Forced Differential Equations: Problems to Impact Intuition

Abstract: How should our students think about external forcing in the differential equation setting, and how can we help them gain intuition? To address this question, we share a variety of problems and projects that explore the dynamics of the undamped forced spring-mass system. We provide a sequence of discovery-based exercises that foster physical and mathematical intuition about polynomial forcing, as we build tools and techniques (including Green's function) to explore the amazing behavior of $y''(t) + y(t) = \tan(t)$. We encourage the insightful use of a Computer Algebra System, and provide a paired supplemental Maple Worksheet[16].

Keywords: ordinary differential equations, variation of parameters, method of undetermined coefficients, forced oscillator, spring-mass system, mathematical modeling, bounded forcing, polynomial drivers, unbounded forcing, Green's function

1 MOTIVATION AND GOALS

Many textbooks [1, 14, 15] choose the ordinary differential equation (ODE)

$$y''(t) + y(t) = \tan(t) \tag{1}$$

as a common example when introducing variation of parameters in the second order ODE setting. This example is chosen because the solution is relatively easily determined by the method of variation of parameter, but is not amenable to the method of undetermined coefficients.

This article is motivated by the solutions and interpretations of (1). While the mathematical solution process is fairly straightforward, intuition about the problem and its solution is lacking. The function $\tan(t)$ is usually referred to as an external forcing function. Since this equation is frequently used to model a simple mass-spring system (or a driven oscillator, more generally), a natural question is: “Is this problem physical? Are the solutions just mathematical, or are they related to physical reality as we model a mass-spring system?” But these questions lead us to other, more basic ones: “Are these forcing terms intuitive?” and then to: “Are any forcing terms intuitive? If so, what is the intuition? How does forcing influence the solution?”

There are many online resources that can be used in the classroom to explore differential equations, including [19, 12, 20, 13], that allow a quick and accessible way to gain intuition. These experiments and visualizations can provide confidence to students in their understanding of the mathematical solutions, and allow them to explain observable features using their mathematical knowledge. Harder to find, though, are demonstrations and tools which explore the contribution of external forcing, and that build intuition about the effects of external forces.

The ideas presented here should be viewed as part of the important and challenging interpretation stage of the solution process. Through a sequence of discovery-based projects for undergraduate students, presented via polynomial forcing functions of increasing degree, we provide a scaffold to explore the questions above. In the process, the students will build intuition and motivation about solving ODEs that lead to a better understanding about the more advanced topic of Green’s function. We concentrate on the numerous questions (and some answers) suitable for students in class and for those considering undergraduate ODE research projects. We chose to study the forced undamped spring-mass system, where a physical interpretation of the solution is easily accessible. Although well suited to the study of oscillators, the idea is applicable to many other forced, physical situations.

2 ASSUMPTIONS

We will assume that the student already has viable techniques for obtaining the solution of the initial-value forced problem

$$y''(t) + y(t) = f(t); \quad y(0) = y_0,\, y'(0) = v_0. \tag{2}$$

We will use ICs to reference the initial conditions $y(0) = y_0$ and $y'(0) = v_0$, y_h to refer to homogenous solutions, and y_p to refer to particular solutions. We occasionally use $y_h(t; \alpha, \beta)$ as needed to reference the explicit dependence of y_h on the ICs $y(0) = \alpha$ and $y'(0) = \beta$.

We encourage the use of a Computer Algebra System (CAS) such as Maple or Mathematica, which makes the solution of (2) easy to generate, manipulate, and visualize. A Maple worksheet [16] (and a pdf of this worksheet) is provided as an online supplement to assist the instructor and/or student as they navigate this article.

3 FORCED SPRING-MASS SYSTEM

The basic spring-mass system, as with many other physical systems, is easily modeled by applying Newton's Second Law. This law asserts that the force applied to an object is directly proportional to the rate of change of the momentum of the body. If the mass is constant in the momentum expression (which is mass times velocity), then the one dimensional form of Newton's Second Law takes the familiar form

$$\frac{d}{dt}(mv) = ma = F = \sum_i F_i \tag{3}$$

where F represents the sum of external forces F_i, m the mass, and a the acceleration.

