

# Analysis of an adjoint problem approach to the identification of an unknown diffusion coefficient

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**Abstract.** An inverse problem for the identification of an unknown coefficient in a quasilinear parabolic partial differential equation is considered. We present an approach based on utilizing adjoint versions of the direct problem in order to derive equations explicitly relating changes in inputs (coefficients) to changes in outputs (measured data). Using these equations it is possible to show that the coefficient to data mappings are continuous, strictly monotone and injective. The equations are further exploited to construct an approximate solution to the inverse problem and to analyze the error in the approximation. Finally, results of some numerical experiments are displayed.

**Keywords.** Inverse problems, quasilinear parabolic equations, parameter identification, adjoint method, ill-posed problems

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## 1. Introduction

Using partial differential equations to model physical systems is one of the oldest activities in applied mathematics. A complete model requires certain state inputs in the form of initial and/or boundary data together with what might be called structure inputs such as coefficients or source terms which are related to the physical properties of the system. Obtaining a unique solution for the associated well posed problem constitutes what we will call solving the direct problem. Solving the direct problem permits the computation of various system outputs of physical interest. On the other hand, when some of the required inputs are not available we may instead be able to determine the missing inputs from outputs that are measured rather than computed by formulating and solving an appropriate inverse problem. In particular, when the missing inputs are one or more unknown coefficients in the partial differential equation, the problem is called a coefficient identification problem. The identification of a diffusion coefficient in a quasilinear diffusion equation is chosen here as a prototype coefficient identification problem that has been approached by various methods.

The most common technique for identifying an unknown coefficient from some measured output is the method of output least squares [1, 4, 8, 9, 10]. Here the unknown coefficient,  $C$ , is chosen from an appropriate space  $K$  and the output,  $\Phi[C]$ , is computed by solving the direct problem. One defines an error functional,  $J[C] = \|\Phi[C] - f\|_F^2$ , comparing the computed output to the measured value,  $f$ , in the norm of the output space,  $F$ , and seeks to minimize  $J$  over  $K$ . Output least squares (OLS) methods are very general and can be efficiently programmed for computer implementation. Typically there are problems with lack of uniqueness, convergence to false minima, and instability under parameter mesh refinement, although a skillful user may be able to incorporate a priori information about the solution into the parametric description of the unknown coefficient in order to lessen some of these difficulties [1, 9]. Since the connection between the inputs and outputs is expressed only indirectly through the solver, general information about an input to output mapping is not readily available from OLS methods.

An alternative to coefficient identification by output least squares is the so called equation error method [3, 6, 7, 11, 12]. Here the measured overspecification is used as input to the differential equation in the direct problem which is viewed then as an equation for the unknown coefficient. This equation expresses a direct relationship between the unknown coefficient values and the measured data values. Since the relationship is frequently quite complicated, it is not easy to discern from it properties of an input to output mapping. Equation error methods are quite problem dependent and produce varying degrees of success.

The method described in this paper is based on an integral equation relating changes in the unknown coefficient to corresponding changes in the measured output. The integral equation is derived by exploiting a problem which is adjoint to the direct problem, an idea close to the techniques often used to estimate sensitivity in the OLS approach [8, 9]. However, this integral equation provides information about the input/output mapping itself rather than the error functional. It is possible then to prove that the input to output map is continuous, monotone and injective. Moreover, it is shown that when the input/output map is restricted to a (finite dimensional) space of polygonal coefficients, it is explicitly invertible. This observation provides the basis for a method for numerically approximating the unknown coefficient. It is shown that a unique polygonal approximation to the unknown coefficient is obtained by solving a triangular system of linear algebraic equations. Error estimates show that the accuracy of the approximation is limited by the precision of the data measurements so that there is an optimal attainable accuracy but exact determination of the coefficient is never possible.

Results of a few numerical experiments are provided here to illustrate the working of the method. A more extensive presentation of numerical experiments will be included in a later publication.

## 2. Analysis of the Direct and Inverse Problems

Consider the following IBVP for a quasilinear conduction/diffusion equation on the domain  $Q_T = \{0 < x < 1, 0 < t < T\}$ ,

$$\begin{aligned} \partial_t u(x, t) &= \partial_x(D(u)\partial_x u(x, t)) = \partial_{xx} B(u(x, t)) & \text{on } Q_T \\ u(x, 0) &= f(0) & 0 < x < 1 \\ u(0, t) &= f(t); \quad \partial_x u(1, t) = 0, & 0 < t < T. \end{aligned} \quad (2.1)$$

Here 
$$B(u) = \int_{f(0)}^u D(s) ds$$

and we suppose

$$f \in C^1[0, T] \text{ and } f'(t) > 0 \text{ for } t > 0 \quad (2.2)$$

For  $f$  satisfying (2.2), we let  $J = [f(0), f(T)]$ , and then suppose that for positive constants,  $D_* \leq D^*$  and  $K$ ,

$$\begin{aligned} D_* \leq D(u) &\leq D^*, & \text{for } u \in J & \text{(i)} \\ |D(\mu_2) - D(\mu_1)| &\leq K|\mu_2 - \mu_1| \quad \forall \mu_1, \mu_2 \in J & \text{(ii)} \end{aligned} \quad (2.3)$$

Note that any polygonal function (i.e., a continuous and piecewise linear function) satisfies (2.3i) and that any function satisfying (2.3ii) is bounded away from zero and has at most finitely many zeroes on  $J$ .

Given  $f$  satisfying (2.2) and  $D(u)$  satisfying (2.3), the so called direct problem (2.1) has a unique weak solution  $u = u(x, t)$  satisfying

$$\begin{aligned} u &\in L^2[0, T : H^1(0, 1) \cap C[0, T : L^2(0, 1)] \\ \partial_t u &\in L^2[0, T : H^{-1}(0, 1)]. \end{aligned}$$

Here, we consider the inverse problem in which the coefficient  $D = D(u)$  is to be identified from measured output data. There are a variety of output measurements that are experimentally feasible in any given physical setting; we are going to base our identification on one or the other of the following observations at the boundary,

$$\begin{aligned} g(t) &= -D(u(0, t))\partial_x u(0, t) \quad \text{or} \\ h(t) &= u(1, t), \quad 0 < t < T \end{aligned}$$

If we denote the class of uniformly positive, Lipschitz coefficients  $D$  satisfying (2.3) by  $W(J)$ , then for a fixed  $f$  satisfying (2.2), we can define mappings

$$\begin{aligned} \Phi \text{ and } \Psi &: W(J) \longrightarrow L^2[0, T] \\ \Phi[f, D] &= g \\ \Psi[f, D] &= h, \end{aligned}$$

which assign to a coefficient  $D$  from  $W(J)$ , the flux data,  $g$  or the function value data,  $h$ , obtained by solving the direct problem (2.1) with inputs  $f$  and  $D$ . Then solving the inverse problem will amount to inverting these mappings.

We begin with a result about the IBVP (2.1).

**Lemma 2.1.** Suppose  $f$  and  $D$  satisfy (2.2) and (2.3) and let  $u = u(x, t)$  denote the corresponding solution of (2.1). Then

- a) for each  $t \in (0, T)$ ,  $f(0) \leq u(x, t) \leq f(t)$ ,  $0 \leq x \leq 1$   
b)  $\partial_x u(x, t) < 0$  a.e. on  $Q_T$

*Proof.* It follows from (2.1) that

$$\begin{aligned} \partial_t[f(t) - u(x, t)] - \partial_{xx}[B(f(t)) - B(u(x, t))] &= f'(t) \quad \text{on } Q_T \\ f(0) - u(x, 0) &= 0 \quad 0 < x < 1 \\ f(t) - u(0, t) &= 0, \quad 0 < t < T, \\ \partial_x[f(t) - u(1, t)] &= 0, \quad 0 < t < T. \end{aligned}$$

Then we multiply the equation by an arbitrary test function,  $\psi(x, t)$ , and integrate by parts,

$$\begin{aligned} & - \int \int_{Q_T} [(f - u)\partial_t\psi + (B(f) - B(u))\partial_{xx}\psi] \, dxdt \\ & + \int_0^1 (f - u)\psi|_{t=0}^{t=T} \, dx \\ & - \int_0^T [\psi\partial_x(B(f) - B(u)) - (B(f) - B(u))\partial_x\psi]_{x=0}^{x=1} \, dt \\ & = \int \int_{Q_T} f'(t)\psi \, dxdt. \end{aligned}$$

Note that  $B(f(t)) - B(u(x, t)) = k(x, t)(f - u)$  where we define  $k(x, t) = D(\mu(x, t))$  for  $\mu(x, t)$  between  $f(t)$  and  $u(x, t)$ . Next we require  $\psi(x, t)$  to solve the adjoint problem,

$$\begin{aligned} \partial_t\psi(x, t) + k(x, t)\partial_{xx}\psi(x, t) &= F(x, t) && \text{in } Q_T, \\ \psi(x, T) &= 0, && 0 < x < 1, \\ \psi(0, t) &= 0, \quad \partial_x\psi(1, t) = 0, && 0 < t < T, \end{aligned}$$

for a smooth function,  $F(x, t)$ . Then the integral expression above reduces to

$$- \int \int_{Q_T} (f - u) F(x, t) \, dxdt = \int \int_{Q_T} f'(t)\psi(x, t) \, dxdt \quad (2.4)$$

