Counting m-coloring classes of knots and links

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August 9, 2005

Abstract

Two Fox m-colorings of a knot or link K are said to be equivalent if they differ only by a permutation of colors. The set of equivalence classes of m-colorings under this relation is the set $C_m(K)$ of Fox m-coloring classes of K. We develop a combinatorical formula for $|C_m(K)|$ for any knot or link K that depends only on the m-nullity of K. As a practical application, we determine the m-nullity, and therefore the value of $|C_m(P_{(p,q,r)})|$, for any any (p, q, r) pretzel link $P_{(p,q,r)}$.

1 Introduction

Tricolorability is a rather coarse knot invariant (it only divides the category of knots into two types), but it nonetheless has useful applications; one simple example is that tricolorability can be used to determine whether a link is non-splittable. The set of integers m for which a knot is m-colorable (the "coloring number set") is a more powerful invariant. To get an even stronger invariant we can consider for each m the number of different ways that a knot can be m-colored; this invariant is used in [1] to distinguish knots with identical HOMFLYPT polynomials. The current literature (for example, [3]) often considers the total number of m-colorings, including trivial m-colorings and m-colorings that differ only by a permutation of colors. In this paper we instead consider the number of nontrivial m-colorings up to permuation; that is, we consider the number of m-coloring classes $C_m(K)$ of a knot K (see Section 3). Although m-colorability is defined for any integer $m > 2$, in this paper we will focus only on the cases where m is prime.

In Section 2 we give a quick overview of Fox m-colorability, and discuss the number of ways that a knot can be m-colored. In Section 3 we introduce the notion of m-coloring classes, that is, m-colorings that differ by more than just permutations of the colors, and develop a combinatorical formula for the number of such colorings based only on the m-nullity of the knot. In Section 4

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§This work was supported by NSF grant number NSF-DMS 0243845.

we focus our attention on (p, q, r) pretzel knots, using determinants and m-nullity to describe the number of m-coloring classes for any (p, q, r) pretzel knot and prime number m. Specifically, we show that for any (p, q, r) pretzel knot K, if m divides Δ, then the m-nullity of K is either 2 or 3, depending on whether or not any of p, q, and r are relatively prime to m. This will mean that every m-colorable (p, q, r) pretzel knot must have either one or four m-coloring classes. A similar result (but counting *all* m-colorings) can be found in [3].

2 Definitions and Background

We begin by presenting some basic definitions and results concerning colorability of knots. (See also [4].) Our definition of m-colorability is Fox's original notion of m-colorability. Throughout this paper, the only m-colorings we will consider are those where m is prime.

Definition 1 *A knot K is* m-colorable *if each strand in a projection of K can be assigned a number (called a "color") from the set $\{0, 1, \cdots, m-1\}$ so that (i) at least two colors are used, and (ii) at each crossing, if x and y are the colors of the understrands and z is the color of the overstrand, then $x+y-2z \equiv 0 \bmod m$.*

Given a knot projection with a labeling of its crossings and strands, the associated *crossing matrix* C is the matrix whose (i, j) entry is 1 if strand j is an understrand at crossing i, and -2 if strand j is an overstrand at crossing i. A matrix obtained by removing one row and one column from a crossing matrix will be called a *minor crossing matrix*. Every minor crossing matrix of a knot K has the same determinant, which we denote $\det(K)$. The system of equations represented by the minor crossing matrix of a knot K has a solution mod m if and only if $\det(K) \equiv 0 \bmod m$. Therefore:

Theorem 1 *Suppose m is prime. A knot K is m-colorable if and only if m divides* $\det(K)$.

Each vector in the mod m nullspace of the crossing matrix C corresponds to an m-coloring of K. The *m-nullity* of a knot K is the dimension of the nullspace of the crossing matrix of K. (Note: Some authors use the mod m nullity of the *minor* crossing matrix rather than the mod m nullity of the full crossing matrix.) We can see immediately that if a knot has m-nullity 1, it is not m-colorable, because in this case, the only m-colorings are the trivial colorings, that is, those represented by multiples of the vector $(1, 1, \ldots, 1, 1)$.

The m-nullity of a knot K determines the number of m-colorings of K, regardless of projection or number of strands. Suppose K is a knot with m-nullity $n > 2$. Then the system of equations describing the m-colorability conditions at each crossing of K will have n free variables. This means that there is a set of n strands of K that can be assigned any color in $0, 1, 2, ..., m-1$ (as long as they are not all assigned the same color). We call this a set of *significant* *strands* for that projection of K. The colors for the remaining strands of the projection will be determined by the colors of the significant strands. Theorem 2 describes a formula for the number of m-colorings of a knot with m-nullity n (see [4]).

Theorem 2 *A knot with m-nullity n has $m^n - m$ nontrivial m-colorings.*

PROOF: Let K be a knot with m-nullity n. This implies that K has n significant strands. Because each strand can be assigned any of m different colors, there exist m^n different ways to assign colors to the strands such that the condition $x + y - 2z \equiv 0 \bmod m$ is satisfied. However, this number includes all the trivial m-colorings, that is, the m-colorings in which the entire knot is colored with only one color. Because an m-coloring must use at least two colors, we subtract the m trivial colorings, leaving us with $m^n - m$ nontrivial m-colorings. QED

Example 1 *For example, the trefoil knot has* 3*-nullity* 2*, and thus* $3^2 - 3 = 6$ *different 3-colorings. In this example,* any *two strands can be chosen as the significant strands. The six 3-colorings in Figure 1 correspond to the six different ways that we can assign colors from* $\{0, 1, 2\}$ *to the two significant strands (without both strands being the same color).*

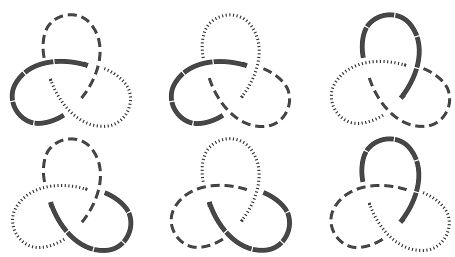


Figure 1: The six 3-colorings of the trefoil knot.

