Abstract

We classify by elementary methods the \( p \)-colorability of torus knots, and prove that every \( p \)-colorable torus knot has exactly one nontrivial \( p \)-coloring class. As a consequence, we note that the two-fold branched cyclic cover of a torus knot complement has cyclic first homology group.

MR Subject Classifications: 57M27, 05C15

1 Introduction

Our first result is a theorem specifically determining the \( p \)-colorability of any \((m, n)\) torus knot. It has been previously shown that a \((m, m - 1)\) torus knot is always \( p \)-colorable for \( p \) equal to \( m \) or \( m - 1 \) depending on which is odd (see [6] and [13]). Another proven result is that a \((2, n)\) torus knot is always \( p \)-colorable for \( p \) equal to \( n \) and a \((3, n)\) torus knot is always 3-colorable if \( n \) is even [13]. A result similar to ours was also stated as a lemma without proof in [3].

Our second result shows that any \( p \)-colorable \((m, n)\) torus knot has only one nontrivial \( p \)-coloring class. A general result investigating colorings of torus knots by finite Alexander quandles appears in [2]. Our result is a special instance of this result; however, we present a proof using only elementary techniques. \( p \)-coloring classes have also previously been investigated in relationship to pretzel knots by [4]. An immediate corollary of this result is that any nontrivial \( p \)-coloring of the standard braid representation of \( T_{m,n} \) must use all \( p \) colors. Distribution of colors in \( p \)-colorings of knots has been previously investigated with the Kauffman-Harary Conjecture, which examines the distribution of colors in a \( p \)-coloring of an alternating knot with prime determinant. Asaeda, Przytcki, and Sikora prove the Harary-Kauffman Conjecture is true for pretzel knots and Montesinos knots in [1]. Another immediate corollary is that the first homology groups of certain branched cyclic covers of torus knot complements (see [5], [11]) are cyclic.

*Dept. of Mathematics and Statistics, James Madison University, MSC 1911, Harrisonburg, VA 22807; taal@math.jmu.edu.
2 Notation

Given a prime \( p > 2 \) and a projection of a knot \( K \) with strands \( s_1, s_2, \ldots, s_r \), a \( p \)-coloring is an assignment \( c_1, c_2, \ldots, c_r \) of elements of \( \mathbb{Z}_p \) to the strands of the projection that satisfies the condition that at each crossing where \( s_i, s_j \) are the undercrossing strands and \( s_k \) is the overcrossing strand, \( c_i + c_j - 2c_k = 0 \mod p \).

A \( p \)-coloring is said to be nontrivial if at least two distinct “colors” in \( \mathbb{Z}_p \) are used.

\( p \)-colorability is invariant under Reidemeister moves and is thus a knot invariant, so a knot \( K \) is \( p \)-colorable if its projections admit non-trivial \( p \)-colorings.

Equivalently, a knot \( K \) is \( p \)-colorable if there exists an onto homomorphism from the knot group \( \pi_1(S^3 - K) \) of \( K \) to the dihedral group

\[
D_{2p} = \langle a, b \mid a^2 = 1, b^p = 1, abab = 1 \rangle.
\]

The knot group \( G \) can be expressed via the Wirtinger presentation, as follows. Given a projection of \( K \), define loops \( x_1, x_2, \ldots, x_r \) around the strands \( s_1, s_2, \ldots, s_r \), respectively, following the right hand rule. At each crossing where \( s_i \) terminates, \( s_j \) originates, and \( s_k \) is the overcrossing strand we have a relation \( R_j \) that is either of the form \( x_j = x_k^{-1}x_i x_k \) or \( x_j = x_k x_i x_k^{-1} \), depending on whether the sign of the crossing is positive or negative (see Section D of chapter 3 of [12]). With this notation, the Wirtinger presentation for the knot group of \( K \) is:

\[
\pi_1(S^3 - K) = \langle x_1, x_2, \ldots, x_r \mid R_1, R_2, \ldots, R_r \rangle.
\]

It is a simple exercise to prove that an assignment of colors \( c_1, c_2, \ldots, c_r \in \mathbb{Z}_p \) is a proper \( p \)-coloring of a projection of \( K \) if and only if the map \( \theta : \pi_1(S^3 - K) \to D_{2p} \) defined by \( \theta(x_i) = ab^{c_i} \) is an onto homomorphism (see p.122 of [8]).