The smoothness of  $k(x, t)$  and  $F(x, t)$  imply that the strong maximum principle can be applied to the adjoint problem to conclude that if the function  $F(x, t)$  is positive in  $Q_T$ , then  $\psi(x, t) < 0$  in  $Q_T$ . Since  $f$  satisfies (2.2), it follows that for every function  $F(x, t)$  which is positive in  $Q_T$  the right side of (2.4) is negative, which is to say, for every  $F(x, t)$ , smooth and positive in  $Q_T$ ,

$$\int \int_{Q_T} (f - u) F(x, t) \, dxdt > 0.$$

But this is just the assertion that  $f(t) - u(x, t)$  is positive in the sense of distributions on  $Q_T$ . Given the smoothness of the solution  $u(x, t)$  this means  $f(t) > u(x, t)$  almost

everywhere on  $Q_T$ . Applying the same reasoning to  $u(x, t) - f(0)$ , we arrive at the expression

$$- \int \int_{Q_T} (u(x, t) - f(0)) F(x, t) dx dt = \int_0^T B(f(t)) \partial_x \psi(0, t) dt.$$

where we again use that  $\psi(x, t) < 0$  in  $Q_T$  if the function  $F(x, t)$  is positive in  $Q_T$ . Now this fact, together with the adjoint boundary conditions imply that  $\partial_x \psi(0, t) < 0$ , for  $0 < t < T$ . Then the conclusion follows as before. This completes the proof of (a).

To prove (b), multiply both sides of (2.1) by  $\partial_x \phi(x, t)$  for an arbitrary test function  $\phi(x, t)$  and use integration by parts to arrive at

$$\begin{aligned} 0 &= \int \int_{Q_T} \partial_x u [\partial_t \phi + D(u) \partial_{xx} \phi] dx dt \\ &\quad + \int_0^T \phi \partial_t u|_{x=0}^{x=1} dt - \int_0^1 \phi \partial_x u|_{t=0}^{t=T} dx \\ &\quad - \int_0^T \partial_x \phi \partial_x B(u)|_{x=0}^{x=1} dt. \end{aligned}$$

Now require that  $\phi(x, t)$  satisfies the adjoint problem

$$\begin{aligned} \partial_t \phi(x, t) + D(u(x, t)) \partial_{xx} \phi(x, t) &= F(x, t) && \text{in } Q_T, \\ \phi(x, T) &= 0, && 0 < x < 1, \\ \partial_x \phi(0, t) &= 0, \quad \phi(1, t) = 0, && 0 < t < T. \end{aligned}$$

Then the preceding integral expression reduces to

$$\int \int_{Q_T} \partial_x u(x, t) F(x, t) dx dt = \int_0^T \phi(0, t) f'(t) dt$$

The maximum principle can be applied to the adjoint problem to conclude that  $\phi(x, t) < 0$  in  $Q_T$  if the continuous function  $F(x, t)$  is positive in  $Q_T$ . In particular,  $\phi(0, t) < 0$  for  $0 < t < T$  and since  $f$  satisfies (2.2), it follows that for every function  $F(x, t)$  which is positive in  $Q_T$  the right side of the expression is negative. Then it follows as in the proof of part a) that  $\partial_x u(x, t) < 0$  almost everywhere in  $Q_T$ .  $\square$

The results of this lemma are crucial to the proof of,

**Lemma 2.2.** Suppose  $f$  satisfies (2.2) and  $D_1, D_2$  both satisfy (2.3). Then if  $D_1(u) > D_2(u)$  for  $u \in J = [f(0), f(T)]$  it follows that,

- (a)  $\Phi[f, D_1](t) > \Phi[f, D_2](t), \quad 0 < t < T,$
- (b)  $\Psi[f, D_1](t) < \Psi[f, D_2](t), \quad 0 < t < T.$

*Proof.* For  $w \in J$ , let  $B'_j(w) = D_j(w), j = 1, 2.$ , and let  $u_1, u_2$  denote the solutions for the direct problem with coefficients  $D_1, D_2$ , respectively. Then

$$\partial_t(u_1 - u_2) - \partial_{xx}(B_1(u_1) - B_1(u_2)) = \partial_{xx}(B_1(u_2) - B_2(u_2))$$

and, for an arbitrary test function  $\phi = \phi(x, t)$ , and arbitrary  $\tau$ ,  $0 < \tau \leq T$ ,

$$\begin{aligned} & \int_0^\tau \int_0^1 [\partial_t(u_1 - u_2) - \partial_{xx}(B_1(u_1) - B_1(u_2))] \phi \, dx \, dt \\ &= \int_0^\tau \int_0^1 \phi \partial_{xx}(B_1(u_2) - B_2(u_2)) \, dx \, dt. \end{aligned}$$

Apply integration by parts on the left side of this equation,

$$\begin{aligned} & \int_0^\tau \int_0^1 [\partial_t(u_1 - u_2) - \partial_{xx}(B_1(u_1) - B_1(u_2))] \phi \, dx \, dt \\ &= - \int_0^\tau \int_0^1 (u_1 - u_2) \{ \partial_t \phi + D_1(\mu(x, t)) \partial_{xx} \phi \} \, dx \, dt \\ & \quad + \int_0^1 (u_1 - u_2) \phi \Big|_{t=0}^{t=\tau} \, dx \\ & \quad - \int_0^\tau [\phi \partial_x (B_1(u_1) - B_1(u_2)) - \partial_x \phi (B_1(u_1) - B_1(u_2))]_{x=0}^{x=1} \, dt, \end{aligned}$$

and on the right side as well,

$$\begin{aligned} & \int_0^\tau \int_0^1 \phi \{ \partial_{xx}(B_1(u_2) - B_2(u_2)) \} \, dx \, dt \\ &= \int_0^\tau [\phi \partial_x (B_1(u_2) - B_2(u_2))]_{x=0}^{x=1} \, dt \\ & \quad - \int_0^\tau \int_0^1 (D_1(u_2) - D_2(u_2)) \partial_x \phi \partial_x u_2 \, dx \, dt \end{aligned}$$

where for all  $(x, t) \in Q_\tau$ ,  $\mu(x, t)$  lies between  $u_1(x, t)$  and  $u_2(x, t)$  such that for  $(x, t) \in Q_\tau$

$$B_1(u_1(x, t)) - B_1(u_2(x, t)) = D_1(\mu(x, t)) [u_1(x, t) - u_2(x, t)].$$

Then we obtain the following integral expression,

$$\begin{aligned} & - \int_0^\tau \int_0^1 (u_1 - u_2) \{ \partial_t \phi + D_1(\mu(x, t)) \partial_{xx} \phi \} \, dx \, dt \\ & \quad + \int_0^1 (u_1 - u_2) \phi \Big|_{t=0}^{t=\tau} \, dx \\ & \quad - \int_0^\tau [\phi \partial_x (B_1(u_1) - B_2(u_2)) - \partial_x \phi (B_1(u_1) - B_1(u_2))]_{x=0}^{x=1} \, dt \\ &= - \int_0^\tau \int_0^1 (D_1(u_2) - D_2(u_2)) \partial_x \phi \partial_x u_2 \, dx \, dt. \end{aligned}$$

The boundary and initial conditions of the direct problem cause this expression to reduce to,

$$\begin{aligned}
& - \int_0^\tau \int_0^1 (u_1 - u_2) \{ \partial_t \phi + D_1(\mu(x, t)) \partial_{xx} \phi \} dx dt \\
& + \int_0^1 (u_1 - u_2)(x, \tau) \phi(x, \tau) dx \\
& + \int_0^\tau \phi(0, t) \partial_x (B_1(u_1) - B_2(u_2)) dt \\
& - \int_0^\tau \partial_x \phi(1, t) [B_1(u_1) - B_1(u_2)] dt \\
= & - \int_0^\tau \int_0^1 (D_1(u_2) - D_2(u_2)) \partial_x \phi \partial_x u_2 dx dt
\end{aligned} \tag{2.5}$$

Now require the arbitrary function  $\phi(x, t)$  to solve the so-called g-adjoint problem,

$$\begin{aligned}
\partial_t \phi + D_1(\mu(x, t)) \partial_{xx} \phi &= 0 && \text{in } Q_\tau \\
\phi(x, \tau) &= 0 && 0 < x < 1, \\
\phi(0, t) &= \theta(t), && 0 < t < \tau, \\
\partial_x \phi(1, t) &= 0, && 0 < t < \tau,
\end{aligned} \tag{2.6}$$

where  $\theta(t) = F(\tau - t)$  and  $F$  is any function satisfying (2.2). Then (2.5) reduces to

$$\int_0^\tau \theta(t) [g_1(t) - g_2(t)] dt = \int_0^\tau \int_0^1 (D_1(u_2) - D_2(u_2)) \partial_x \phi \partial_x u_2 dx dt. \tag{2.7}$$

