Simplicity is not Simple: Tessellations and Modular Architecture

The 2000 MathFest in Los Angeles was an extravaganza of mathematical talks, short courses and exhibits. By the third day, we needed a break from math and decided to check out the city. Of course, after being immersed in mathematics for so long, we noticed it everywhere. In particular, while walking down a street in a remote neighborhood, we happened upon a rhombic dodecahedron in a shop window. It was a hanging lamp, in a gallery full of fantastical hip furniture. We went in, fully expecting that our shabby clothing would result in a cool reception (we later discovered that the chairs sold for $3000 apiece). Much to our surprise, we were shown enthusiastically around the shop by a man who turned out to be the designer, Gregg Fleishman. He took us into his workshop, and showed us how he had been using rhombic dodecahedra to develop a modular building system. Despite his claims that he was not very good at math, it became clear as we talked that mathematics was the foundation for much of his work.

In this article, we'll introduce you to Fleishman's work, modular architecture more generally, and talk about how various architectural considerations can be described in mathematical terms. Along the way, we'll discuss and prove some basic facts about polyhedra and tessellations.

What is Modular Architecture?

Modular Architecture is any building system in which a few standardized components used to build a structure on a scale much larger than the components. Ideally, the parts should be easy to duplicate and simple to assemble by one person. To get a sense of what is meant by this, consider first the mathematically simpler problem faced by a company that wants to break up a large workroom into smaller identical workstations. Generally, rather than having a construction crew come in and build walls, the company will buy a set of identical dividers and connectors which fit together to create the cubicles. They buy as many as they need to complete their project, and the parts can be assembled without specialized crews.

Mathematically, the problem involved is one of breaking up a planar region into subregions using a finite number of identical pieces. These subregions should fit together so that they cover the whole plane without overlapping, except on edges. Such a method of subdivision is called a *tiling* or a *tessellation* of the Euclidean plane.

There are many kinds of tessellations, from the one given by simple square bathroom tiles to tessellations involving lizards or demons in the artwork of M. C. Escher to the non-periodic Penrose tilings of the plane. The simplest kinds of planar tessellations are those like the bathroom tiles where the subregions are identical, regular, convex polygons. In these tessellations, all angles and sides are identical. Architecturally, this means that if you were to use such a tessellation to construct office workstations, you would only need one basic type of panel and one basic type of connector. The resulting cubicles would all be equal sized, and arranged in a regular pattern.

Planar tessellations and Floor Plans

The fact that squares and rectangles *tessellate* the plane explains why they are used so often in architecture. Using squares or rectangles, floor plans can be constructed that have no wasted space between rooms. Of the *regular* polygons (convex polygons where all sides have equal length and all angles are equal) there are only three that tessellate the plane: triangles, squares and hexagons (Figure 1).

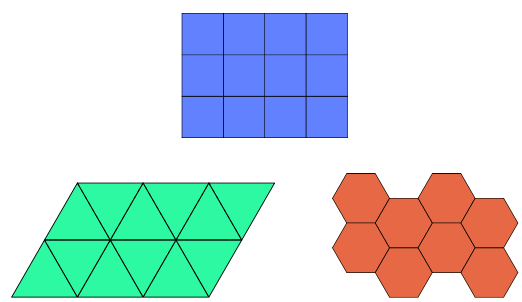

Figure 1: The three regular tessellations.

It is not hard to prove that these are the only three tessellating regular polygons. Th following proof is taken from Keith Critchlow's book *Order in Space*, The Viking Press, 1970.

Proof. Let P be a regular polygon with $n > 2$ sides. Then each interior angle of P measures $\alpha = 180 - \frac{360}{n}$ degrees. If P tessellates the plane then each vertex of the tessellation will be surrounded by $k(n) = \frac{360}{\alpha} = 2 + \frac{4}{n-2}$ polygons. It now suffices to determine which integers $n > 2$ will make $k(n)$ an integer. If $n > 7$ then $0 < \frac{4}{n-2} < 1$ and thus $k(n)$ could not be an integer. Checking the remaining possible values for n we see that $k(3) = 6$, $k(4) = 4$, $k(5) = 2\frac{4}{3}$, and $k(6) = 3$. Thus only 3-, 4-, and 6-sided regular polygons can tessellate the plane. ∎

Although squares are more often used than hexagons in architecture today, the hexagon has a mathematical advantage over both the square and the triangle. A hexagonal tessellation has the most efficient perimeter-to-area ratio, and thus will require the least amount of wall material per square foot of floor space.