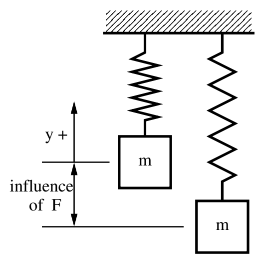

Figure 1. System with zero forcing and with constant negative forcing.

We now quickly develop a model for a mass suspended vertically from a spring, with gravity taken to be negative and pointing downward, but with no frictional damping (see Figure 1). At equilibrium, the spring force will balance the gravitational force, leaving us with a zero sum for these forces; and we label the equilibrium position $y = 0$. If the mass is displaced from equilibrium (by an initial condition, for example), then the spring provides a restoring force. This force is written $-ky$ when modeled using Hooke's Law, and is proportional to and in the opposite direction of the displacement. The complete model is now

$$my''(t) = -ky(t); \quad y(0) = y_0,\, y'(0) = v_0. \tag{4}$$

Additional external forces, such as the constant negative forcing as in Figure 1, would join the $-ky$ term. Rearranging slightly, and scaling (by non-dimensionalizing, or by setting $\sqrt{k/m} = 1$, or by prescribing m and k to be 1 in their dimensional units), the system results in an initial value ODE with external force $F(t)$:

$$y''(t) + y(t) = F(t); \quad y(0) = y_0,\, y'(0) = v_0.$$

More modeling details and a discussion of oscillator dynamics (typically under periodic forcing and with a focus on frequency coupling) can be found in many classical mechanics/dynamics texts including [18, 17]. In addition, [10] is an open source text which treats scaling.

4 PROBLEMS AND QUESTIONS FOR STUDENTS

We now present a sequence of problems to explore external forcing, beginning with polynomial expressions. The approach requires that students first understand the influence of ICs, which is developed in 4.1. Constant forcing is introduced in 4.2 to demonstrate a connection between the particular solution (as might be generated via undetermined coefficients) that we call the *baseline* solution, and a change of variables that eliminates the external forcing. Exercises concerning increasing degree polynomial external forces that may be used to approximate $\tan(t)$ will lead to intuition about $y''+y = \tan(t)$ in 4.6, where the external forcing is unbounded in finite time. Along the way, students encounter the concept of Green's function. Immediately following questions, comments and suggestions for the instructor may be contained within brackets ([$\ldots$]).

4.1 (Un)Forced: $y''(t) + y(t) = 0$, $y(0) = y_0$, $y'(0) = v_0$

The goal of these exercises is to understand the influence of initial conditions. Figure 2 shows y_h for a total of nine distinct (y_0, v_0) pairs, and may be useful when discussing ICs.