An argument similar to the one used in the proof of the previous lemma, applied to (2.6), shows that the assumption on the adjoint input,  $\theta$ , implies  $\partial_x \phi(x, t) < 0$  on  $Q_\tau$ . Since  $\partial_x u_2 < 0$  on  $Q_T$  and  $D_1(u_2) > D_2(u_2)$  it follows that the right side of the last expression is positive. Since (2.7) holds for all  $\theta(t) = F(\tau - t)$ , such that  $F$  satisfies (2.2), it follows that

$$\begin{aligned}
& g_1(t) - g_2(t) > 0 \text{ for } 0 < t < T \\
i.e. & \quad g_1(t) = \Phi[f, D_1](t) > \Phi[f, D_2](t) = g_2(t).
\end{aligned}$$

To see that this is true, note first that if  $D_1(u) > D_2(u)$  for  $u \in J$ , then existence of an interval  $(0, t_1)$  with  $g_1(t) \leq g_2(t)$  for  $0 < t < t_1$  is precluded by (2.7) simply by choosing  $\tau = t_1$ . Suppose then that there exists  $t_2 > t_1 > 0$  such that  $g_1(t) \geq g_2(t)$  for  $0 < t < t_1$  and  $g_1(t) \leq g_2(t)$  for  $t_1 < t < t_2$ . Then choosing  $\tau = t_2$  in (2.7) implies that for any admissible  $\theta(t)$ ,

$$\begin{aligned}
\int_{t_1}^{t_2} \theta(t) [g_1(t) - g_2(t)] dt &= \int_{t_1}^{t_2} \int_0^1 (D_1(u_2) - D_2(u_2)) \partial_x \phi \partial_x u_2 dx dt \\
&+ \int_0^{t_1} \int_0^1 (D_1(u_2) - D_2(u_2)) \partial_x \phi \partial_x u_2 dx dt \\
&- \int_0^{t_1} \theta(t) [g_1(t) - g_2(t)] dt.
\end{aligned}$$

Now choose  $\theta(t)$  so as to have

$$\int_0^{t_1} \int_0^1 (D_1(u_2) - D_2(u_2)) \partial_x \phi \partial_x u_2 dx dt \geq \int_0^{t_1} \theta(t) [g_1(t) - g_2(t)] dt$$

in order to obtain a contradiction with (2.7).

Suppose now that we choose  $\phi$  in (2.5) to solve a problem different from (2.6). This problem will be called the h-adjoint problem,

$$\begin{aligned} \partial_t \phi + D_1(\mu(x, t)) \partial_{xx} \phi &= 0 && \text{in } Q_\tau, \\ \phi(x, \tau) &= 0, && 0 < x < 1, \\ \phi(0, t) &= 0, && 0 < t < \tau, \\ D(\mu(1, t)) \partial_x \phi(1, t) &= \beta(t), && 0 < t < \tau. \end{aligned} \tag{2.8}$$

Here, choose  $\beta(t) = F(\tau - t)$  where  $F$  is any function satisfying (2.2). Then (2.5) reduces to

$$\begin{aligned} &\int_0^\tau D(\mu(1, t)) \partial_x \phi(1, t) (u_1(1, t) - u_2(1, t)) dt \\ &= \int_0^\tau \int_0^1 (D_1(u_2) - D_2(u_2)) \partial_x \phi \partial_x u_2 dx dt, \end{aligned}$$

or

$$\int_0^\tau \beta(t) [h_1(t) - h_2(t)] dt = \int_0^\tau \int_0^1 (D_1(u_2) - D_2(u_2)) \partial_x \phi \partial_x u_2 dx dt. \tag{2.9}$$

In this case, the hypotheses on  $\beta(t)$  imply that  $\partial_x \phi(x, t) > 0$  on  $Q_\tau$  and since  $\partial_x u_2 < 0$  and  $D_1(u_2) > D_2(u_2) \quad \forall u_2 \in J$ , it follows that the right side of (2.9) is negative. Since this holds with  $\beta(t) = F(\tau - t)$  for any  $F$  satisfying (2.2), it follows that

$$\Psi[f, D_1](t) = u_1(1, t) < u_2(1, t) = \Psi[f, D_2](t) \quad \text{for } 0 < t < \tau.$$

Finishing the argument as in the previous case, we see that this holds for  $\tau \leq T$ .  $\square$

The conclusions of lemma 2.2 assert that input to output mappings  $\Phi$  and  $\Psi$  are monotone mappings. More precisely, the mapping  $\Phi$  is *isotone* while the mapping  $\Psi$  is an *antitone* mapping.

Now suppose  $D_1(u_1)$  and  $D_2(u_1)$  are any two coefficients, both satisfying (2.3). Let  $u_1(x, t), u_2(x, t)$  denote the solutions of (2.1) when the coefficient is, respectively,  $D_1(u)$  and  $D_2(u)$ , and for  $i = 1, 2$ , let

$$\begin{aligned} g_i(t) &= \Phi[f, D_i] && \text{and} \\ h_i(t) &= \Psi[f, D_i], && 0 < t < T. \end{aligned}$$

Now choose the data in the adjoint problems (2.6) and (2.8) as,

$$\phi(0, t) = \theta(t) = \frac{g_1(t) - g_2(t)}{\|g_1 - g_2\|_{L^2[0, T]}}, \quad \text{in (2.6)}$$



and

$$D_1(\mu(1, t))\partial_x\psi(1, t) = \beta(t) = \frac{h_1(t) - h_2(t)}{\|h_1 - h_2\|_{L^2[0, T]}} \quad \text{in (2.8).}$$

It follows at once from (2.7) that

$$\begin{aligned} \|g_1 - g_2\|_{L^2[0, T]} &\leq \left| \int_0^T \int_0^1 (D_1(u_2) - D_2(u_2)) \partial_x u_2 \partial_x \phi \, dx \, dt \right| \\ &\leq C \|D_1 - D_2\|_\infty \end{aligned}$$

and from (2.9) that

$$\begin{aligned} \|h_1 - h_2\|_{L^2[0, T]} &\leq \left| \int_0^T \int_0^1 (D_1(u_2) - D_2(u_2)) \partial_x u_2 \partial_x \psi \, dx \, dt \right| \\ &\leq C \|D_1 - D_2\|_\infty \end{aligned}$$

Evidently, this is just the assertion that  $\Phi$  and  $\Psi$  are continuous as a function of  $D$  from  $W(J)$  into  $L^2[0, T]$ ; i.e.,

$$\begin{aligned} \|g_1 - g_2\|_{L^2[0, T]} &= \|\Phi(f, D_1) - \Phi(f, D_2)\|_{L^2[0, T]} \leq C \|D_1 - D_2\|_\infty \\ \|h_1 - h_2\|_{L^2[0, T]} &= \|\Psi(f, D_1) - \Psi(f, D_2)\|_{L^2[0, T]} \leq C \|D_1 - D_2\|_\infty. \end{aligned}$$

Having shown that  $\Phi$  and  $\Psi$  are continuous and strictly monotone, one is encouraged to believe that this inverse problem is not so badly ill posed and that  $\Phi$  and  $\Psi$  might be continuously invertible. Such a strong result seems to be unlikely without a simple ordering on the domain and range of these maps but it is at least true that the input/output maps  $\Phi$  and  $\Psi$  are injective as the following lemma shows.

**Lemma 2.3.** For a fixed  $f$  satisfying (2.2) and coefficients  $D_1, D_2 \in W(J)$  let  $g_k(t) = \Phi[f, D_k]$  and  $h_k(t) = \Psi[f, D_k t]$ , for  $k = 1, 2$ .

Then

- a)  $\Phi[f, D_1] = \Phi[f, D_2], 0 < t < T$ , implies  $D_1(u) = D_2(u)$  for  $u \in J$ .
- b)  $\Psi[f, D_1] = \Psi[f, D_2], 0 < t < T$ , implies  $D_1(u) = D_2(u)$  for  $u \in J$ .