3 m-coloring classes

Notice that the six 3-colorings of the trefoil knot in Figure 1 are all permutations of each other. We will now examine whether a knot can be colored in two or more ways that are not simply permutations of each other.

Let $G_m(K)$ be the set of all possible nontrivial m-colorings for a knot K. We wish to count the number of m-colorings in $G_m(K)$ that differ by more than just a permutation of colors. We can think of m-colorings of K as maps from the set of strands of K to $\mathbb{Z}^m$, and consider two m-colorings to be equivalent if their maps differ by a permuation; this partitions $G_m(K)$ into equivalence classes of m-colorings. We will denote the set of these m-coloring classes of K by $C_m(K)$. Two m-colorings are said to be *equivalent* if they are in the same

m-coloring class, and *fundamentally different* if they are in different m-coloring classes.

Example 2 *Main Theorem 1 will guarantee that* $|C_3(9_{35})| = 4$. *Representatives of the four 3-coloring classes of* 9_{35} *are shown in Figure 2. Note that none of these 3-colorings is a permutation of another.*

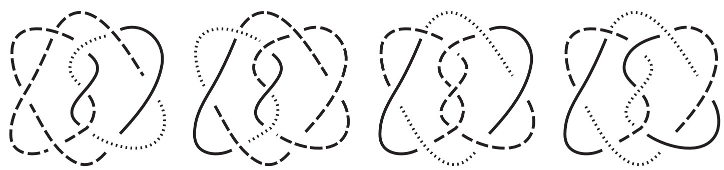


Figure 2: The four 3-coloring classes of the 9_{35} knot.

To find the number of m-coloring classes of a knot, we will separate the possible m-colorings into "types," and then count how many m-coloring classes are possible of each "type." A knot with m-nullity n will have n significant strands that can each be colored with any number $0, 1, 2, \ldots, m-1$ (as long as at least two colors are used). A choice of these m-colorings corresponds to a *coloring vector* $(c_1, c_2, \ldots, c_n)$. For example, a knot with 5-nullity 4 could have coloring vectors $(0, 2, 1, 4)$, $(1, 1, 1, 4)$, or $(3, 3, 0, 3)$. We will say that two coloring vectors (for m-nullity n) are of the same *type* if they are the same up to permutations of the order and choice of colors. For example, $(1, 1, 1, 4)$ and $(3, 3, 0, 3)$ are of the same type, but $(0, 2, 1, 4)$ and $(1, 1, 1, 4)$ are not.

We can classify a type of coloring vector using a *type vector* $\vec{d} = \langle d_1, d_2, \ldots, d_n \rangle$ that records the sizes of the sets of strands that are similarly colored. Each d_i in the type vector describes a number of strands that are assigned the same color. For example, the coloring vector $(3, 3, 0, 3)$ is made up of a set of 3 strands of the same color (3), and a set of 1 strand of a different color (0). Therefore we say $(3, 3, 0, 3)$ is of type $\langle 3, 1, 0, 0 \rangle$. (We append zeroes so that the type vector has length $n = 4$.)

Definition 2 *Suppose K is a knot with m-nullity n. A* type vector *for K is a vector $\vec{d} = \langle d_1, d_2, \ldots, d_n \rangle$ in $\mathbf{Z}_+^n$ satisfying the following three conditions:*

i) $d_{i+1} \leq d_i < n$ *for* $1 \leq i \leq n-1$*;*

ii) $\sum_{k=1}^{n} d_i = n$*;*

iii) At least $n - m$ of the d_i are zero.

The set of all type vectors for m-nullity n is denoted $D_{n,m}$.

Part (i) of Definition 2 ensures that the entries d_i are decreasing (to avoid repetition), and omits the type of m-coloring where all n of the significant strands are assigned the same color (or the trivial coloring). We require that the entries d_i sum to n in part (ii) since the sum of the d_i will be the number of significant strands. Part (iii) is only relevant if $n > m$. The number of non-zero entries in $\vec{d}$ represents the number of colors that are used. An m-coloring can use at most m colors, so if $n > m$ then at least $m - n$ of the d_i must be zero.

Our formula for the number of m-colorings of a knot with m-nullity n will also require the following definition.

Definition 3 *The* k-count *of a type vector* $\vec{d} = \langle d_1, d_2, \ldots, d_n \rangle$*, denoted* $s_k(\vec{d})$*, is the number of entries of* $\vec{d}$ *which are equal to* k*: for* $0 \leq k < n$*,*

$$s_k(\vec{d}) := \left| \{ d_i \mid d_i = k, 1 \leq i \leq n \} \right| .$$

Note that we will never need to consider $s_n(\vec{d})$, because any coloring whose type has $s_n(\vec{d}) \neq 0$ will be the trivial coloring. In this notation, part (iii) of Definition 2 says that $s_0 \leq m - n$. A few examples: The type vector $\langle 2, 1, 0 \rangle$ has $s_0 = 1$, $s_1 = 1$, and $s_2 = 1$. The type vector $\langle 1, 1, 1 \rangle$ has $s_0 = 0$, $s_1 = 3$, and $s_2 = 0$. A type vector with $s_1 = n$ corresponds to m-colorings where all the significant strands are colored differently.