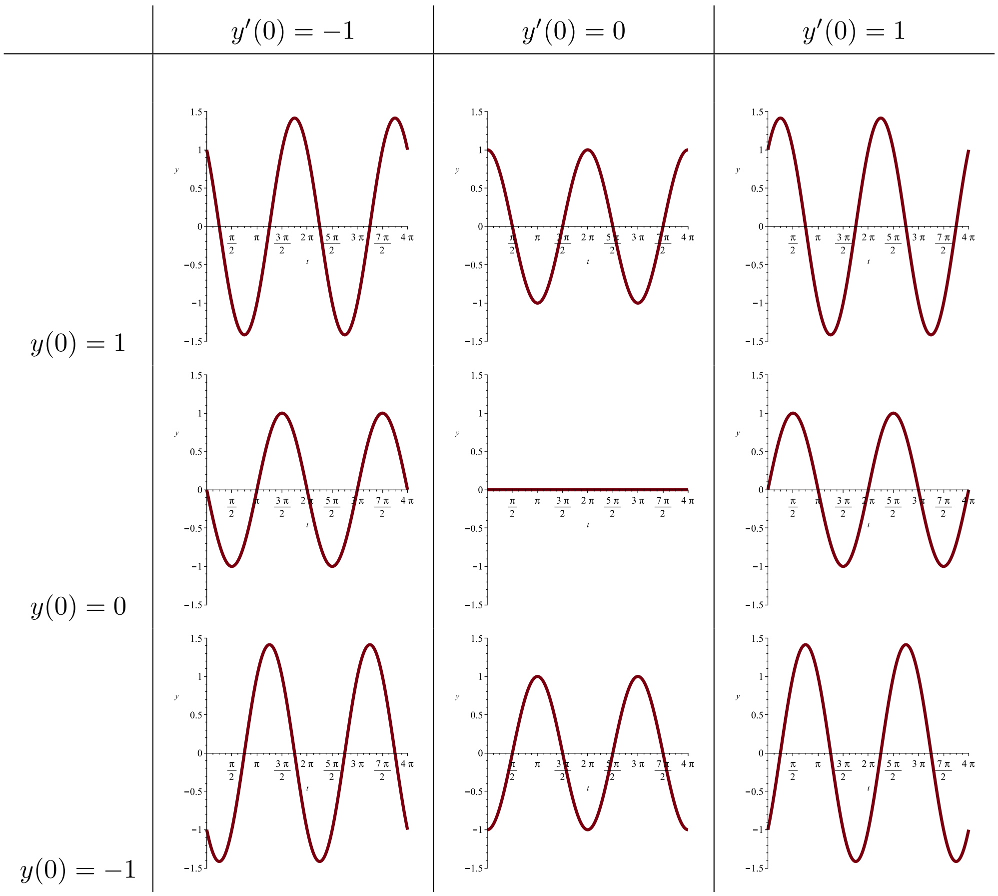

Figure 2. No external force: $y'' + y = 0$ with $y(0) \in \{-1, 0, 1\}$ and $y'(0) \in \{-1, 0, 1\}$

4.1.1 Exercises for students:

- Predict the solution based on the observable physical behavior after watching an in-class demonstration, simulation, or video online [*e.g.* [19, 13], or even [6] for a smartphone project].
- Plot several solutions using a CAS for a variety of ICs. Have the students note any observable trends. [See supplemental MAPLE Worksheet[16].]
- Talk about what happens to y_h as initial position y_0 and/or the initial velocity v_0 are varied.
- Fill in some missing panels in Figure 2, or add a new column or row. [Start with the trivial solution, and build from there.]
- Plot many solutions for a particular parameter range: say $\{y_0 = 0, v_0 \in [-1, 1]\}$ or $\{y_0 \in [0, 1], v_0 = 0\}$.

4.2 Constant Forcing: $y''(t) + y(t) = F_0$, $y(0) = y_0$, $y'(0) = v_0$

How can students think about external forcing? Our favorite image is to consider the external force as a gravitational force - such as might result by adding a (possibly variable) weight to the system, or being able to tune gravity, as suggested both by the modeling process and Figure 1.

We begin with constant forcing, where intuition is based on an additional constant gravitational force. A basic change of variables, $z = y - F_0$, is used to recast the forced problem as an unforced one. Students' understanding of ICs will help them make sense of the necessary shift in ICs for the recast problem. We will call $y_p = F_0$ the *baseline* solution for this problem. Note: It is usual in a PDE setting to homogenize the IC and/or BCs first, and then deal with nonhomogeneous PDE – we are doing just the opposite!