*Proof.* Suppose first that  $D_1(f(0)) = D_2(f(0))$ . Now, since  $D_1$  and  $D_2$  both satisfy (2.3), their difference satisfies (2.3) and if these functions are not identical on  $J$  then there exists a positive time  $t_1$ ,  $0 < t_1 \leq T$ , where the difference,  $D_1(f(t)) - D_2(f(t))$  is of one sign on  $[0, t_1]$ . Then lemma 2.1(a) implies  $D_1(u_2(x, t)) - D_2(u_2(x, t))$  is of one sign on  $(0, 1) \times (0, t_1)$ . Using the identity (2.7), we have

$$\begin{aligned} &\int_0^{t_1} \int_0^1 (D_1(u_2(x, t)) - D_2(u_2(x, t))) \partial_x u_2 \partial_x \phi \, dx \, dt \\ &= \int_0^{t_1} (g_1(t) - g_2(t)) \theta(t) \, dt, \end{aligned}$$

where  $\phi$  solves (2.6) with  $\tau = t_1$ . Then the hypotheses imply the right side of this equation vanishes; i.e.,

$$\int_0^{t_1} \int_0^1 (D_1(u_2) - D_2(u_2)) \partial_x u_2 \partial_x \phi \, dx \, dt = 0,$$

and this holds independent of the data  $\theta(t)$  chosen as input to the adjoint problem. It is clearly possible to choose  $\theta(t)$  so that  $\partial_x \phi < 0$  on  $(0, 1) \times (0, t_1)$  and in view of lemma 2.1(b) it is also the case that,  $\partial_x u_2 < 0$  on  $(0, 1) \times (0, t_1)$ . Then the vanishing integral above has an integrand which is of one sign over the domain of integration and vanishes on no positive measure subset of the domain. This contradiction is in opposition to the assumption that  $D_1$  and  $D_2$  are not identical.

If we suppose  $D_1(f(0)) \neq D_2(f(0))$  then it follows that either there is a smallest time  $t_1, 0 < t_1 < T$ , where the difference  $D_1(f(t)) - D_2(f(t))$  is zero, or else  $t_1 = T$  and the difference is of one sign on  $[0, T]$ . In either case, it is evident that  $D_1(f(t)) - D_2(f(t))$  is of one sign on  $[0, t_1], 0 < t_1 \leq T$ , and the argument can be completed as before. A similar argument, using the identity in (2.9), establishes conclusion(b).  $\square$

Formally, we can write,

$$\begin{aligned} (\Phi[f, D_1] - \Phi[f, D_2], \theta)_{L^2} &\stackrel{def}{=} (\delta\Phi[D_1, D_2] \Delta D, \theta)_{L^2} \\ &= \langle \Delta D, {}^t \delta\Phi[D_1, D_2] \theta \rangle_{W(J) \times W(J)^*}. \end{aligned}$$

In view of (2.7),

$$\begin{aligned} (\Phi[f, D_1] - \Phi[f, D_2], \theta)_{L^2} &= \int_0^T (g_1(t) - g_2(t)) \theta(t) \, dt \\ &= \int_0^T \int_0^1 (D_1(u_2) - D_2(u_2)) \partial_x u_2 \partial_x \phi \, dx \, dt, \\ &= \langle \Delta D, {}^t \delta\Phi[D_1, D_2] \theta \rangle_{W(J) \times W(J)^*} \end{aligned}$$

Similarly,

$$\begin{aligned} (\Psi[f, D_1] - \Psi[f, D_2], \beta)_{L^2} &\stackrel{def}{=} (\delta\Psi[D_1, D_2] \Delta D, \beta)_{L^2} \\ &= \langle \Delta D, {}^t \delta\Psi[D_1, D_2] \beta \rangle_{W(J) \times W(J)^*}, \end{aligned}$$

and, referring to (2.9),

$$\begin{aligned} (\Psi[f, D_1] - \Psi[f, D_2], \beta)_{L^2} &= \int_0^T (h_1(t) - h_2(t)) \beta(t) \, dt \\ &= \int_0^T \int_0^1 (D_1(u_2) - D_2(u_2)) \partial_x u_2 \partial_x \psi \, dx \, dt \\ &= \langle \Delta D, {}^t \delta\Psi[D_1, D_2] \beta \rangle_{W(J) \times W(J)^*} \end{aligned}$$

Evidently, (2.7),(2.9) provide realizations for  ${}^t \delta\Phi[D_1, D_2]$  and  ${}^t \delta\Psi[D_1, D_2]$ , the Gateaux derivatives with respect to  $D$  of the mappings  $\Phi$  and  $\Psi$ . It will be shown in the next section that  ${}^t \delta\Phi[D_1, D_2]$  and  ${}^t \delta\Psi[D_1, D_2]$  are invertible in an approximate sense. More precisely we will devise a restriction of the coefficient to data maps that induces a mapping from  $\mathbb{R}$  into  $\mathbb{R}$ . The restriction inherits the strict monotonicity and continuity

from the coefficient to data map hence the restriction defines a homeomorphism from its domain onto its range. Inversion of this mapping leads to an approximate inverse for the coefficient to data map.

### 3. The Approximate Solution of the Inverse Problem

We consider the inverse problem in which the coefficient  $D = D(u)$  is to be identified from data which is assumed to be recorded at fixed nodes  $0 = t_0 < t_1 < \dots < t_N = T$  in the interval  $[0, T]$ :

$$\text{data}(f, g) \begin{cases} f(t_k) = \mu_k \\ g(t_k) = -D(\mu_k) \partial_x u_1(0, t_k) = \gamma_k, \end{cases} \quad k = 0, 1, \dots, N$$

We are also interested in the identification of  $D = D(u_1)$  based on the alternative data,

$$\text{data}(f, h) \begin{cases} f(t_k) = \mu_k \\ h(t_k) = u_1(1, t_k) = \eta_k, \end{cases} \quad k = 0, 1, \dots, N$$

More precisely, we are going use one or the other of these data sets to construct a polygonal (i.e. piecewise linear and continuous) approximation to the unknown coefficient  $D(u)$ . The data set,  $f_k = f(t_k)$ ,  $k = 0, 1, \dots, N$ , is assumed to be given at fixed nodes which define a partition,  $0 = t_0 < t_1 < \dots < t_N = T$ , of the interval  $I = [0, T]$ . This partition of  $I$  will be called the "inner mesh". We then define an associated (but coarser) partition of  $J = [f(0), f(T)]$ , the domain of the coefficient  $D$ . This partition will be called the "outer mesh" and is given by  $f(0) = \mu_0 < \mu_1 < \dots < \mu_M = f(T)$ , ; i.e.,  $\mu_0 = f_0$ , and  $\mu_M = f_N$  and for each  $j = 1, \dots, M < N$ , we have  $\mu_j = f_k$  for some  $k \geq j$ .

It is necessary for the outer mesh to be coarser than the inner mesh since on each subinterval in the outer mesh, we will need to compute interior values of the solution  $u(x, t)$ , for the direct problem in order to be able to evaluate the integrals which appear in the identities used in the identification. Between two outer mesh knots  $\mu_j = f(t_k)$  and  $\mu_{j+1}$ , there must occur several inner mesh knots and this fact prevents the outer mesh from being made arbitrarily fine in order to improve the accuracy of the identification.

We can now consider a family of polygonal functions,  $\hat{D}$ , associated with the partition of  $J$ . Each member of the family is characterized by its values at the nodes  $\mu_k$ ; i.e., for  $d_k = \hat{D}(\mu_k)$ ; More precisely, we define

$$\hat{D}(u) = \sum_{k=1}^M [d_{k-1} \rho_{k-1}(u) + d_k \lambda_k(u)] \quad (3.1)$$

where

$$\rho_k(u) = \begin{cases} \frac{\mu_{k+1} - u}{\mu_{k+1} - \mu_k} & \text{if } \mu_k \leq u \leq \mu_{k+1} \\ 0 & \text{otherwise} \end{cases} \quad 1 \leq k \leq M-1$$

$$\lambda_k(u) = \begin{cases} \frac{u - \mu_{k-1}}{\mu_k - \mu_{k-1}} & \text{if } \mu_{k-1} \leq u \leq \mu_k \\ 0 & \text{otherwise} \end{cases} \quad 1 \leq k \leq M.$$

Equivalently, we could write  
for  $1 \leq k \leq M$ ,

$$\hat{D}(u) = d_{k-1} \rho_{k-1}(u) + d_k \lambda_k(u) \text{ for } \mu_{k-1} \leq u \leq \mu_k \quad (3.2)$$

We will introduce several notations:

- $\hat{D}(u) = P_M[d_0, d_1, \dots, d_M]$  denotes the polygonal coefficient given by (3.1) based on nodal values  $[d_0, d_1, \dots, d_M]$ .
- $u(x, t; D, f)$  denotes the solution of the direct problem (2.1) with coefficient  $D$  and data,  $f$ .
- $\phi(x, t, D, \theta)$  denotes the solution of the adjoint problem (2.6) with coefficient  $D(x, t) \stackrel{\text{def}}{=} D(\mu(x, t))$  and data,  $\theta(t)$ .
- $\psi(x, t, D, \beta)$  denotes the solution of the adjoint problem (2.8) with coefficient  $D(x, t) \stackrel{\text{def}}{=} D(\mu(x, t))$  and data,  $\beta(t)$ .