Proof. A square with side length E has area E^2 and perimeter $4E$, so the perimeter of a square with area A is $4\sqrt{A}$. An (equilateral) triangle with side length E has area $\frac{\sqrt{3}}{4}E^2$, so a triangle with area A will have perimeter $3E = 2(3)^{\frac{3}{4}}\sqrt{A} \approx 4.559\sqrt{A}$. Finally, a hexagon with side length E has area $A = \frac{3\sqrt{3}}{2}E^2$ and thus perimeter $6E = 2(2)^{\frac{1}{2}}(3)^{\frac{1}{4}}\sqrt{A} \approx 3.722\sqrt{A}$. ■

For example, a square room enclosing 100 square feet requires a perimeter of 40 feet. A triangular room with the same square footage would need a bit more perimeter, about 45.59 feet. The most efficient floor unit is the hexagon, which requires a perimeter of only about 37.22 feet to enclose 100 square feet of area.

A workspace which is closer to a circle in shape also wastes less space in inaccessible corners. An office worker sitting in a swivel chair in the middle of an office can reach everything within a circular area whose radius is that worker's reach. A worker in a hexagonal office can more easily reach everything in the workspace than a worker in a square or rectangular cubicle. Further, hexagonal workstations can be arranged to fill circular or odd-shaped spaces. For all of these reasons, some companies are turning to hexagonal cubicle systems (Figure 2). The Silicon Valley company Nokia used hexagonal cubicles from Herman Miller, Inc. to fit eleven workstations and a couch area in a circular space where only six to eight rectangular workstations could have fit. Employees feel that they have more room and a more flowing office space with this hexagonal system, according to the report *Resolve connects with Nokia* on the Herman Miller, Inc., website at www.hermanmiller.com/us/pdfs/resolve/nokia_cs.pdf, 2000.

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Figure 2: A hexagonal office cubicle system by Herman Miller, Inc.

Despite the mathematical efficiency of hexagons, in real life they are often not as convenient as square or rectangular floor shapes. One problem is the difficulty in placing hallways and larger classrooms in hexagonal systems. For example, one could build a high school with identical hexagonal classrooms that tile very efficiently, but hallways and larger rooms like cafeterias or libraries could be difficult to place in such a floor plan (see William Blackwell's text *Geometry in Architecture*, John Wiley & Sons, 1984). Another problem is that so much architecture is already rectangular that other shapes often don't fit as well in the available spaces.

Modular Buildings

Office cubicle systems are a two-dimensional version of modular building systems. Gregg Fleishman is interested in creating entire buildings, not just floor plans within existing buildings. This means he is interested in ways of subdividing three-dimensional space. Designing a house, for instance, means creating an enclosed space and separating off rooms within that possibly multi-floored structure. The goal is still to do this with only a few basic parts.

Consider how you could build a backyard clubhouse. If you had five identical square pieces of plywood, you could build a simple cube-shaped clubhouse with a dirt floor. Rectangular prisms ("boxes") are the most common basic solid used in architecture. Cubes are the simplest of them in that all the faces are equivalent. Hence even a couple of grade school kids can build one. A solid where every face is the same regular polygon and all vertices (corners) are alike is called a *regular* or *Platonic* solid. There are exactly five of these: the tetrahedron, the cube, the octahedron, the dodecahedron and the icosahedron. This can be proved in a few ways. Perhaps the prettiest uses a number called the Euler characteristic.