Two \( p \)-colorings \( c_1, c_2, \ldots, c_r \) and \( d_1, d_2, \ldots, d_r \) of a projection of \( K \) are said to be equivalent, or in the same \( p \)-coloring class if for all \( 1 \leq i, j \leq r \), \( c_i = c_j \) if and only if \( d_i = d_j \); in this case we say that the two \( p \)-colorings differ only by a permutation of the colors. This definition of \( p \)-coloring classes corresponds directly to the mod \( p \) rank discussed in chapter 3 of [9].

3 \( p \)-Colorability of Torus Knots

Let \( T_{m,n} \) represent the torus knot characterized by the number of times \( m \) that it circles around the meridian of the torus and the number of times \( n \) that it circles around the longitude of the torus. \( T_{m,n} \) has one component if and only if \( m \) and \( n \) are relatively prime. It is well-known that every knot is the closure of some braid (see chapter 3 of [9]). For example, the trefoil knot \( T_{3,2} \) is the closure of the braid \( (\sigma_1 \sigma_2)^2 \) shown in Figure ??, where \( \sigma_i \) represents a crossing where string \( i - 1 \) crosses over string \( i \). In general, the knot \( T_{m,n} \) can be realized as the closure of the braid word \( (\sigma_1 \sigma_2 \cdots \sigma_{m-2} \sigma_{m-1})^n \).

Since \( T_{m,n} \) is equivalent to \( T_{n,m} \), the following theorem completely characterizes the \( p \)-colorability of torus knots. Note that if \( T_{m,n} \) has one component then \( m \) and \( n \) cannot both be even. Results similar to those in Theorem 1 were
stated without proof by Asami and Satoh in [3]. We present an elementary proof here.

**Theorem 1.** Suppose $T_{m,n}$ is a torus knot and $p$ is prime.

i) If $m$ and $n$ are both odd, then $T_{m,n}$ is not $p$-colorable.

ii) If $m$ is odd and $n$ is even, then $T_{m,n}$ is $p$-colorable if and only if $p|m$.

**Proof.** If $p$ is prime, then a knot $K$ is $p$-colorable if and only if $p$ divides $\det(K)$ (see chapter 3 of [9]). We will show that the determinant of $T_{m,n}$ is

$$\det(T_{m,n}) = \begin{cases} 1, & \text{if } m \text{ and } n \text{ are both odd} \\ m, & \text{if } m \text{ is odd and } n \text{ is even.} \end{cases}$$

By [10] we have $\det(K) = |\Delta_K(-1)|$, where $\Delta_K(t)$ is the Alexander polynomial of $K$. The Alexander polynomial for $T_{m,n}$ is given by (see part C of chapter 9 of [5])

$$\Delta_{T_{m,n}}(t) = \frac{(t^{mn} - 1)(t - 1)}{(t^m - 1)(t^n - 1)}.$$

Therefore if $m$ and $n$ are both odd, we have $\Delta_{T_{m,n}}(-1) = \frac{(-2)(-2)}{(-2)(-2)} = 1$. If $m$ is odd and $n$ is even, then by L’Hôpital’s rule we have $\Delta_{T_{m,n}}(-1) = \frac{\frac{(mn+1)+mn-1}{(m+n)^2-m+n}}{m} = m$.  

\[ \square \]

### 4 $p$-Coloring Classes of Torus Knots

Our second theorem is a special case of a result found by Asami and Kuga in [2]. They prove that if a knot $T_{m,n}$ can be $p$-colored using a finite Alexander quandle, it has a total of $p^2$ trivial and non-trivial colorings. If $T_{m,n}$ cannot be colored by such a quandle, then it has only the $p$ trivial colorings. It is important to note that Asami and Kuga only consider the total number of all $p$-colorings without distinguishing between equivalent colorings, while we consider equivalence classes of $p$-colorings, or $p$-coloring classes. Also note that the proof of Theorem 2 will show that every non-trivial $p$-coloring of the standard braid projection of $T_{m,n}$ must use all $p$ colors.
Figure 2: The action of \( \phi \) on the \( j \)'th color array of \( T_{m,n} \).

**Theorem 2.** If \( p \) is prime and \( T_{m,n} \) is \( p \)-colorable, then \( T_{m,n} \) has only one nontrivial \( p \)-coloring class.