We are finally in a position to describe our first main result, a combinatorical formula for the number of m-coloring classes of a knot with m-nullity n. Note that every knot with m-nullity n will have the same list of type vectors, with the same k-counts. Therefore the formula in Main Theorem 1 depends only on the prime number m and the m-nullity n of the knot.

Main Theorem 1 *Suppose K is a knot with m-nullity n. Then K has $|C_m(K)|$ m-coloring classes, given by the formula:*

$$|C_m(K)| = \sum_{\vec{d} \in D_{n,m}} \frac{n!}{(d_1!\, d_2! \cdots d_n!)\,(s_1!\, s_2! \cdots s_n!)}$$

Before proving Main Theorem 1, let us consider an example.

Example 3 *If K has 5-nullity 4, then $D_{n,m}$ consists of four type vectors: $\langle 3, 1, 0, 0 \rangle$, $\langle 2, 2, 0, 0 \rangle$, $\langle 2, 1, 1, 0 \rangle$, and $\langle 1, 1, 1, 1 \rangle$. If we use these vectors (and the corresponding k-counts) in the formula in Main Theorem 1, we obtain the sum:*

$$\frac{4!}{(3!\,1!\,0!\,0!)(1!\,0!\,1!)} + \frac{4!}{(2!\,2!\,0!\,0!)(0!\,2!\,0!)} + \frac{4!}{(2!\,1!\,1!\,0!)(2!\,1!\,0!)} + \frac{4!}{(1!\,1!\,1!\,1!)(4!\,0!\,0!)} = 14.$$

Therefore, if K has 5-nullity 4, then $|C_5(K)| = 14$. One knot with 5-nullity 4 is the composition of three Figure-8 (or 4_1) knots, shown in Figure 3. Using the four boldly marked strands as the set of significant strands, the $4_1\#4_1\#4_1$ knot can be colored 14 fundamentally different ways.

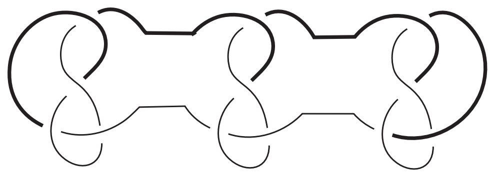


Figure 3: The nullity of $4_1\#4_1\#4_1$ is 4, and $|C_5(4_1\#4_1\#4_1)| = 14$.

In fact, any knot K with m-nullity 4 will have $|C_m(K)| = 14$, provided that $4 \leq m$. Table 1 lists the number of m-coloring classes of any knot with m-nullity n provided that $n \leq m$. If $n > m$, then K has more significant strands than colors, which affects the number of possible m-coloring classes of K (for example, see Table 2; notice that this table differes from Table 1 only when $n > 3$).

m-nullity n	$\lvert C_m(K)\rvert$
1	0
2	1
3	4
4	14
5	51
6	202

Table 1: $|C_m(K)|$ given $n \leq m$.

3-nullity n	$\lvert C_3(K)\rvert$
1	0
2	1
3	4
4	13
5	40
6	121

Table 2: $|C_3(K)|$ given 3-nullity n.

PROOF: Suppose K is a knot with m-nullity n. Then K has n significant strands, which we can label with (not necessarily distinct) colors $a_1, a_2, \ldots, a_n$. It is unimportant what specific color each strand is assigned; what matters is how many different ways we can assign colors to the strands so that a given type vector is satisfied. Before developing the general formula, we will focus on a few specific examples.

For example, the type vector $\langle 3, 2, 1, 0, 0, 0\rangle$ for a set of six significant strands represents a color assignment where three strands are one color, two more strands are another color, and the remaining strand is a third color. We will now count the number of arrangements of the strands a_i that satisfy this type vector. We must first choose three out of the six strands to be assigned the first color; there are $\binom{6}{3} = 20$ ways to do this. Similarly, there are $\binom{3}{2} = 3$ ways to choose two strands (from the remaining three) to be the second color. Finally, there is only one strand left to be assigned the remaining color ($\binom{1}{1} = 1$ choices). Therefore, there are $20 \cdot 3 \cdot 1 = 60$ strand arrangements satisfying the type vector. For example, two of these 60 arrangements are:

$$(a_1 = a_2 = a_3) \neq (a_4 = a_5) \neq (a_6),$$

$$(a_1 = a_4 = a_5) \neq (a_2 = a_6) \neq (a_3).$$

The counting method used above will overcount the number of arrangements if there are two or more equal entries in the type vector. For example, consider the type vector $\langle 2,2,1,0,0\rangle$ for a set of five significant strands, and the following two arrangements:

$$(a_1 = a_3) \neq (a_2 = a_5) \neq (a_4);$$
$$(a_2 = a_5) \neq (a_1 = a_3) \neq (a_4).$$

The counting method predicts that there will be $\binom{5}{2}\binom{3}{2}\binom{1}{1} = 30$ different arrangements for this type vector. However, this counts the two arrangements above differently, although they are obviously the same. Since there are two sets of the same size in the arrangements for this type vector, we must divide our result by 2; therefore there will be 15 different arrangements for the type vector $\langle 2,2,1,0,0\rangle$. If there are three sets of the same size, we must divide by $3 \cdot 2$; if there are s sets of the same size, we must divide by $s!$. We must divide for *every* multiple set, so if we had type vector $\langle 2,2,1,1,1,0,0\rangle$, we would have to divide our result by $2! \cdot 3!$. Notice that the k-counts from Definition 3 are exactly what we need to measure how many sets of the same size occur in any type vector.