Proof. Given a surface, S, a *decomposition* of S is a way of cutting S up into pieces so that each piece can be flattened into a planar region with no holes. The pieces are called the *faces* of the decomposition, the lines along which the faces were cut apart are called the *edges*, and the corners where edges connect are called the *vertices.*

The Euler characteristic of S, denoted $\chi(S)$, is then given by the formula $\chi(S) = F - E + V$, where F is the number of faces, E is the number of edges, and V is the number of vertices in any decomposition of the surface (see figure ??). It is a theorem that this number does not depend on how the surface is decomposed, and also doesn't change if you stretch or bend the surface (the Euler Characteristic is thus an example of a *topological invariant*). For a proof of this, see pp. 29–32 of *Algebraic Topology: An Introduction*, W.S. Massey, Graduate Texts in Mathematics

56, Springer-Verlag 1967.

Every Platonic solid gives a decomposition of the sphere, which you can see by imagining inflating the solid until its faces bulge out to a sphere. Using the tetrahedron, for example, we get a decomposition with four faces, twelve edges and eight vertices, so we know that χ(sphere)$= 4 - 12 + 8 = 2$. So we know that for any Platonic solid, we have the equation $F - E + V = 2$.

Now we can use the defining properties of Platonic solids to show there can only be five of them. Since the faces of a Platonic solid must be regular n-sided polygons for some $n \geq 3$, we know that the total number of face-edge intersections must be n times the number of faces, since each face meets n edges. Also, each edge meets two faces, so we get a second equation $nF = 2E$. Similarly, since each vertex is regular, each vertex meets the same number of edges, call it m, while each edge meets two vertices. Thus we get a third equation coming from counting the number of edge-vertex intersections, $mV = 2E$. Each vertex must meet at least 3 edges, so m is also at least 3.

Solving for F and V in terms of E in the second and third equations and plugging into the first equation, we obtain:

$$\frac{2E}{n} - E + \frac{2E}{m} = 2.$$

Rearranging this, we get:

$$\frac{1}{n} + \frac{1}{m} = \frac{2}{E} + \frac{1}{2}.$$

This tells us that $1/n + 1/m > 1/2$. Since n and m must each be at least 3, this gives us five possibilities. The first, n=3 and m=3, gives a solid in which three (since m=3) equilateral triangles (since n=3) meet at each vertex, that is, the tetrahedron. When n=3 and m=4, we get the octahedron, n=3 and m=5, gives the icosahedron, n=4 and m=3 gives the cube, and finally, n=5 and m=3, gives the dodecahedron. ■

If cubes and rectangular prisms are so easy to build with and so common in architecture, what is the motivation for exploring different systems? We asked Mr. Fleishman about this, who explained, "Cubes are useful, they are part of the solution I have arrived at. I am not against the cube at all. But it has drawbacks. First, cubes have a low volume to surface-area ratio. Of all three-dimensional shapes, the sphere has the highest. The closer you get to approximating a sphere, the better structural economy you have." This is akin to the perimeter to area issue for cubicles. An ideal structure would enclose a large amount of space with a small amount of building material.

"Another issue is the shape of the space. If you use other shapes, like the rhombic dodecahedron, you can get a higher and more interesting ceiling and roof. Finally, the closer a building is to spherical, the better it is able to withstand high winds and other weather."

These types of considerations led modular building pioneer Buckminster Fuller to start experimenting with geodesic domes in the 1940's. "Fuller generally started with the icosahedron and then subdivided each face into smaller triangles," explained Fleishman. "Of course, the object that results is no longer a Platonic solid, and can be difficult to build. You have to use a great number of rods of various lengths that intersect at various angles. Building large domes requires using a computer to keep track of where the different length rods go." Domes (large ones) have nevertheless been popular. Two of the best known examples are the Epcot Center at Disneyworld and the NSF Amundsen-Scott South Pole Station (to see pictures of these architectural domes, visit www.disneyworld.go.com/ waltdisneyworld/parksandmore/attractions and www.stanford.edu/group/zarelab/antartica.slide21.html).

Fleishman's personal goals have led him to experiment with different types of modular building systems. "One of my goals has always been to deal with a way of building the house, and I spent a long time on dome design. Geodesic dome architecture has to do with building a frame, then covering it, but covering the frame is frequently difficult. Also, a sphere is one object, and it is difficult to connect to others like it, so to make a large house, the only choice is to make a large sphere, and then you still have all of the interior walls to construct. In residential structures I have found it makes sense to go to a panel structure." Thus Fleishman has been working on creating a system of modular panels which can attach to create a variety of buildings with several joined smaller rooms rather than a single large dome.