**Proof.** If \( T_{m,n} \) is \( p \)-colorable, then by Theorem 1 can assume without loss of generality that we have \( m \) odd, \( n \) even, and \( p \mid m \). Given a \( p \)-coloring of \( T_{m,n} \) in the standard \( m \)-stand braid projection, let its \( j \)'th color array be the element of \((\mathbb{Z}_p)^m\) whose \( i \)'th component is the color of the \( i \)'th strand of the braid representation of \( T_{m,n} \) after \( j \) cycles. The map \( \phi : (\mathbb{Z}_p)^m \rightarrow (\mathbb{Z}_p)^m \) defined by

\[
\phi(c_0, c_1, \ldots, c_{m-1}) = (2c_0 - c_1, 2c_0 - c_2, \ldots, 2c_0 - c_{m-1}, c_0)
\]

describes the transition from the \( j \)'th to the \( (j + 1) \)'st color array of \( \alpha \) according the rules of \( p \)-colorability, as seen in Figure 1. Note that a \( p \)-coloring of \( T_{m,n} \) is entirely determined by its initial color array, and that to have a proper \( p \)-coloring it is necessary and sufficient that \( \phi^n \) fixes this initial color array.

Now since \( p \) divides the number \( m \) of braid strands in our projection of \( T_{m,n} \), we can consider the \( 0 \)'th color array that consists of the colors \( 0, 1, \ldots, p - 1 \) listed in order \( \frac{m}{p} \) times:

\[
C_0 = (0, 1, \ldots, p - 2, p - 1, 0, 1, \ldots, p - 2, p - 1, \ldots, p - 2, p - 1).
\]

Under the action of \( \phi \), the \( 1 \)'st color array is clearly

\[
\phi(C_0) = (p - 1, p - 2, \ldots, 1, 0, p - 1, p - 2, \ldots, 1, 0, \ldots, 1, 0),
\]

and the \( 2 \)'nd color array is

\[
\phi^2(C_0) = (0, 1, \ldots, p - 2, p - 1, 0, 1, \ldots, p - 2, p - 1, \ldots, p - 2, p - 1).
\]

Since \( \phi^2 \) fixes \( C_0 \) and \( n \) is even, we know that \( \phi^n \) fixes \( C_0 \) and thus the initial color array \( C_0 \) induces a nontrivial \( p \)-coloring.

Note that the \( p \)-coloring constructed above has the property that its initial color array \( c_0, c_1, c_2, \ldots, c_{m-1} \) has constant variance of 1, since \( c_{j+1} - c_j = 1 \) for all \( 0 \leq j \leq m \) (with indices mod \( m \)). It can be shown by elementary, but tedious, methods that if \( T_{m,n} \) is a one-component link then any \( p \)-coloring of its standard \( m \)-strand braid projection will have constant variance (not necessarily
equal to 1), and that all such $p$-colorings are equivalent to the $p$-coloring constructed above. Thankfully, the reviewer for this paper suggested a much more elegant method of proving that up to equivalence there can be no more than one nontrivial $p$-coloring of $T_{m,n}$, as follows.

Seeking a contradiction, suppose that there are two non-equivalent nontrivial $p$-colorings $c_1, c_2, \ldots, c_r, d_1, d_2, \ldots, d_r$ of a projection of $T_{m,n}$ with strands $s_1, s_2, \ldots, s_r$. We will show that these colorings induce what we can think of as a $\mathbb{Z}_p \oplus \mathbb{Z}_p$-coloring $(c_1, d_1, \ldots, c_r, d_r)$ of the strands of the projection in the sense that we have an onto homomorphism from the knot group $\theta$ from $\pi_1(S^3 - K)$ to the generalized Dihedral group $D = \langle a, b \mid a^2 = 1, b \in \mathbb{Z}_p \oplus \mathbb{Z}_p, abab = 1 \rangle$.