4.2.1 Exercises for students:

- Does the equilibrium point move up or down under the external force $F(t) = 1$? [Up in our vertical frame.]
- What is an intuitive way to think about the external force $F(t) = \pm 1$? [Adjusting gravity or taking/adding weight to the system.]
- Use the method of undetermined coefficients, variation of parameters, or a CAS generate a solution.

  $$[y(t) = \underbrace{(y_0 - F_0)\cos(t) + (v_0)\sin(t)}_{y_h(t;y_0-F_0,v_0)} + \underbrace{F_0}_{y_p}]$$
- What is the change of variable for z to transform $y'' + y = 1$ into a homogenous problem in z? [Let $z(t) = y(t) - 1$. For linear problems in general, let $z(t) = y(t) - F_0$. F_0 can shift the solution up or down.]
- What is the impact that this constant external force has on the initial conditions in the new z homogenous problem? [The ICs become $z(0) = y_0 - F_0$ with $z'(0) = v_0$.]
- Compare the non-homogeneous problem $y'' + y = F_0$, $y(0) = y_0$, $y'(0) = v_0$ with the homogeneous problem $z'' + z = 0$, $z_0 = y_0 - F_0$, $z_0' = v_0$ where $z(t) = y(t) - F_0$. [The dynamics of z should look just like the dynamics of y, but shifted to the new equilibrium F_0, with a corresponding change in amplitude of the *cosine* term. The dynamics of z will be an oscillation about F_0, which is just y_p.]
- Have the students work through a specific example like $y''+y = -2$, $y(0) = 1$, $y'(0) = 0$. They should plot in order: y_p, z, $y = y_p + y_h$ in order to see the effect of $F_0 = -2$. [See supplemental Maple Worksheet[16].]

4.3 Linear Forcing: $y'' + y = F_0 + F_1 t$, $y(0) = y_0$, $y'(0) = v_0$

What if the external forcing is linear? Again we use a baseline solution to understand the forcing. What does $z(t)$ need to be? Letting $z(t) = y(t) - y_p$ transforms the original ODE to the homogeneous ODE $z'' + z = 0$, $z(0) = y_0 - F_0$, $z'(0) = v_0 - F_1$. The initial conditions for z are the original ICs for y offset by the values of the baseline y_p and y_p' at $t = 0$.

4.3.1 Exercises for students:

- Explain how a time dependent external force like $F(t) = -1 + t$ can be imagined. [Gravity pulling down more strongly for $t < 1$ and becoming anti-gravity (pushing up) when $t > 1$.]
- Use the method of undetermined coefficients, variation of parameters, or a CAS generate a solution.
  $$[y(t) = \underbrace{(y_0 - F_0)\cos(t) + (v_0 - F_1)\sin(t)}_{y_h(t;y_0-F_0,v_0-F_1)} + \underbrace{F_0 + F_1 t}_{y_p}.]$$
- Find a baseline (a viable particular solution y_p) curve for $y'' + y = 1 + t$. Why is the baseline important? [With $y_0 = F_0$ and $v_0 = F_1$ the baseline solution is $y_p = F_0 + F_1 t$.]
- Find the change of coordinates that makes $y'' + y = 1 + t$ homogeneous, and interpret the new ICs? [Let $z = y - (1 + t)$. Intuitively, ICs capture a snapshot of the system at a specific time. Note the shift of ICs in two panels of Figure 3.]
- Find the solution to the transformed homogeneous problem $z'' + z = 0$, and plot it with the baseline curve. [See supplemental Maple Worksheet[16].]
- How are y_h and y related? [See Figure 3.]
- What happens to the solution $y(t)$ in the limit as $t \to \infty$? [y_p drags $y(t)$ to infinity, too.]

