2z+5z^3+4z^5+z^7$	2	5	9^2_{23*}	$2z-4z^3$	2	-4
9^2_{24}	$-3z+2z^3-1z^5$	-3	2	9^2_{24*}	$3z+6z^3+3z^5$	3	6
9^2_{25}	$-2z^3+z^5$	0	-2	9^2_{26}	$2z-z^3-2z^5$	2	1
9^2_{26*}	$-2z+2z^3-z^5$	-2	0	9^2_{27}	$-3z^3-2z^5$	0	-3
9^2_{28}	$2z+3z^3+2z^5$	2	1	9^2_{28*}	$-2z-2z^3+z^5$	-2	0
9^2_{29}	$2z+7z^3+3z^5$	2	9	9^2_{29*}	$2z+6z^3+4z^5+z^7$	2	8
9^2_{30}	$2z+3z^3-z^5$	2	5	9^2_{30*}	$2z+6z^3+3z^5$	2	8
9^2_{31}	$-z^3-3z^5-z^7$	0	-2	9^2_{32}	$-z^3-2z^5$	0	-1
9^2_{33}	$-z^3-2z^5$	0	-1	9^2_{34}	$z-2z^3-3z^5-z^7$	1	-2
9^2_{34*}	$z+2z^3+2z^5$	1	2	9^2_{35}	$z+2z^3+3z^5+z^7$	1	2
9^2_{35*}	$-z+2z^3+2z^5$	-1	2	9^2_{36}	$-2z^3-2z^5$	0	-2
9^2_{37}	$-2z^3-3z^5-z^7$	0	-2	9^2_{38}	$2z+z^3+2z^5$	2	1
9^2_{38*}	$2z-2z^5$	2	0	9^2_{39}	$-z-3z^3-3z^5-z^7$	-1	-2
9^2_{39*}	$z-z^3-2z^5$	1	-2	9^2_{40}	$3z+7z^3$	3	4
9^2_{40*}	$3z+3z^3+3z^5+z^7$	3	0	9^2_{41}	$3z^3+3z^5+z^7$	0	3
9^2_{41*}	$3z^3+z^5$	0	3	9^2_{42}	$z-2z^5-z^7$	1	0
9^2_{42*}	$z+2z^5$	1	0	9^2_{43}	$2z+4z^3+z^5$	2	6
9^2_{43*}	$2z+9z^3+6z^5+z^7$	2	11	9^2_{44}	$-2z^5-z^7$	0	0
9^2_{45}	$2z+7z^3+2z^5$	2	9	9^2_{46}	$-2z^3$	0	-2
9^2_{47}	$-z^3$	0	-1	9^2_{48}	$2z+3z^3$	2	5
9^2_{48*}	$2z+6z^3+2z^5$	2	8	9^2_{49}	$3z+9z^3+6z^5+z^7$	3	6
9^2_{49*}	$3z+z^3$	3	-2	9^2_{50}	$z+3z^3+z^5$	1	2

L	Conway	$lk(L)$	$\lambda(L)$	L	Conway	$lk(L)$	$\lambda(L)$
$9^2_{50^*}$	$z - z^3$	1	-2	9^2_{51}	$3z + 3z^3 + z^5$	3	0
$9^2_{51^*}$	$3z + 7z^3 + 2z^5$	3	4	9^2_{52}	$z - z^3 - z^5$	1	-2
$9^2_{52^*}$	$z + 3z^3$	1	2	9^2_{53}	$4z + 9z^3 + 6z^5 + z^7$	4	9
$9^2_{53^*}$	$4z + z^3$	4	1	9^2_{54}	$-z + 2z^3$	-1	2
$9^2_{54^*}$	$-z - 2z^3 - z^5$	-1	-2	9^2_{55}	$z^3 + z^5$	0	1
9^2_{56}	$-z^3 - z^5$	0	-1	9^2_{57}	$2z - z^3 - z^5$	2	-1
$9^2_{57^*}$	$2z + 2z^3 + z^5$	2	2	9^2_{58}	$2z + z^3 + z^5$	2	1
$9^2_{58^*}$	$2z - z^5$	2	0	9^2_{59}	$2z + 5z^3 + z^5$	2	7
$9^2_{59^*}$	$2z + 8z^3 + 6z^5 + z^7$	2	10	9^2_{60}	$2z + 5z^3 + z^5$	2	7
$9^2_{60^*}$	$2z + 4z^3 + z^5$	2	6	9^2_{61}	$4z + 5z^3 + z^5$	4	5

References

[1] C. Adams The Knot book, W.H. Freeman and Company, New York, 1994

[2] B. Lindstrom and H. Zetterstrom, "Borromean Circles are Impossible", *The American Mathematical Monthly* April 1991

[3] T. Farmer, "Cylinder and Cone Cutting", *The College Mathematics Journal*, Vol 28, No. 2, March 1997

[4] D. Rolfsen, Knots and Links, Publish or Perish, Berkeley, 1976

[5] P. Cromwell, E. Beltrami, and M. Rampichini, 'The Borromean Rings", *The Mathematical Intelligencer*, Vol 20, No. 1, 1998

[6] C. Livingston, "Enhanced Linking Numbers", *The American Mathematical Monthly*, Vol 110, No. 5, May 2003

[7] H. Doll, J. Hoste, "A Tabulation of Oriented Links", *Mathematics of Computation*, Vol 57, No. 196, October 1991, Pages 747-761

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