In the notation above, and writing $b = (b_1, b_2)$, define $\theta(x_i) = a(b_i^{c_i}, b_i^{d_i})$. We will show that $\theta$ is onto by showing that $a, (b_1, 1)$, and $(1, b_2)$ are in its image. By non-equivalence there must exist some $i, j$ such that either $c_i = c = j$ but $d_i \neq d_j$, or $d_i = d_j$ but $c_i \neq c_j$. Without loss of generality we will assume the former. With this $i, j$ it is a simple exercise to show that $\phi(x_i x_j) = (1, b_2^{d_j - d_i})$. Since $d_j - d_i \neq 0$ and $p$ is prime, some power of this element is $(1, b_2)$. Now since $c_1, c_2, \ldots, c_r$ is a nontrivial $p$-coloring there must exist some $k$ such that $c_i \neq c_k$, and for this $i, k$ we have $\phi(x_i x_k) = (b_1^{d_k - c_i}, b_2^{c_k - d_i})$. The product of this element with $(1, b_2)^{-d_k + d_i}$ is $(b_1^{d_k - c_i}, 1)$, and again since $c_k - c_i \neq 0$ and $p$ is prime, some power of this is $(b_1, 1)$. We now see immediately that $a$ is in the image of $\theta$ since $\theta(x_i)(b_1^{-c_i}, b_2^{d_i}) = a$.

The existence of this onto map $\theta$ provides a contradiction to there being two non-equivalent $p$-colorings, as follows. It is well-known (see p.58 of [5]) that $\pi_1(S^3 - T_{m,n}) = \langle x, y \mid x^m = y^n \rangle$, and that the center $Z$ of this group is generated by $x^m$. Since $\theta$ is onto and thus carries centers into centers, $\theta(Z)$ is contained in the center of $D$, which is trivial since $p$ is odd. Therefore $Z \subseteq \ker \theta$, and thus the map $\theta$ factors through the group $\langle x, y \mid x^m = 1, y^m = 1 \rangle$. This induces a map $\beta: \langle x, y \mid x^m = 1, y^m = 1 \rangle \to D$, which must be onto since $\theta$ is onto. But there can be no onto homomorphism from a free product of two cyclic groups to a group whose presentation requires at least three generators. Therefore there can be only one nontrivial $p$-coloring class for $T_{m,n}$, namely the one we constructed above.  

Notice that the proof of Theorem 2 shows that any $p$-coloring of the standard minimal projection of a torus knot must use all $p$ colors. In particular, this gives another proof that torus knots of the form $T_{p,2}$ for $p$ an odd prime satisfy the Kauffman-Harary conjecture (6.2 in [7]); such torus knots are alternating with determinant $p$, and the least number of colors needed to nontrivially color a minimal projection of $T_{p,2}$ will be equal to the crossing number $p$.

The reviewer for this paper pointed out to the authors that another immediate consequence of our elementary result in Theorem 2 is that the first homology groups of certain $q$-fold branched cyclic covers of torus knot complements are cyclic. The requirement that this homology group be cyclic in the two-fold case has been suggested as a weaker hypothesis for the Kauffman-Harary conjecture (p.7 of [1]). Given a torus knot $T_{m,n}$, let $\hat{C}_{m,n}^q$ denote the $q$-fold branched cyclic cover of $S^3 - T_{m,n}$ (see 8.18 in [5]).
**Corollary 3.** If $q = 2 + kmn$ for some nonnegative integer $k$, then the homology group $H_1(\hat{C}_{m,n}^q)$ is cyclic.

**Proof.** By 14.8 in [5], $T_{m,n}$ is $p$-colorable for some prime $p$ if and only if $p$ divides $|H_1(\hat{C}_{m,n}^2)|$. For each such prime $p$, Theorem 2 shows that there is only one nontrivial $p$-coloring class, which in turn guarantees that the 2-fold branched cyclic cover of the knot complement contains only one subgroup of order $p$ (see Section 3 of [11]). Since this is true for all primes $p$ that divide $|H_1(\hat{C}_{m,n}^2)|$, the result follows in the $q = 2$ (i.e. $k = 0$) case. The general result now follows from the fact that the $q$-fold coverings have period $mn$ (see 6.15 in [5]), i.e. $\hat{C}_{m,n}^q \cong \hat{C}_{m,n}^{q+kmn}$ for any nonnegative integer $k$. \hfill $\Box$

The authors of this paper would like to thank the reviewer, who made extremely detailed comments and in addition suggested a more streamlined argument and a way to extend our original results (Corollary 3). This work was supported by NSF grant number NSF-DMS 0243845.

**References**