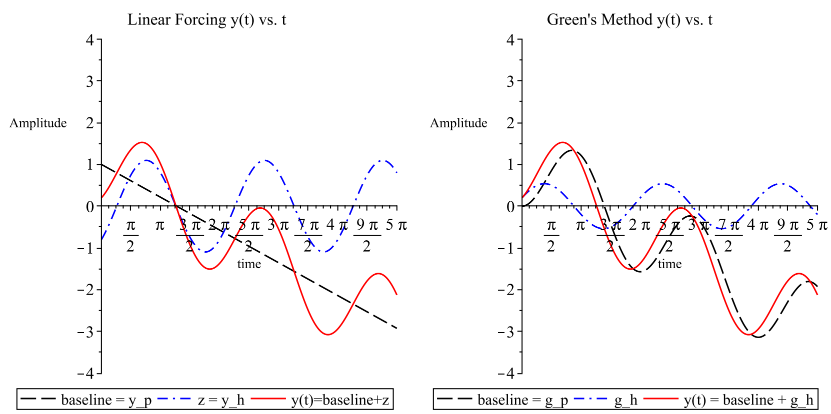

Figure 3. Linear forcing with $F(t) = 1 - \frac{1}{4}t$ and $y_0 = 0.2$ and $v_0 = 0.5$. (a) Plot of y_h, z, and y for linear forcing, (b) Plot of g_h, g_p, and y for linear forcing using Green's function.

4.4 Quadratic Forcing: $y'' + y = F_0 + F_1 t + F_2 t^2$, $y(0) = y_0$, $y'(0) = v_0$

Does anything change with a quadratic external force? The particular solution is more involved since $F''(t) \not\equiv 0$, but the argument is similar. Again, a change of variables suggested by the particular solution gives the standard homogeneous ODE $z'' + z = 0$, although the ICs become slightly more complicated because they reflect the complexity of the linear part of y_p.

4.4.1 Exercises for students:

- Explain how this quadratic force might be imagined. [Another gravity analogy.]
- Use the method of undetermined coefficients, variation of parameters, or a CAS generate a solution. [See supplemental Maple Worksheet[16].]

- What are the choices of y_0 and v_0 that give just a baseline solution? [$y_0 = F_0 - 2F_2, v_0 = F_1$.]
- Find a change of variables to transform this forced problem into a homogenous one. [Use $y_p = -2F_2 + F_0 + F_1 t + F_2 t^2$.]
- Describe the solution y in terms of the baseline and the homogeneous solution. [The shifted homogeneous solution y_h sits on top of the particular solution.]
- What happens to the solution $y(t)$ in the limit as $t \to \infty$? [y_p drags y to infinity.]
- What if $F(t) = F_0 + F_1 t + F_2 t^2 + F_3 t^3$? Can you describe the solution? [The general solution is

  $$\underbrace{C_1 \cos(t) + C_2 \sin(t)}_{y_h} + \underbrace{[(-2F_2 + F_0) + (-6F_3 + F_1)t + F_2 t^2 + F_3 t^3]}_{y_p},$$

  with full solution

  $$\cos(t)(y_0 - (-2F_2 + F_0)) + \sin(t)(v_0 - (-6F_3 + F_1)) + y_p.$$

  The graph is a cubic shift from the original solution.]

4.5 Reflection, and Green's function!

Have you noticed that the form of the particular solution makes it increasingly difficult to understand the behavior of the homogenous solution? Is there a different way to identify a particular solution and homogeneous solution? Recall for students what has been done, and ask if they see a way to avoid the shift in the ICs. You might remind them that they started with $y'' + y = F(t)$ and split this problem into two problems:

$$y_h'' + y_h = 0, \text{ with } y(0) = y_0 - y_p(0), y'(0) = v_0 - y_p'(0) \text{ and} \tag{5}$$

$$y_p'' + y_p = F(t) \text{ with } \textbf{no} \text{ prescribed ICs.} \tag{6}$$

We're hopeful that students might come up with this new way to split the problem (maybe after a little coaxing):

$$g_h'' + g_h = 0, \text{ with } g_h(0) = y_0, g_h'(0) = v_0 \text{ and} \tag{7}$$

$$g_p'' + g_p = F(t), \text{ with } g_p(0) = 0, g_p'(0) = 0, \tag{8}$$

which is essentially a Green's function approach! Green's functions are the classic tool to study the influence of forcing on boundary and initial condition linear problems, but are normally reserved as a topic in advanced differential equations courses. Here the idea of Green's function arises quite naturally. We will use g, g_h and g_p to denote these solutions.

4.5.1 Exercise for students:

- Analyze the initial value ODES considered in sections 4.2-4.4 using the analysis suggested by (7) and (8). Which method provides more intuition? [See Figure 3. We prefer the first method in this case, although we will see that both are valuable.]