For a given  $f(t)$  satisfying (2.2), an unknown coefficient  $D = D(u)$  satisfying (2.3) and measured flux data  $g(t) = \Phi[f, D]$ , we assume there is a fixed outer partition,  $\Pi = \{0 = \mu_0 < \mu_1 < \dots < \mu_M = f(T)\}$  of  $J$ . Then we will define a polygonal coefficient approximation to  $D$  by the following recursive algorithm based on  $(f, g) - \text{data}$ ,  $\{f(t), g(t)\}$  :

- $d_0$  is assumed to be given
- for  $k = 1, 2, \dots, d_k$  is determined from  $d_0, d_1, \dots, d_{k-1}$  by
- $(d_k - d_{k-1}) \int_{T_{k-1}}^{T_k} \lambda_k(u_2) \partial_x u_2 \partial_x \phi dx dt = - \int_{T_{k-1}}^{T_k} (g(t) - g_2(t)) \theta(t) dt,$  (3.3)

where

$$\begin{aligned} D_1(u) &= P_M[d_0, d_1, \dots, d_{k-1}, d_k] \\ D_2(u) &= P_M[d_0, d_1, \dots, d_{k-1}, d_{k-1}] \\ u_2(x, t) &= u(x, t; D_2, f), \\ g_2(t) &= -D_2(f(t)) \partial_x u_2(0, t) \quad 0 \leq x \leq 1, \quad 0 \leq t \leq T_k, \\ \phi(x, t) &= \phi(x, t; D_1, f(T-t)), \quad \text{for } 0 \leq x \leq 1, \quad 0 \leq t \leq T_k. \end{aligned}$$

The approximation of  $D(u)$  based on  $(f, h) - \text{data}$ ,  $\{f(t), h(t)\}$ , is analogous. We can show then,

**Lemma 3.1.** For  $f(t)$  satisfying (2.2), for coefficient  $D$  satisfying (2.3) and for a fixed partition,  $\Pi = 0 = \mu_0 < \mu_1 < \dots < \mu_M = f(T)$  of  $J$ , let the nodal values  $[d_0, d_1, \dots, d_M]$  be determined by the algorithm (3.3). Then for  $k = 1, 2, \dots, M$ ,

$$|D(\mu_k) - d_k| \leq C |\mu_k - \mu_{k-1}| \quad (3.4)$$

*Proof.* We are going to assume that the initial nodal value,  $D(\mu_0) = D(f(0)) = d_0$ , is known and that the remaining values  $d_1, \dots, d_M$  are determined by the algorithm (3.3). Consider first, the value  $d_1$ . If we apply the identity (2.7) with  $\tau = T_1$ , and

- on  $J_1 = [\mu_0, \mu_1]$ ,  $D_1(u) = P_M[d_0, d_1]$ , and  $D_2(u) = P_M[d_0, d_0]$ ,
- on  $Q_1 = (0, 1) \times (0, T_1)$   $u_1(x, t) = u(x, t; D_1, f)$  and  $u_2(x, t) = u(x, t; D_2, f)$

then we have

$$\int_0^{T_1} \int_0^1 (D_1(u_2) - D_2(u_2)) \partial_x u_2 \partial_x \phi \, dx \, dt = \int_0^{T_1} (g(t) - g_2(t)) \theta(t) \, dt.$$

Here  $g(t)$  is the measured flux data and  $g_2(t)$  is the output generated by solving (2.1) with the coefficient  $D(u) = D_2(u)$ ; i.e.,  $g_2 = \Phi[f, D_2]$ . The functions  $\theta(t)$  and  $\phi(x, t)$  denote the data and solution respectively for the g-adjoint problem. Since the function  $f(t)$  in the direct problem satisfies (2.2), it follows from Lemma 2.1(a) that  $u_2$  satisfies

$$f(0) = \mu_0 \leq u_2(x, t) \leq \mu_1 = f(T_1) \quad \text{for } (x, t) \in (0, 1) \times (0, T_1).$$

Then according to (3.2) for  $u \in J_1 = \mu_0 \leq u \leq \mu_1$ ,

$$\begin{aligned} D_1(u) &= d_0 \rho_0(u) + d_1 \lambda_1(u) \\ D_2(u) &= d_0 \rho_0(u) + d_0 \lambda_1(u) = d_0 \quad . \end{aligned}$$

and so  $D_1(u_2) - D_2(u_2) = (d_1 - d_0) \lambda_1(u_2)$ .

Note that for each nodal value,  $\mu_k$ ,  $0 \leq k \leq M$ , we have  $u_2(x_k(t), t) = \mu_k$  along some curve  $x = x_k(t)$ , with  $x_k(0) = \mu_k$  and  $x_k(\tau_k) = 1$  for some  $\tau_k > \tau_{k-1} > \dots > \tau_1 > 0$ . Examples of such curves are shown in Figure (1).

Then we have  $u_2(x(t), t) = \mu_0$  along a curve  $x = x_0(t)$ , with  $x_0(0) = 0$  and  $x_0(\tau_1) = 1$  for some  $\tau_1 > 0$ . We suppose further that  $T_1$  is sufficiently small that  $0 < x_0(T_1) < 1$ . Then

$$\lambda_1(u_2(x, t)) = \left\{ \begin{array}{ll} \frac{\mu_0 - u_2(x, t)}{\mu_0 - \mu_1} & \text{if } 0 \leq x \leq x_0(t), 0 \leq t \leq T_1 \\ 0 & \text{if } x > x_0(t), 0 \leq t \leq T_1 \end{array} \right\}$$

and the integral identity reduces to

$$(d_1 - d_0) \int_0^{T_1} \int_0^{x_0(t)} \lambda_1(u_2) \partial_x u_2 \partial_x \phi \, dx \, dt = \int_0^{T_1} (g(t) - g_2(t)) \theta(t) \, dt;$$

i.e.,

$$d_1 = d_0 + \frac{\int_0^{T_1} (g(t) - g_2(t)) \theta(t) \, dt}{\int_0^{T_1} \int_0^{x_0(t)} \lambda_1(u_2) \partial_x u_2 \partial_x \phi \, dx \, dt}.$$

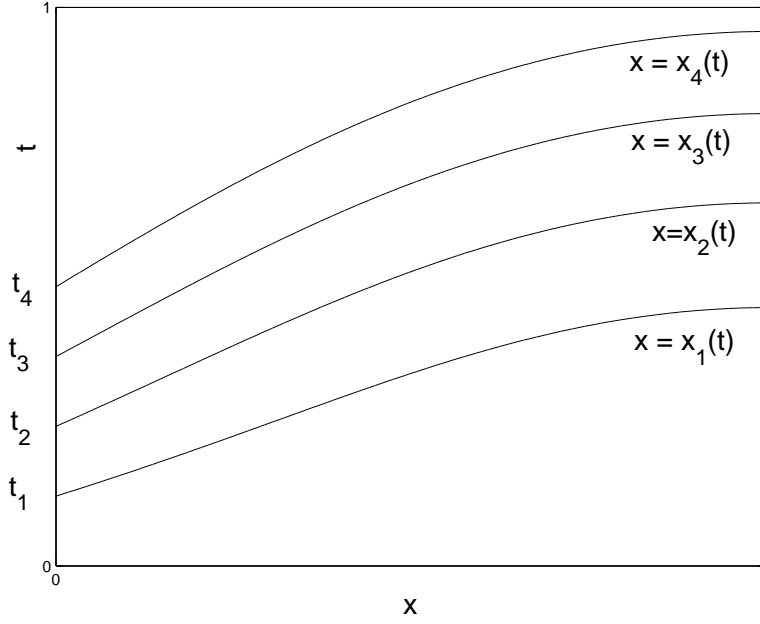


Figure 1:

This equation defines the first unknown nodal value  $d_1$ . Now we will establish the relationship between  $d_1$  and  $D(\mu_1)$ . It follows from (2.7) that

$$\begin{aligned}
 & \int_0^{T_1} \int_0^{x_0(t)} (D(u_2) - D_2(u_2)) \partial_x u_2 \partial_x \phi \, dx \, dt \\
 &= \int_0^{T_1} (g(t) - g_2(t)) \theta(t) \, dt \\
 &= (d_1 - d_0) \int_0^{T_1} \int_0^{x_0(t)} \lambda_1(u_2) \partial_x u_2 \partial_x \phi \, dx \, dt.
 \end{aligned}$$