Where does this leave us mathematically? Well, we are still hoping to create a structure using only a few different types of simple pieces, so we want to use Platonic solids if possible, or at least shapes with some of the same regularity properties, like being built of regular polygons, or having all faces the same. In addition, we now want to be able to connect rooms to each other in a space-filling way. So what we need is a three-dimensional version of a tessellation involving regular or semi-regular solids.

Space-filling solids

This leads us back to cubes and rectangular prisms. There is a reason besides simplicity that they are the most common solid used in architecture: they tessellate 3-space. That is, a collection of rectangular prisms of the same size can be arranged to fill 3-dimensional space so that there no gaps in between. Although these are the most common architectural building blocks, they are not the only solids that fill space. However, they are the only regular polyhedron which does (top of Figure 3). Thus to find other polyhedra that tessellate 3-space, we need to relax the regularity conditions. There are a couple of ways to do this. Either you can relax the condition that the faces all be the same regular polygons, or you can require the faces to be identical, but allow them not to be regular polygons.

The first possibility leads us to *Archimedean* solids: polyhedra whose faces are regular polygons (possibly of more than one variety) where every vertex is identical. There are thirteen Archimedean polyhedra, and of the thirteen the only one that fills 3-space is the truncated octahedron (bottom left of Figure 3). See Alan Holden's book *Shapes, Space, and Symmetry*, Columbia University Press, 1971.

If we replace the vertices of an Archimedean polyhedron by faces and replace the faces by vertices we get the *Archimedean dual* of the polyhedron. This gives us the second possibility; the faces will be identical, but they may not be regular polygons. Of the thirteen Archimedean duals, only one tessellates three space: the rhombic dodecahedron (bottom right of Figure 3). See *Order in Space*, Keith Critchlow, The Viking Press, 1970.

In general, there are only eight space-filling polyhedra, three of which are described above. Of these eight, the truncated octahedron (a polyhedron with six square faces and eight hexagonal faces) fills space most efficiently, *i.e.* has the most volume for the least surface area. The rhombic dodecahedron (a twelve-sided polyhedron whose faces are diamonds) is the second most efficient. The cube ranks a measly sixth out of eight in surface area to volume efficiency (rankings can be found in Critchlow's *Order in Space*, The Viking Press, 1980). By manipulating surface area and volume formulas we can prove that the truncated octahedra is slightly more efficient than the rhombic dodecahedron and significantly more efficient than the cube.

Proof. The surface area formula for each of these solids will be some multiple of the square E^2 of the edge length, and the volume formula will be some multiple of E^3. Thus the function writing surface area in terms of volume will be of the form $SA(V) = \alpha V^{\frac{2}{3}}$ for some constant α. Table 1 lists the volume and surface areas for the cube, rhombic dodecahedron (R.D.) and truncated octahedron (T.O.) in terms of their edge length E, and an approximate value for the associated coefficient α. ■

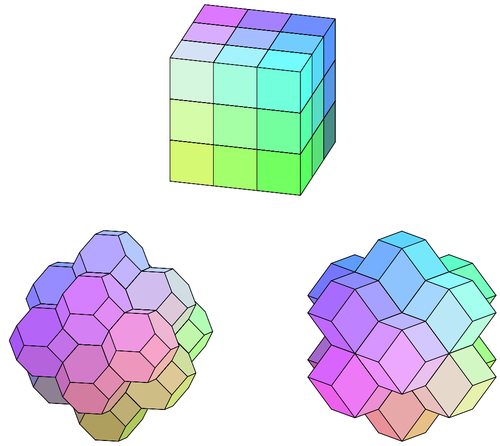

Figure 3: The only regular polyhedron that tessellates 3-space is the cube. The only Archimedean polyhedron that fills space is the truncated octahedron (bottom left). The only space-filling Archimedean dual is the rhombic dodecahedron (bottom right).

Table 1: Volume and surface area formulae for three space-filling solids. Smaller values of α indicate more efficient space-filling.