4.6 Infinite Force: $y'' + y = \tan(t)$, $y(0) = y_0$, $y'(0) = v_0$

Figure 4 contains a plot of the *baseline* solution ($y(0) = 0$ and $y'(0) = -1$) in bold, along with solution curves for a sequence of initial positions with fixed velocity $y'(0) = -1$. Surprisingly, the system acts in some ways as if it were damped! We think this is a very interesting problem, and hope students do, too!

4.6.1 Exercises for students:

- Describe a model where this type of forcing might be relevant. [A near field magnetic force with an inverse square singularity.]
- What happens to the forcing at $t \to \pi/2$, and how do you think this will influence the solution? [Forcing becomes infinite, and it seems

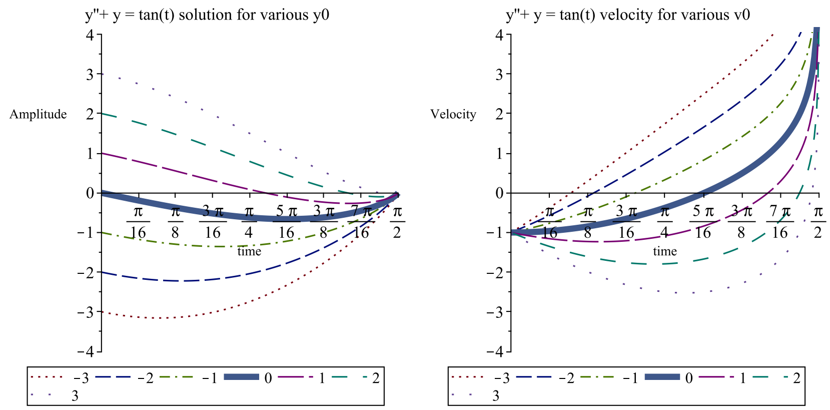

Figure 4. Amplitude and velocity for the solution of $y'' + y = \tan(t)$ with $y(0) \in \{-3, 3\}$ and $y'(0) = -1$. The legend indicates values of $y(0)$, and baseline curve is in bold for $y(0) = 0$.

like the solution should, too. But it doesn't!]

- Construct and plot $y(t)$ (the amplitude) and $y'(t)$ (the velocity) for a variety of ICs.

$$[y(t) = \underbrace{y_0 \cos(t) + \sin(t)(v_0 + 1)}_{y_h(t; y_0, v_0+1)} + \underbrace{\left(-\cos(t) \ln\left(\left|\frac{1 + \sin(t)}{\cos(t)}\right|\right)\right)}_{y_p}$$

with baseline y_p.]

- Where is this solution valid? Can it be extended? [Defined on the interval $(-\pi/2, \pi/2)$, but can be "filled" to be continuously extended. The derivative is discontinuous, however.]
- Do these solutions match original intuition, particularly at $t = \pi/2$? [For arbitrary velocity v_0, $y(\pi/2) = v_0 + 1$. Interactive plots or a sequence of plots might work well here.]
- Identify and use the homogenous and particular solutions to explain the fact that $y(\pi/2) = v_0 + 1$ for arbitrary ICs. [$y_h(\pi/2) = v_0 + 1$ and $y_p(\pi/2) = 0$, so it is only y_h (and the ICs) that determine the

solution at $t = \pi/2$.]

- Explain why $y'(\pi/2) \to \infty$ for any choice of ICs. [$y_h'(\pi/2) = -y_0$, $y_p'(\pi/2) \to \infty$ and so y_p drives the solution to infinity!]
- Can $y'' + y = M(t)$ where $M(t)$ is the Maclaurin polynomial approximation for $\tan(t)$ be used to understand $y'' + y = \tan(t)$? Is there anything to learn from the sequence of particular solutions? Do the ideas of Green's function help? [See supplemental MAPLE Worksheet[16].]
- Identify and explore additional questions that the students have about this solution. [For example: Does this solution represent a possible physical scenario? Why is the particular solution bounded? Can a series representation be used to approximate the forcing? How does this undamped system manage to have features of a damped system?]