Let  $\hat{D}_M(u)$  denote the polygonal coefficient on the partition  $\Pi$  which satisfies  $\hat{D}_M(\mu_k) = D(\mu_k)$  for all  $k$ . Note that this coefficient does not, in general, generate the given measured data,  $g(t)$ , and is not then the polygonal coefficient with nodal values  $\{d_k\}$  generated by the algorithm. However, these coefficients are related as follows,

$$\begin{aligned}
 & \int_0^{T_1} \int_0^{x_0(t)} (D(u_2) - D_2(u_2)) \partial_x u_2 \partial_x \phi \, dx \, dt \\
 &= \int_0^{T_1} \int_0^{x_0(t)} (D(u_2) - \hat{D}_M(u_2)) \partial_x u_2 \partial_x \phi \, dx \, dt \\
 & \quad + \int_0^{T_1} \int_0^{x_0(t)} (\hat{D}_M(u_2) - D_2(u_2)) \partial_x u_2 \partial_x \phi \, dx \, dt \\
 &= \int_0^{T_1} \int_0^{x_0(t)} (D(u_2) - \hat{D}_M(u_2)) \partial_x u_2 \partial_x \phi \, dx \, dt
 \end{aligned}$$

$$+ (D(\mu_1) - d_0) \int_0^{T_1} \int_0^{x_0(t)} \lambda_1(u_2) \partial_x u_2 \partial_x \phi \, dx \, dt.$$

and by combining these two expressions it follows that

$$\begin{aligned} & (d_1 - D(\mu_1)) \int_0^{T_1} \int_0^{x_0(t)} \lambda_1(u_2) \partial_x u_2 \partial_x \phi \, dx \, dt \\ &= \int_0^{T_1} \int_0^{x_0(t)} (D(u_2) - \hat{D}_M(u_2)) \partial_x u_2 \partial_x \phi \, dx \, dt \\ &\leq \max_{J_1} |D - \hat{D}_M| \left| \int_0^{T_1} \int_0^{x_0(t)} \partial_x u_2 \partial_x \phi \, dx \, dt \right| \end{aligned}$$

Now

$$\max_{J_1} |D - \hat{D}_M| = |D(\mu_*) - \hat{D}_M(\mu_*)| \quad \text{for some } \mu_* \in J_1.$$

But

$$\begin{aligned} |D(\mu_*) - \hat{D}_M(\mu_*)| &\leq |D(\mu_*) - D(\mu_0)| + |D(\mu_0) - \hat{D}_M(\mu_*)| \\ &\leq K |\mu_* - \mu_0| + |\hat{D}_M(\mu_0) - \hat{D}_M(\mu_*)|. \end{aligned}$$

$$\text{In addition, } |\hat{D}_M(\mu_0) - \hat{D}_M(\mu_*)| \leq K |\mu_* - \mu_0|,$$

and

$$|D(\mu_*) - \hat{D}_M(\mu_*)| \leq 2K |\mu_* - \mu_0|.$$

Then

$$|d_1 - D(\mu_1)| \leq 2K \frac{\left| \int_0^{T_1} \int_0^{x_0(t)} \partial_x u_2 \partial_x \phi \, dx \, dt \right|}{\left| \int_0^{T_1} \int_0^{x_0(t)} \lambda_1(u_2) \partial_x u_2 \partial_x \phi \, dx \, dt \right|} |\mu_* - \mu_0|.$$

Since it is clear that for some  $\lambda_1^*$ ,  $0 < \lambda_1^* < 1$ ,

$$\int_0^{T_1} \int_0^{x_0(t)} \lambda_1(u_2) \partial_x u_2 \partial_x \phi \, dx \, dt = \lambda_1^* \int_0^{T_1} \int_0^{x_0(t)} \partial_x u_2 \partial_x \phi \, dx \, dt$$

we find

$$1 \leq \frac{\left| \int_0^{T_1} \int_0^{x_0(t)} \partial_x u_2 \partial_x \phi \, dx \, dt \right|}{\left| \int_0^{T_1} \int_0^{x_0(t)} \lambda_1(u_2) \partial_x u_2 \partial_x \phi \, dx \, dt \right|} \leq \frac{1}{\lambda_1^*} < \infty.$$

Then,

$$|d_1 - D(\mu_1)| \leq \frac{2K}{\lambda_1^*} |\mu_* - \mu_0| \leq C_1 |\mu_1 - \mu_0|.$$

This is the result (3.4) for  $k = 1$ .

In determining the succeeding values  $d_k$ , we assume  $d_0, d_1, \dots, d_{k-1}$  are known and we let,

- on  $[\mu_0, \mu_k]$ ,  $D_1(u) = P_M[d_0, d_1, \dots, d_{k-1}, d_k]$ ,  
and  $D_2(u) = P_M[d_0, d_1, \dots, d_{k-1}]$ ,
- on  $Q_k = (0, 1) \times (0, T_k)$ ,  $u_1(x, t) = u(x, t; D_1, f)$ ,  
and  $u_2(x, t) = u(x, t; D_2, f)$

Then  $D_1(u)$  and  $D_2(u)$  are identical on  $[\mu_0, \mu_{k-1}]$  and only differ on  $J_k = [\mu_{k-1}, \mu_k]$  where we have

$$\begin{aligned} D_1(u) &= d_{k-1} \rho_{k-1}(u) + d_k \lambda_k(u) & \text{for } \mu_{k-1} \leq u \leq \mu_k, \\ D_2(u) &= d_{k-1} \rho_{k-1}(u) + d_{k-1} \lambda_k(u) = d_{k-1} & \text{for } \mu_{k-1} \leq u \leq \mu_k. \end{aligned}$$

Then

$$\begin{aligned} & \int_0^{T_k} \int_0^1 (D_1(u_2) - D_2(u_2)) \partial_x u_2 \partial_x \phi \, dx \, dt \\ &= \int_{T_{k-1}}^{T_k} \int_0^1 (D_1(u_2) - D_2(u_2)) \partial_x u_2 \partial_x \phi \, dx \, dt \\ &= (d_k - d_{k-1}) \int_{T_{k-1}}^{T_k} \int_0^{x_{k-1}(t)} \lambda_k(u_2) \partial_x u_2 \partial_x \phi \, dx \, dt. \end{aligned}$$

and we have

$$\begin{aligned} & (d_k - d_{k-1}) \int_{T_{k-1}}^{T_k} \int_0^{x_{k-1}(t)} \lambda_k(u_2) \partial_x u_2 \partial_x \phi \, dx \, dt \\ &= \int_{T_{k-1}}^{T_k} (g(t) - g_2(t)) \theta(t) \, dt, \end{aligned}$$

as prescribed by (3.3). Now we proceed as in the first part of the proof to show that

$$|d_k - D(\mu_k)| \leq C |\mu_k - \mu_{k-1}|.$$

The proof of the analogous result based on the data  $\{f(t_k), h(t_k)\}$  proceeds similarly.  $\square$

For  $d_0$  fixed and  $d_1 > 0$ , let  $P_1(d_1)(u) = d_0 \rho_0(u) + d_1 \lambda_1(u)$  for  $u \in J_1$ . Then  $P_1$  is a mapping from  $[0, \infty]$  into a one dimensional subspace of  $W(J_1)$ . It follows from (3.3) in the case  $k = 1$  that

$$\begin{aligned} & \langle \Delta D(u_2), {}^t \delta \Phi [P_1(d_1), P_1(d_0)](\theta) \rangle \\ &= \langle (d_1 - d_0) \lambda_1(u_2), {}^t \delta \Phi [P_1(d_1), P_1(d_0)](\theta) \rangle \\ &= (d_1 - d_0) \int_0^{T_1} \int_0^{x_0(t)} \lambda_1(u_2) \partial_x u_2 \partial_x \phi \, dx \, dt \end{aligned}$$

This means that the double integral in the expression above is a representation for the derivative with respect to the parameter  $d$ , of the coefficient-to-data mapping,  $\Phi$ , restricted to the one dimensional subspace of  $W(J_1)$ . Since the double integral can be shown to be nonzero, it follows that the restricted input/output mapping is locally approximately invertible. Theorem 3.1 asserts that, if we are given the data,  $\{f(t_k), g(t_k)\}$  or  $\{f(t_k), h(t_k)\}$ , then we can compute the nodal values  $\{d_k\}$  which reproduce the measured data in the sense of (3.3) and that these nodal values approach the nodal values of the “true coefficient”  $D(u_1)$ , as the mesh size of the outer mesh decreases. However, this conclusion ignores certain difficulties:



- it is not possible to know the coefficient  $D_1(\mu(x, t))$  in the adjoint problems since  $D_1$  is the coefficient we wish to identify and  $\mu$  is an indeterminate point between  $u_1$  and  $u_2$ . This means we can only approximate the solution to the adjoint problem and this will have an influence on the conclusions of theorem 3.1.
- the integrals in the identity can only be approximated by numerical integrations for which there is only a limited degree of refinement possible. This may further interfere with the agreement between  $d_k$  and  $D(\mu_k)$ .

We will consider both of these effects, starting with the effect of the approximate adjoint solution.

Note first, that the algorithm (3.3) asserts that in determining the nodal value  $\mu_k$ , it is necessary to solve the adjoint problem only on the strip  $S_k = \{(0, 1) \times (T_{k-1}, T_k)\}$ . Let  $\hat{\phi}(x, t)$  denote the adjoint solution we compute using a convenient approximation for the unknown coefficient  $D_1(\mu(x, t))$  on this strip. For example, suppose the coefficient in the g-adjoint problem is chosen to have the known constant value,  $d_{k-1}$ ; i.e.,

$$D_1(\mu(x, t)) = d_{k-1} \quad \mu(x, t) \in J_k = [\mu_{k-1}, \mu_k].$$

Then if we replace  $\phi$  in (3.3) by  $\hat{\phi}(x, t)$ , we can denote the resulting computed nodal value by  $\hat{d}_k$ . Note that with this choice for the coefficient, there is now no difficulty in solving the adjoint problem (2.6) for  $\hat{\phi}$  on the strip,  $(0, 1) \times [T_{k-1}, T_k]$  and proceeding to compute  $\hat{d}_k$  using (3.3). It remains to be seen how the values  $\hat{d}_k$  compare to the values  $d_k$ . We begin with a lemma.