Applying this counting method to a general type vector $\vec{d} = \langle d_1, d_2, \ldots, d_n\rangle$, we see that there are at most

$$\binom{n}{d_1}\binom{n-d_1}{d_2}\binom{n-d_1-d_2}{d_3}\cdots\binom{n-d_1-d_2-\cdots-d_{n-1}}{d_n} \tag{1}$$

different arrangements. The expression in (1) may overcount the number of arrangements, depending on the k-counts $s_k(\vec{d})$. To make an accurate count of the arrangements, we must divide the expression in (1) by the product of the factorials of the k-counts $s_k = s_k(\vec{d})$:

$$\frac{1}{s_1!\, s_2!\, s_3! \cdots s_{n-1}!}$$

(Note that if $s_i(\vec{d}) = 1$, then we divide by $1!$, which does not change the expresssion.)

To find the total possible number $|C_m(K)|$ of m-coloring classes of K, we must add together the number of arrangements corresponding to each type vector. This process results in the following formula:

$$|C_m(K)| = \sum_{\vec{d}\in D_{n,m}} \left(\prod_{1\le i\le n,\, d_i\neq 0} \binom{n-\sum_{j=1}^{i} d_{j-1}}{d_i} \cdot \prod_{1\le k<n} \frac{1}{s_k!} \right). \tag{2}$$

It now only remains to simplify the equation in (2) and show it is equivlant to the formula in Main Theorem 1. We begin by considering the part of (2) that comes from expression 1. It is easy to show that:

$$\prod_{1\le i\le n\; d_i\neq 0} \binom{n-\sum_{j=1}^{i} d_{j-1}}{d_i} = \frac{n!}{(d_1! d_2! \cdots d_n!)(n-d_1-d_2-\cdots-d_n)!}.$$

Since $\sum_{k=1}^{n} d_i = n$, the second factor in the denominator of the expression above is $(n-n)! = 0! = 1$. Thus we have:

$$\prod_{1\leq i\leq n\ d_i\neq 0} \binom{n-\sum_{j=1}^{i} d_{j-1}}{d_i} = \frac{n!}{d_1!d_2!d_3!\cdots d_n!}.$$

Therefore, the equation in (2) can be written as:

$$|C_m(K)| = \sum_{\vec{d}\in D_{n,m}} \frac{n!}{(d_1!d_2!d_3!\cdots d_n!)(s_1!s_2!s_3!\cdots s_n)}.$$

QED

Given a type vector $\vec{d}$, the corresponding term in the sum from Main Theorem 1 counts the number of possible strand arrangements that satisfy the type vector $\vec{d}$. To count the total number of m-colorings, including those that differ only by permutations, we must multiply each of the terms in the sum from Main Theorem 1 by $\frac{m!}{(m-(n-s_0))!}$, where $s_0 = s_0(\vec{d})$ is the 0-count for the type vector $\vec{d}$. This means that the total number of m-colorings of a knot with m-nullity n is given by the expression:

$$\sum_{\vec{d}\in D_{n,m}} \left(\frac{m!}{(m-(n-s_0))!} \cdot \frac{n!}{(d_1!d_2!d_3!\cdots d_n!)(s_1!s_2!s_3!\cdots s_n)} \right). \tag{3}$$

It is interesting to note that this complicated expression is in fact equal to $m^n - m$, the formula from Theorem 2.

4 Pretzel Knots

As a practical application, we will now consider Main Theorem 1 in the context of pretzel knots.

A *twist* is a part of a knot comprised of two strands twisted together monotonically one or more times; for example, the knot in Figure 4 has three twists. The top right strand is called the *northeast* strand, the bottom right is called the *southeast* strand, and so on. A twist is *positive* if its northeast strand is an overstrand; otherwise it is *negative*. An twist is *odd* if it has an odd number of crossings; otherwise it is *even*. For any integers p, q, and r, the *pretzel knot* $P_{(p,q,r)}$ is the knot or link composed of three twists of size p, q, and r joined together such that each northeast strand connects to the adjacent northwest strand, and each southeast strand connects to the adjacent southwest strand (we consider the first and last twists to be adjacent). For example, the knot $P_{(5,4,7)}$ in Figure 4 consists of three positive twists with 5, 4, and 7 crossings.

$P_{(p,q,r)}$ is a knot if at most one twist is even, a 2-component link if two twists are even, and a 3-component link if all three twists are even. The results in this paper apply to both pretzel knots and pretzel links, but from this point forward we will refer to both pretzel knots and pretzel links simply as pretzel knots.

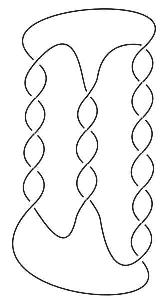


Figure 4: A $(5, 4, 7)$ pretzel knot.

Equations (4) and 5 describe basic properties of pretzel knots. Their proofs are elementary and we do not include them here. For any integers p, q, and r we have:

$$P_{(p,q,r)} = P_{(p,r,q)} = P_{(r,p,q)} = P_{(r,q,p)} = P_{(q,p,r)} = P_{(q,r,p)}. \tag{4}$$

$$P_{(p,q,r)} = P_{(-p,-q,-r)}. \tag{5}$$

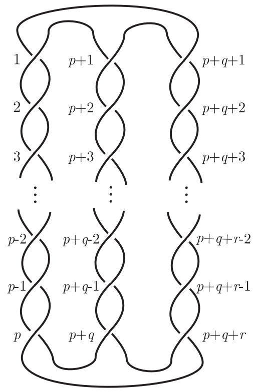
These equations show that to prove a theorem for all (p, q, r) pretzel knots, it suffices to prove two special cases: the case where p, q, and r are all positive, and the case where exactly one of p, q, and r is negative. We will make use of this fact when we prove Main Theorem 2.