4.7 Other ideas

Here are some additional questions students might enjoy:

- Consider $y'' + y = F_0 + F_1 t + F_2 t^2 + ... + F_n t^n$ in terms of what they now know. Why is it that the even terms affect only the $\cos(t)$ term and the odd terms affect only the $\sin(t)$ term?
- Consider the initial value ODE $y''+y = \tan(t+\epsilon)$; $y(0) = y_0$, $y'(0) = v_0$. Analyze similarly and compare the results with the $\tan(t)$. What happens in your solution as $\epsilon \to 0$?
- When solving $y'' + y = \tan(t)$ by hand with variation of parameters, the solution should involve absolute values. Most CAS systems generate the solution without absolute values. What is the difference in these two solutions? Since $\tan(t)$ is an odd function, does the CAS solution make sense? Are both solutions continuous? Are

both solutions differentiable? What is the domain of each solution? Does the extension of the CAS solution oscillate?

- $y'' + y = \sec(t)$. Can students make sense of this model physically? Can they make sense of it mathematically? Encourage them to generate some solutions using variation of parameters for (8). [Another infinite force in finite time.]
- $y''+y = \cos(\omega t)$. (Beats and resonance!) Can students make sense of this model physically? Can they make sense of it mathematically? Ask students to generate and compare intuition from both the naïve analysis allowed by (5) and (6) to the analysis based on (7) and (8)? What happens as $\omega \to 1$? What happens as $\omega \to 2$? [A classic! Try the interactive plot on the supplemental MAPLE Worksheet[16].]
- Convolution and Green's Function: If we seek a particular solution to $y''(t) + y(t) = f(t); \quad y(0) = 0, y'(0) = 0$ via variation of parameters, we see

$$y_p(t) = -\cos t \int_0^t f(\tau)\, \sin\tau \, d\tau + \sin t \int_0^t f(\tau)\, \cos\tau \, d\tau,$$

  which (after combining the two integrals and using a double angle formula) can be written in Green's functional form

$$y_p(t) = \int_0^t f(\tau)\, \sin(t-\tau)\, d\tau, \tag{9}$$

  which is a *convolution* of Green's function (here it is $\sin(t)$) with the forcing $f(t)$. Notice that the variation of parameters method produces a solution to (6). This functional form has many nice properties, and could be the start of many beautiful projects on Green's function and provide intuition about the convolution operator.
- Consider the initial value ODE, $y''+y = \dfrac{1}{(1-t)^r}$; $y(0) = y_0$, $y'(0) = v_0$ using the variation of parameters formula or Green's integral (9)

for values of r with $1 \leq r \leq 2$. For what values of r does the solution exist as $t \to 1$, and why?

5 CONCLUSIONS

We have demonstrated that even for a simple second order ODE, external forcing functions can be quite complicated to understand and can lead to interesting questions and phenomenon. The mathematics can question the physics and vice versa. Both students and instructors will be surprised by what they learn. We were (and are) still amazed that a solution is able to maintain a bounded position even as forcing becomes unbounded, as evidenced in the $\tan(t)$ forcing case!

Looking for the baseline solution in an externally forced ODE can help develop insight and intuition. Considering the particular solution derived using the method of undetermined coefficients as a baseline for the dynamics leads to a mathematically intuitive idea, and is one that works remarkably well in practice. An alternate choice of a particular solution for the baseline is one that is related to Green's function, and corresponds to solving the ODE with non-homogeneous forcing, but with homogeneous initial conditions. In both cases, splitting of the problem into two parts allows for a fairly detailed analysis of the solution, and provides intuition into the ingredients of the problem at hand. The use of Computer Algebra Systems is encouraged, and viewed as a tool of discovery and insight rather than just as a calculator. We hope that the discovery-based process detailed in this article and supplemental MAPLE Worksheet[16] will allow instructors of differential equations to foster intuition and confidence in their students.

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