**Lemma 3.2.** Let  $f(t)$  satisfy (2.2), let coefficient  $D$  satisfy (2.3) and let  $\Pi$  denote a fixed partition,  $\Pi = \{\mu_k = f(T_k) : k = 0, 1, \dots, M\}$  of  $J$ . For  $k$  between 1 and  $M$  consider the following adjoint problem,

$$\begin{aligned} \partial_t \phi(x, t) + c \partial_{xx} \phi(x, t) &= 0, & x \in S_k \\ \phi(x, T_k) &= 0, & x \in (0, 1) \\ \phi(0, t) &= f(T_k - t) & t \in (T_{k-1}, T_k) \\ \partial_x \phi(1, t) &= 0, & t \in (T_{k-1}, T_k) \end{aligned}$$

Suppose  $\{\phi_i, c_i\}$ ,  $i = 1, 2$  denote two solutions to the adjoint problem corresponding to distinct choices of the coefficient  $c$ . In particular, suppose  $\phi_1 = \phi(x, t, c_1, \theta)$  for the constant  $c_1 = d_{k-1}$ , while  $\phi_2 = \phi(x, t, c_2, \theta)$  corresponding to the choice,  $c_2(x, t) = D(\mu(x, t))$ , where  $\mu(x, t)$  denotes a function that is continuous on the strip  $S_k = (0, 1) \times (T_{k-1}, T_k)$  with values in  $J_k = [\mu_{k-1}, \mu_k]$ . Then

$$\|\partial_x(\phi_1 - \phi_2)\|_{L^2(S_k)} \leq C |\mu_k - \mu_{k-1}|$$

*Proof.* Begin by noting that  $\Delta\phi = \phi_1 - \phi_2$  satisfies,

$$\begin{aligned} \partial_t(\Delta\phi) + c_1 \partial_{xx}(\Delta\phi) &= (c_2 - c_1) \partial_{xx} \phi_2, & (x, t) \in S_k \\ \Delta\phi(x, T_k) &= 0, & x \in (0, 1) \\ \Delta\phi(0, t) &= 0, & t \in (T_{k-1}, T_k), \\ \partial_x(\Delta\phi)(1, t) &= 0, & t \in (T_{k-1}, T_k), \end{aligned}$$

and if  $\psi$  denotes an arbitrary test function, then

$$\int \int_{S_k} \{\partial_t(\Delta\phi) + c_1 \partial_{xx}(\Delta\phi)\} \partial_x \psi \, dx dt = \int \int_{S_k} \{-\Delta c \partial_{xx} \phi_2\} \partial_x \psi \, dx dt.$$

Integration by parts yields,

$$\begin{aligned} \int \int_{S_k} \partial_t(\Delta\phi) \partial_x \psi \, dx dt &= \int \int_{S_k} \partial_x(\Delta\phi) \partial_t \psi \, dx dt \\ &+ \int_0^1 \Delta\phi \partial_x \psi|_{t=0}^{t=T} \, dx \\ &- \int_{T_{k-1}}^{T_k} \Delta\phi \partial_x \psi|_{x=0}^{x=1} \, dt, \end{aligned}$$

and

$$\begin{aligned} \int \int_{S_k} \partial_{xx}(\Delta\phi) \partial_x \psi \, dx dt &= - \int \int_{S_k} \partial_x(\Delta\phi) \partial_{xx} \psi \, dx dt \\ &+ \int_{T_{k-1}}^{T_k} \partial_x(\Delta\phi) \partial_x \psi|_{x=0}^{x=1} \, dt, \end{aligned}$$

so

$$\begin{aligned} &\int \int_{S_k} \partial_x(\Delta\phi) [\partial_t \psi - c_1 \partial_{xx} \psi] \, dx dt + \int_0^1 \Delta\phi \partial_x \psi|_{t=T_{k-1}}^{t=T_k} \, dx \\ &- \int_{T_{k-1}}^{T_k} \Delta\phi \partial_x \psi|_{x=0}^{x=1} \, dt + c_1 \int_{T_{k-1}}^{T_k} \partial_x(\Delta\phi) \partial_x \psi|_{x=0}^{x=1} \, dt \\ &= \int \int_{S_k} \{-\Delta c \partial_{xx} \phi_2\} \partial_x \psi \, dx dt \end{aligned}$$

Now choose the test function  $\psi$  to satisfy

$$\begin{aligned} \partial_t \psi - c_1 \partial_{xx} \psi &= \partial_x(\Delta\phi), & (x, t) \in S_k \\ \psi(x, T_{k-1}) &= 0, & x \in (0, 1) \\ \partial_x \psi(0, t) &= 0, & t \in (T_{k-1}, T_k) \\ \psi(1, t) &= 0, & t \in (T_{k-1}, T_k) \end{aligned}$$

Then the previous integral identity reduces to

$$\int \int_{S_k} [\partial_x(\Delta\phi)]^2 \, dx dt = \int \int_{S_k} (c_2 - c_1) \partial_{xx} \phi_2 \partial_x \psi \, dx dt.$$

Now,  $\psi$  is the solution to a linear problem with constant coefficients so it can be expressed in terms of a Green's function,  $\Gamma(x, t)$ ,

$$\psi(x, t) = \int_{T_{k-1}}^t \int_0^1 \Gamma(x - y, t - \tau) \partial_x(\Delta\phi)(y, \tau) \, dy d\tau, \quad (x, t) \in S_k,$$

and

$$\partial_x \psi(x, t) = \int_{T_{k-1}}^t \int_0^1 \partial_x \Gamma(x - y, t - \tau) \partial_x (\Delta \phi)(y, \tau) dy d\tau.$$

Then for all  $(x, t) \in S_k$ ,

$$\begin{aligned} |\partial_x \psi(x, t)| &\leq \int_{T_{k-1}}^t \int_0^1 |\partial_x \Gamma(x - y, t - \tau) \partial_x (\Delta \phi)(y, \tau)| dy d\tau, \\ &\leq \left( \int_{T_{k-1}}^{T_k} \int_0^1 |\partial_x \Gamma(x - y, t - \tau)|^2 dy d\tau \right)^{1/2} \cdot \left( \int_{T_{k-1}}^{T_k} \int_0^1 |\partial_x (\Delta \phi)(y, \tau)|^2 dy d\tau \right)^{1/2} \end{aligned}$$

and

$$\max_{(x,t) \in S_k} |\partial_x \psi(x, t)| \leq C \|\partial_x (\Delta \phi)\|_{L^2(S_k)}.$$

Then it follows that

$$\begin{aligned} \int_{S_k} \int [\partial_x (\Delta \phi)]^2 dx dt &= \left| \int_{S_k} \int (c_2 - c_1) \partial_{xx} \phi_2 \partial_x \psi dx dt \right| \\ &\leq \max_{S_k} |\Delta c(x, t)| \int_{S_k} \int |\partial_{xx} \phi_2 \partial_x \psi| dx dt \\ &\leq \max_{S_k} |\Delta c(x, t)| \|\partial_{xx} \phi_2\|_{L^1} \|\partial_x \psi\|_{\infty} \end{aligned}$$

and

$$\|\partial_x (\Delta \phi)\|_{L^2(S_k)} \leq C \max_{S_k} |\Delta c(x, t)|.$$

Also

$$\begin{aligned} \max_{S_k} |\Delta c(x, t)| &= \max_{S_k} |d_{k-1} - D(\mu(x, t))| \\ &\leq |d_{k-1} - D(\mu_{k-1})| + \max_{S_k} |D(\mu_{k-1}) - D(\mu(x, t))| \\ &\leq 2K |\mu_k - \mu_{k-1}|. \end{aligned}$$

Then, it follows that,

$$\|\partial_x (\Delta \phi)\|_{L^2(S_k)} \leq C |\mu_k - \mu_{k-1}|.$$

□

Now we will use this estimate in considering the effect of using the approximate adjoint solution in the determination of the first nodal value,  $d_1$ . It follows from (3.3) that the difference between the value,  $d_1$ , computed using the correct but unknown adjoint solution and the value,  $\hat{d}_1$ , computed using an incorrect but computable adjoint solution is given by,

$$\begin{aligned} \hat{d}_1 - d_1 &= \frac{\int_0^{T_1} (g(t) - g_2(t)) \theta(t) dt}{\int_0^{T_1} \int_0^{x_0(t)} \lambda_1(u_2) \partial_x u_2 \partial_x \hat{\phi} dx dt} \\ &\quad - \frac{\int_0^{T_1} (g(t) - g_2(t)) \theta(t) dt}{\int_0^{T_1} \int_0^{x_0(t)} \lambda_1(u_2) \partial_x u_2 \partial_x \phi dx dt} \\ &= \frac{(g - g_2, \theta)}{II(\hat{\phi})} - \frac{(g - g_2, \theta)}{II(\phi)} \end{aligned}$$

$$= (g - g_2, \theta) \left\{ \frac{1}{II(\hat{\phi})} - \frac{1}{II(\phi)} \right\}$$

$$\hat{d}_1 - d_1 = (d_1 - d_0) \left\{ \frac{II(\phi) - II(\hat{\phi})}{II(\hat{\phi})} \right\}$$