If p, q, and r are all positive integers, then we will label the crossings of $P_{(p,q,r)}$ as shown in Figure 5, and the strands as shown in Figure 6. If any of the twists are negative, we require a different labeling. We will choose this labeling so that the connecting strands are labeled the same way as in the all-positive pretzel labeling in Figures 5 and 6. For example, a $P_{(p,q,-r)}$ pretzel knot (where p, q, and r are positive) will be labeled as in Figures 7 and 8.

These choices of labelings result in particularly nice crossing matrices. The matrix for a knot $P_{(p,q,r)}$ with all positive twists is shown in (6). (Note that the crossing matrix, and all of its diagonal blocks, are square, and that each unlabeled entry in the matrix is zero.)

$$\begin{array}{l} 1 \\ 2 \\ \vdots \\ p-1 \\ p \\ p+1 \\ p+2 \\ \vdots \\ p+q-1 \\ p+q \\ p+q+1 \\ p+q+2 \\ \vdots \\ p+q+r-1 \\ p+q+r \end{array} \left(\begin{array}{ccccc|ccccc|ccccc}
-2 & 1 & & & & & & & & & 1 & & & & \\
1 & -2 & 1 & & & & & & & & & & & & \\
 & & \ddots & & & & & & & & & & & & \\
 & & 1 & -2 & 1 & & & & & & & & & & \\
 & & & 1 & -2 & & & & & 1 & & & & & \\
\hline
1 & & & & & -2 & 1 & & & & & & & & \\
 & & & & & 1 & -2 & 1 & & & & & & & \\
 & & & & & & & \ddots & & & & & & & \\
 & & & & & & & 1 & -2 & 1 & & & & & \\
 & & & & & & & & 1 & -2 & & & & & 1 \\
\hline
 & & & & & 1 & & & & & -2 & 1 & & & \\
 & & & & & & & & & & 1 & -2 & 1 & & \\
 & & & & & & & & & & & & \ddots & & \\
 & & & & & & & & & & & & 1 & -2 & 1 \\
 & & & & 1 & & & & & & & & & 1 & -2
\end{array}\right) \tag{6}$$

Similarly, if p, q, and r are positive, then the pretzel knot $P_{(p,q,-r)}$ has the



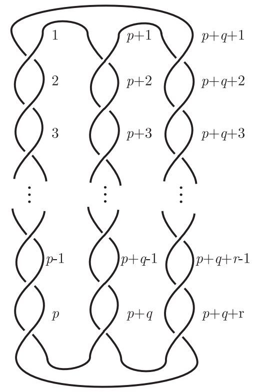
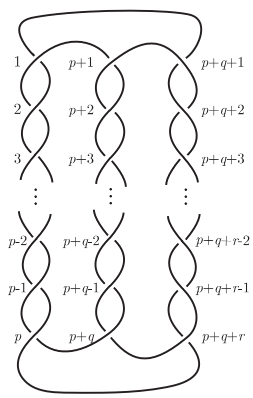


Figure 5: Crossing labeling for a positive-twist pretzel knot.

Figure 6: Strand labeling for a positive-twist pretzel knot.



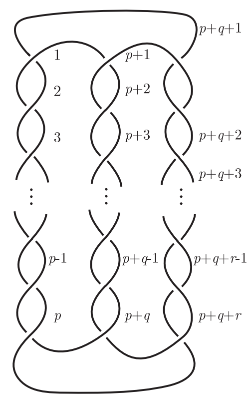


Figure 7: Crossing labeling for a $(p, q, -r)$ pretzel knot.

Figure 8: Strand labeling for a $(p, q, -r)$ pretzel knot.

general crossing matrix shown in (7).

$$
\begin{array}{l}
1 \\ 2 \\ 3 \\ \vdots \\ p-2 \\ p-1 \\ p \\ p+1 \\ p+2 \\ p+3 \\ \vdots \\ p+q-2 \\ p+q-1 \\ p+q \\ p+q+1 \\ p+q+2 \\ p+q+3 \\ \vdots \\ p+q+r-2 \\ p+q+r-1 \\ p+q+r
\end{array}
\left(\begin{array}{ccccccc|ccccccc|ccccccc}
-2 & 1 & & & & & & & & & & & & & 1 & & & & & & \\
1 & -2 & 1 & & & & & & & & & & & & & & & & & & \\
 & 1 & -2 & 1 & & & & & & & & & & & & & & & & & \\
 & & & \ddots & & & & & & & & & & & & & & & & & \\
 & & & 1 & -2 & 1 & & & & & & & & & & & & & & & \\
 & & & & 1 & -2 & 1 & & & & & & & & & & & & & & \\
 & & & & & 1 & -2 & & & & & & & 1 & & & & & & & \\
\hline
1 & & & & & & & -2 & 1 & & & & & & & & & & & & \\
 & & & & & & & 1 & -2 & 1 & & & & & & & & & & & \\
 & & & & & & & & 1 & -2 & 1 & & & & & & & & & & \\
 & & & & & & & & & & \ddots & & & & & & & & & & \\
 & & & & & & & & & & 1 & -2 & 1 & & & & & & & & \\
 & & & & & & & & & & & 1 & -2 & 1 & & & & & & & \\
 & & & & & & & & & & & & 1 & -2 & & & & & & & 1 \\
\hline
 & & & & & & & -2 & & & & & & & 1 & 1 & & & & & \\
 & & & & & & & 1 & & & & & & & & -2 & 1 & & & & \\
 & & & & & & & & & & & & & & & 1 & -2 & 1 & & & \\
 & & & & & & & & & & & & & & & & \ddots & & & & \\
 & & & & & & & & & & & & & & & & & 1 & -2 & 1 & \\
 & & & & & & 1 & & & & & & & & & & & & 1 & -2 & \\
 & & & & & & -2 & & & & & & & & & & & & & 1 & 1
\end{array}\right) \tag{7}
$$