Solid	**Volume**	**Surface Area**	$\approx \alpha$
Cube	E^3	$6\,E^2$	6.0000
R.D.	$\frac{16}{9}\sqrt{3}\,E^3$	$8\sqrt{2}\,E^2$	5.3454
T.O.	$8\sqrt{2}\,E^3$	$(6+12\sqrt{3})\,E^2$	5.3147

For example, a cube-shaped room enclosing 1000 cubic feet would have an edge length of 10 cubic feet and a surface area of $\alpha V^{\frac{2}{3}} = 6(100) = 600$ square feet. To enclose the same volume with a rhombic dodecahedron we would only need about 534.54 square feet of surface area (since $\alpha = 5.3454$). The most efficient space-filler, the truncated octahedron, can fill 1000 cubic feet of volume with a surface area of only 531.47 square feet. Note that the rhombic dodecahedron is nearly as efficient as the truncated octahedron, but the cube is quite inefficient in comparison.

Further Considerations

Fleishman has developed two modular building systems: one based on the rhombic dodecahedron and one based on a semi-regular tiling using rhombicuboctahedrons, cubes and tetrahedrons. “The rhombic dodecahedron is a nice shape because it involves standard angles and lengths. The diagonals of the faces have lengths in the ratio of $1 : \sqrt{2}$, the same as the ratio of the length of the side of a square to its diagonal. The dihedral angles (angles between faces) are 120 degrees. This makes it easier for lay people to machine the parts and build the structures.” Compare this to the truncated octahedron, whose dihedral angles are approximately 125 degrees 16 minutes and 109 degrees 29 minutes. Because the dihedral angles of the rhombic dodecahedron are all equal and because each face has an even number of edges, Fleishman was able to develop a tab and slot connection system. Since the panels themselves interlock in this system (see picture), it eliminates need for a separate connector system.

The system involving the tetrahedron, cube, and rhombicuboctahedron is based on the rhombic dodecahedron system. This tiling of space is achieved by truncating the vertices of the rhombic dodecahedrons. We asked Fleishman how these systems compare to each other. “The advantages of this system over the rhombic dodecahedron system are that instead of the large diamond panel, which most likely needs to be cast or built up in some manner, there are square and rectangular panels, which are easier to cut from stock material. The variety of shapes and ability to rescale the components also permits a wider variety of module, or, basically, room sizes. However, the system also has the disadvantage of having more component parts and not creating as stable an exterior skin as the rhombic dodecahedron system.” At the cost of simplicity, Fleishman has also designed a three way panel connection system for both geometries so that interior walls could be built with the same panels as the walls. Fleishman never stops searching for better solutions; the conversations we had with him while writing this article led him to reconsider the truncated octahedron as a basic unit. He is now working on a new system based on that shape.

Fleishman wants people to be able to build with his panels without extensive training. “Making a good system from only a few pieces which are not too hard to machine or build is quite difficult,” says Fleishman, “Simplicity is not simple.” “In the end, in spite of its complex appearnce which has perhaps prevented others from exploring its potential, the math involved in the rhombic dodecahedron system is very simple. It is amazing what you can do with simple math.”

Housing for the World

We had a great time playing with Fleishman’s small-scale building models, and told him we thought they would make great math toys, and he said he’d be willing to make some if there was interest. But when we asked him how he envisioned his modular buildings being used, he had a much more revolutionary plan. “The plywood panel house is lacking insulation and waterproofing, but it is very easy to assemble and has a lot of potential for variety. My idea is that it has the perfect form for creating low-cost housing in emerging economies. The panel structures could be erected by crews with minimal training to provide interior structure. Exterior surfacing and finishing, like insulation, could be done using indigenous materials and techniques. These structures could provide a global solution to the housing problem.”

For Further Reading

Pictures of Gregg Fleishman’s work and the history of the development of his modular building systems can be found at www.greggfleishman.com. For more information about polyhedra, see the following websites: www.physics.orst.edu/bulatov/polyhedra, www.ics.uci.edu/eppstein/junkyard/tiling.html, and polytopes.wolfram.com.