Regardless of the signs of p, q, and r, it is known (see [2], [3]) that

$$\det(P_{(p,q,r)}) = |pq + pr + qr|.$$

This can also be shown by row-reducing the matrices in (6) and (7). By Theorem 1 this means that:

Theorem 3 *For any integers p, q, and r, $P_{(p,q,r)}$ is m-colorable if and only if m divides $|pq + pr + qr|$.*

By calculating the m-nullity of $P_{(p,q,r)}$ and applying Main Theorem 1, we will arrive at our second main theorem:

Main Theorem 2 *Suppose p, q, and r are integers and m is a prime number that divides $|pq + pr + qr|$.*

i) If at least one of p, q, and r is relatively prime to m, then $|C_m(P_{(p,q,r)})| = 1$.

ii) If none of p, q, and r are relatively prime to m, then $|C_m(P_{(p,q,r)})| = 4$.

Proof: It suffices to show that the m-nullity of $P_{(p,q,r)}$ is 2 when at least one of p, q, and r is relatively prime to m, and that otherwise the m-nullity of $P_{(p,q,r)}$ is 3. In each of parts (i) and (ii) it suffices to consider two cases: when all three twists are positive, and when two twists are positive and one is negative (see Equations 4 and 5).

Case 1: Part (i) when all three twists are positive. Without loss of generality, we can assume that q is relatively prime to m (see Equation 4). We will reduce the crossing matrix C from (6) so that only zeroes appear below the lower diagonal. We begin with the following sequence of row operations:

$$\begin{array}{rcl} R_1 & \longrightarrow & R_1 + \sum_{i=2}^{p+q+r} R_i, \\ R_{p+1} & \longrightarrow & R_{p+1} - \sum_{i=2}^{p} (i-1) R_i, \\ R_{p+q+1} & \longrightarrow & R_{p+q+1} - \sum_{i=p+2}^{p+q} (i-1) R_i. \end{array}$$

These row operations change the matrix C into the matrix shown in (8).

$$\begin{array}{r} 1 \\ 2 \\ \vdots \\ p-1 \\ p \\ p+1 \\ p+2 \\ \vdots \\ p+q-1 \\ p+q \\ p+q+1 \\ p+q+2 \\ \vdots \\ p+q+r-1 \\ p+q+r \end{array} \left[\begin{array}{ccccc|ccccc|ccccc} 0 & 0 & & & & & & & & & & & & & \\ 1 & -2 & 1 & & & & & & & & & & & & \\ & & \ddots & & & & & & & & & & & & \\ & & 1 & -2 & 1 & & & & & & & & & & \\ & & & 1 & -2 & & & & & 1 & & & & & \\ \hline & & & & p & -2 & 1 & \cdots & \cdots & -(p-1) & & & & & \\ & & & & & 1 & -2 & 1 & & & & & & & \\ & & & & & & & \ddots & & & & & & & \\ & & & & & & & 1 & -2 & 1 & & & & & \\ & & & & & & & & 1 & -2 & & & & & 1 \\ \hline & & & & & & & & & q & -2 & 1 & \cdots & \cdots & -(q-1) \\ & & & & & & & & & & 1 & -2 & 1 & & \\ & & & & & & & & & & & & \ddots & & \\ & & & & & & & & & & & & 1 & -2 & 1 \\ & & & & 1 & & & & & & & & & 1 & -2 \end{array} \right] \qquad (8)$$

It remains only to reduce row R_{p+q+r}. We begin by switching this row with row R_{p+1} (because there is a p above the 1 in column C_p):

$$R_{p+q+r} \longleftrightarrow R_{p+1}.$$

The new row R_{p+q+r} can now be reduced as follows:

$$\begin{array}{lcl} R_{p+q+r} & \longrightarrow & R_{p+q+r} - \sum_{i=p+1}^{p+q} (i-1) R_i, \\ R_{p+q+1} & \longrightarrow & R_{p+q+1} + R_{p+q+r}, \\ R_{p+q+r} & \longrightarrow & q \cdot R_{p+q+r}, \\ R_{p+q+r} & \longrightarrow & R_{p+q+r} + p \cdot R_{p+q+1}. \end{array}$$

Note that multiplying by q is valid modulo m because we are assuming that q and m are relatively prime. The final row is now:

$$\left(\begin{array}{ccc|ccc|cccccccc} 0 & \cdots & 0 & 0 & \cdots & 0 & -2(q+p) & p+q & 0 & \cdots & 0 & -pq & p+q+pq \end{array} \right).$$

The remainder of the row reduces fairly easily using the row operation:

$$R_{p+q+r} \longrightarrow R_{p+q+r} + \sum_{i=p+q+2}^{p+q+r-1} (i-p-q)(q+p) R_i.$$

We have now reduced the crossing matrix C to the reduced matrix R shown in (9).

$$\begin{array}{r} 1 \\ 2 \\ \vdots \\ p-1 \\ p \\ p+1 \\ p+2 \\ \vdots \\ p+q-1 \\ p+q \\ p+q+1 \\ p+q+2 \\ \vdots \\ p+q+r-1 \\ p+q+r \end{array} \left[\begin{array}{ccccc|ccccc|ccccc} 0 & 0 & & & & & & & & & & & & & \\ 1 & -2 & 1 & & & & & & & & & & & & \\ & & \ddots & & & & & & & & & & & & \\ & & 1 & -2 & 1 & & & & & & & & & & \\ & & & 1 & -2 & & & & & 1 & & & & & \\ \hline & & & & 1 & & & & & & & & & 1 & -2 \\ & & & & & 1 & -2 & 1 & & & & & & & \\ & & & & & & & \ddots & & & & & & & \\ & & & & & & & 1 & -2 & 1 & & & & & \\ & & & & & & & & 1 & -2 & & & & & 1 \\ \hline & & & & & & & & & q & -2 & 1 & \cdots & \cdots & -(q-1) \\ & & & & & & & & & & 1 & -2 & 1 & & \\ & & & & & & & & & & & & \ddots & & \\ & & & & & & & & & & & & 1 & -2 & 1 \\ & & & & & & & & & & & & & \Delta & \Delta \end{array} \right] \tag{9}$$

Since m divides the determinant $|\Delta| = |pq + pr + qr|$, this matrix R, and thus the matrix C, clearly has m-nullity 2.

Case 2: Part (i) when one twist is negative. Suppose p, q, and r are positive integers, and let $P_{(p,q,r)}$ have labelings as in Figures 7 and 8 and crossing matrix D as in (7). Again we can assume without loss of generality that q is relatively prime to m. The row operations for reducing the crossing matrix D are similar to the row operations for reducing the crossing matrix C in Case 1, and are as follows:

$$
\begin{aligned}
R_1 &\longrightarrow R_1 + \textstyle\sum_{i=2}^{p+q+r} R_i, \\
R_{p+1} &\longrightarrow R_{p+1} - \textstyle\sum_{i=2}^{p}(i-1)R_i, \\
R_{p+q+1} &\longleftrightarrow R_{p+q+2}, \\
R_{p+q+2} &\longrightarrow R_{p+q+2} + 2 \cdot R_{p+q+1}, \\
R_{p+q+1} &\longrightarrow R_{p+q+1} - \textstyle\sum_{i=p+2}^{p+q}(i-p-1)R_i, \\
R_{p+q+r} &\longrightarrow R_{p+q+r} + 2 \cdot R_{p+q+r-1}, \\
R_{p+q+r-1} &\longleftrightarrow R_{p+1}, \\
R_{p+q+r-1} &\longrightarrow R_{p+q+r-1} + -p \cdot R_{p+1} \textstyle\sum_{i=p+2}^{p+q}(i-p)R_i. \\
R_{p+q+r-1} &\longrightarrow R_{p+q+r-1} + R_{p+q+1}, \\
R_{p+q+r-1} &\longrightarrow q \cdot R_{p+q+r-1}, \\
R_{p+q+r-1} &\longrightarrow R_{p+q+r-1} + p \cdot R_{p+q+1}, \\
R_{p+q+r-1} &\longrightarrow R_{p+q+r-1} + \textstyle\sum_{i=p+q+3}^{p+q+r-2}(i-p-q-1)(p+q)R_i, \\
R_{p+q+r-1} &\longleftrightarrow R_{p+q+r}, \\
R_{p+q+r} &\longrightarrow 2 \cdot R_{p+q+r}, \\
R_{p+q+r} &\longrightarrow R_{p+q+r} + (-pr - qr - pq + 2p + 2q) \cdot R_{p+q+r-q}.
\end{aligned}
$$

Note that since q and m are relatively prime in this case, the row operation where we multiply row $R_{p+q+r-1}$ by q is valid modulo m. These row operations result in the reduced matrix shown in (10), which clearly has m-nullity 2.

$$
\begin{array}{r}
1 \\ 2 \\ 3 \\ \vdots \\ p-2 \\ p-1 \\ p \\ p+1 \\ p+2 \\ p+3 \\ \vdots \\ p+q-2 \\ p+q-1 \\ p+q \\ p+q+1 \\ p+q+2 \\ p+q+3 \\ \vdots \\ p+q+r-2 \\ p+q+r-1 \\ p+q+r
\end{array}
\left[\begin{array}{ccccccc|ccccccc|ccccccc}
0 & 0 & & & & & & & & & & & & & & & & & & & \\
1 & -2 & 1 & & & & & & & & & & & & & & & & & & \\
 & 1 & -2 & 1 & & & & & & & & & & & & & & & & & \\
 & & & \ddots & & & & & & & & & & & & & & & & & \\
 & & & 1 & -2 & 1 & & & & & & & & & & & & & & & \\
 & & & & 1 & -2 & 1 & & & & & & & & & & & & & & \\
 & & & & & 1 & -2 & & & & & & & 1 & & & & & & & \\
\hline
 & & & & & & 1 & & & & & & & & & & & & & 1 & -2 \\
 & & & & & & & 1 & -2 & 1 & & & & & & & & & & & \\
 & & & & & & & & 1 & -2 & 1 & & & & & & & & & & \\
 & & & & & & & & & & & \ddots & & & & & & & & & \\
 & & & & & & & & & & 1 & -2 & 1 & & & & & & & & \\
 & & & & & & & & & & & 1 & -2 & 1 & & & & & & & \\
 & & & & & & & & & & & & 1 & -2 & & & & & & & 1 \\
\hline
 & & & & & & & & & & & & & q & 0 & -2 & 1 & & \cdots & & -(q-1) \\
 & & & & & & & & & & & & & & 1 & -3 & 2 & & & & \\
 & & & & & & & & & & & & & & & 1 & -2 & 1 & & & \\
 & & & & & & & & & & & & & & & & & \ddots & & & \\
 & & & & & & & & & & & & & & & & & 1 & -2 & 1 & \\
 & & & & & & & & & & & & & & & & & & 2 & -3 & 1 \\
 & & & & & & & & & & & & & & & & & & & -\Delta & \Delta
\end{array}\right]
\tag{10}
$$

Case 3: Part (ii) when all three twists are positive. Suppose p, q, and r are positive integers and no twist in $P_{(p,q,r)}$ is relatively prime to m. During our reduction of the crossing matrix C for $P_{(p,q,r)}$ in Case 1, we multiplied a row by q. Since m divides q (and in fact, also divides p and r), this is not valid modulo m. The row operations before this step result in the partially reduced matrix shown in (11).

$$
\begin{array}{r}
1 \\ 2 \\ \vdots \\ p-1 \\ p \\ p+1 \\ p+2 \\ \vdots \\ p+q-1 \\ p+q \\ p+q+1 \\ p+q+2 \\ \vdots \\ p+q+r-1 \\ p+q+r
\end{array}
\left[\begin{array}{ccccc|ccccc|ccccc}
0 & 0 & & & & & & & & & 1 & & & & \\
1 & -2 & 1 & & & & & & & & & & & & \\
 & & \ddots & & & & & & & & & & & & \\
 & & 1 & -2 & 1 & & & & & & & & & & \\
 & & & 1 & -2 & & & & & 1 & & & & & \\
\hline
 & & & & 1 & & & \cdots & & & & & & 1 & -2 \\
 & & & & & 1 & -2 & 1 & & & & & & & \\
 & & & & & & & \ddots & & & & & & & \\
 & & & & & & & 1 & -2 & 1 & & & & & \\
 & & & & & & & & 1 & -2 & & & & & 1 \\
\hline
 & & & & & & & & & 0 & -2 & 1 & \cdots & \cdots & 1 \\
 & & & & & & & & & & 1 & -2 & 1 & & \\
 & & & & & & & & & & & & \ddots & & \\
 & & & & & & & & & & & & 1 & -2 & 1 \\
 & & & & & & & & & 0 & -2 & 1 & \cdots & 0 & 1
\end{array}\right] \tag{11}
$$

Notice that the rows R_{p+q+1} and R_{p+q+r} are the same. The following operation is enough to finish reducing the matrix:

$$R_{p+q+r} \longrightarrow R_{p+q+r} + (-R_{p+q+1}).$$

We now have a final row of zeros, and a matrix with m-nullity 3.

Case 4: Part (ii) where one twist is negative. The proof in this case is similar to the proof of Case 3, with the crossing matrix and row reductions used in Case 2. QED

Example 4 *$P_{(3,15,5)}$ has determinant $135 = 3^3 \cdot 5$, and thus is m-colorable for $m = 3$ or $m = 5$. Because $m = 3$ is relatively prime to $r = 5$, $P_{(3,15,5)}$ has 3-nullity 2, and therefore only one 3-coloring class (Figure 9). Similarly, since $m = 5$ is relatively prime to $p = 3$, $P_{(3,15,5)}$ has 5-nullity 2, and only one 5-coloring class (Figure 10).*

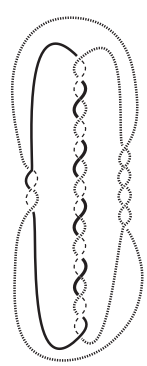


Figure 9: The 3-coloring of $P_{(3,15,5)}$.

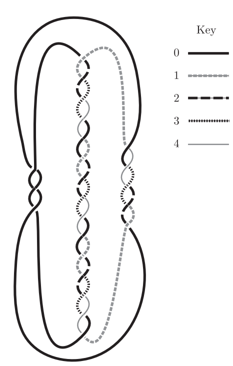


Figure 10: The 5-coloring of $P_{(3,15,5)}$.

Example 5 *$P_{(3,6,9)}$ has determinant $99 = 3^2 \cdot 11$, and thus is thus both 3-colorable and 11-colorable. Because 3 divides all three twists, $P_{(3,6,9)}$ has 3-nullity 3, and therefore four fundamentally different 3-colorings (shown in Figure 11). On the other hand, since 11 is relatively prime to at least one twist (in* *fact, to all of the twists), the 11-nullity of $P_{(3,6,9)}$ is 2. Therefore, $P_{(3,6,9)}$ has only one fundamental 11-coloring (shown in Figure 12).*

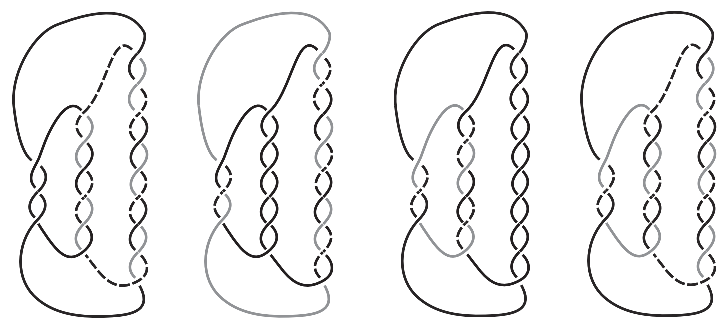


Figure 11: Representatives of the four 3-coloring classes of $P_{(3,6,9)}$.

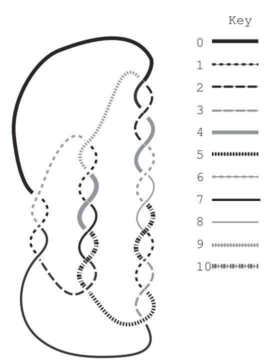


Figure 12: $P_{(3,6,9)}$ has one 11-coloring class.

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