Here

$$II(\hat{\phi}) = \int_0^{T_1} \int_0^{x_0(t)} \lambda_1(u_2) \partial_x u_2 \partial_x \hat{\phi} dx dt.$$

We wish to show that as the outer mesh is refined, the discrepancy  $II(\phi) - II(\hat{\phi})$  that is due to solving the adjoint problem with the wrong coefficient decreases to zero. On the other hand,  $II(\hat{\phi})$  also decreases toward zero as the mesh is refined. To see whether  $II(\hat{\phi})$  decreases more or less rapidly than  $II(\phi) - II(\hat{\phi})$ , it is necessary to examine the asymptotic behavior of  $II(\hat{\phi})$ . We assume that  $x_0(T_1) < 1$  since if this is not the case, we can always refine the outer partition to shrink the width of the strip  $S_1$  so as to make it true. Then the domain of integration for  $II(\hat{\phi})$  is the approximately triangular region  $\{0 \leq x \leq x_0(t), 0 \leq t \leq T_1\}$ . An exact analysis of the asymptotic rate of convergence of  $II(\hat{\phi})$  as  $T_1$  tends to zero is difficult, but if we assume that  $f(t) = At$  for a positive constant  $A$ , then it is possible to solve explicitly for  $u_2(x, t)$  and  $\hat{\phi}(x, t)$ . Using arguments like those in [1], one finds that  $g(t) = -D(u_1(0, t)) \partial_x u_1(0, t)$  and  $g_2(t) = -d_k \partial_x u_2(0, t)$  behave asymptotically like  $\sqrt{t}$ .

This leads to

$$\int_0^{T_1} (g(t) - g_2(t)) \theta(t) dt = \int_0^{T_1} (g(t) - g_2(t)) A(T_1 - t) dt \approx CT_1^{5/2} \quad (3.5)$$

A similar crude estimate for  $\partial_x u_2 \partial_x \hat{\phi}$  on  $0 \leq x \leq 1, 0 \leq t \leq T_1$ , is the following

$$\partial_x u_2 \partial_x \hat{\phi}(x, t) \approx \sqrt{t} m(x) \sqrt{T_1 - t} m(x)$$

where  $m(x)$  denotes a decreasing function with  $m(0) = 1$  and  $m(1) = 0$ . In addition, for  $T_1$  small, one can suppose  $x_0(t) \approx at$  for a positive constant  $a$ , and this leads to

$$II(\hat{\phi}) = \int_0^{T_1} \int_0^{x_0(t)} \lambda_1(u_2) \partial_x u_2 \partial_x \hat{\phi} dx dt \approx \int_0^{T_1} \int_0^{at} \frac{u_2(x, t)}{AT_1} \sqrt{t} m(x) \sqrt{T_1 - t} m(x) dx dt$$

*i.e.*  $II(\hat{\phi}) \approx CT_1^{5/2}$ . (3.6)

Since this estimate (3.6) is rather rough, the quantity  $II(\hat{\phi})$  was computed numerically for a sequence of values for  $T_1$  decreasing to zero. The result of this numerical asymptotic estimate supported the estimate (3.6) which asserts that  $II(\hat{\phi})$  decreases like the  $\frac{5}{2}$  power of  $T_1$  as  $T_1$  tends to zero.

Now

$$\hat{d}_1 - d_1 = (d_1 - d_0) \left\{ \frac{II(\phi) - II(\hat{\phi})}{II(\hat{\phi})} \right\}$$

and,

$$\begin{aligned} \left| II(\phi) - II(\hat{\phi}) \right| &= \left| \int_0^{T_1} \int_0^{x_0(t)} \lambda_1(u_2) \partial_x u_2 [\partial_x \phi - \partial_x \hat{\phi}] dx dt \right| \\ &\leq C(T_1) \|\partial_x(\Delta\phi)\|_{L^2(S_1)} \leq C(T_1) |\mu_1 - \mu_0|. \end{aligned}$$

Also,

$$|d_1 - d_0| = |D(\mu_1) - D(\mu_0)| \leq K |\mu_1 - \mu_0|,$$

and hence

$$\left| \hat{d}_1 - d_1 \right| \leq |d_1 - d_0| \left| \frac{II(\phi) - II(\hat{\phi})}{II(\hat{\phi})} \right| \leq \frac{K C(T_1)}{II(\hat{\phi})} |\mu_1 - \mu_0|^2.$$

Then for  $T_1$  sufficiently small,

$$\begin{aligned} \left| \hat{d}_1 - d_1 \right| &\leq \frac{K C(T_1)}{C T_1^{5/2}} |\mu_1 - \mu_0|^2 \\ &\leq \frac{K f'(\tau)^2}{C} T_1^{-1/2} \text{ for some } \tau > 0. \end{aligned}$$

In general, we have

**Lemma 3.3.** For  $f(t) = At$ ,  $A > 0$ , for coefficient  $D$  satisfying (2.3) and for a fixed partition,  $\Pi = \{\mu_k = AT_k : k = 0, 1, \dots, M\}$  of  $J$ , fix  $k$  between 1 and  $M$ . Let  $\hat{\phi} = \phi(x, t, d_{k-1}, A(T_k - t))$  and  $\phi = \phi(x, t, c, A(T_k - t))$  corresponding to the coefficients,  $d_{k-1}$  and  $c(x, t) = D(\mu(x, t))$ , respectively. Finally, let  $\hat{d}_k$  and  $d_k$  denote the nodal values determined from (3.3) using the values  $[d_0, d_1, \dots, d_{k-1}]$  and the adjoint solutions  $\hat{\phi}$  and  $\phi$ , respectively. Then

$$\left| \hat{d}_k - d_k \right| \leq \frac{K}{II(\hat{\phi})} |\mu_k - \mu_{k-1}|^2 \leq \frac{K f'(\tau)^2}{C} |T_k - T_{k-1}|^{-1/2}.$$

This lemma implies that the error introduced into the identification by solving the adjoint problem with an approximate coefficient has an increasing effect as the outer mesh is refined. As the mesh is refined, the discrepancy  $II(\phi) - II(\hat{\phi})$  does tend to zero like the square of the mesh size. However, as the mesh size tends to zero, we find also that  $II(\hat{\phi})$ , which can be viewed as an approximation to the Gateaux derivative of the mapping  $\Phi$  restricted to a one dimensional subspace of  $W(J_k)$ , tends to zero even faster, (like the  $\frac{5}{2}$  power of the mesh size). It is likely that the means of approximating the adjoint solution could be improved so that  $II(\phi) - II(\hat{\phi})$  would approach zero sufficiently rapidly that  $\left| \hat{d}_k - d_k \right|$  would tend to zero as the mesh size goes to zero. However, the next result will show that such an improvement does not improve the convergence of the approximate solution.

We wish finally to consider the effect of numerical integration errors on the calculation of  $\hat{d}_k$ . We begin by considering  $k = 1$ . We have,

$$\hat{d}_1 = d_0 + \frac{\int_0^{T_1} (g(t) - g_2(t)) \theta(t) dt}{\int_0^{T_1} \int_0^{x_0(t)} \lambda_1(u_2) \partial_x u_2 \partial_x \hat{\phi} dx dt} = d_0 + \frac{I(g - g_2)}{II(\hat{\phi})}$$

and

$$\hat{d}_1^* = d_0 + \frac{I^*(g - g_2)}{II^*(\hat{\phi})}$$

where  $I^*(g - g_2)$  and  $II^*(\hat{\phi})$  denote, respectively, the computed results using the inner mesh to numerically approximate the corresponding exact single and double integrals. Then

$$\begin{aligned}\hat{d}_1^* &= d_0 + \frac{I^*(g - g_2) - I(g - g_2) + I(g - g_2)}{II^*(\hat{\phi}) - II(\hat{\phi}) + II(\hat{\phi})} \\ &= d_0 + \frac{I(g - g_2)}{II(\hat{\phi})} \frac{1 + \varepsilon_1}{1 + \varepsilon_2}\end{aligned}$$

where

$$\varepsilon_1 = \left| \frac{I - I^*}{I} \right| \text{ and } \varepsilon_2 = \left| \frac{II - II^*}{II} \right|.$$

Now

$$\frac{1 + \varepsilon_1}{1 + \varepsilon_2} \approx 1 + \varepsilon_1 + \varepsilon_2$$

so

$$\hat{d}_1^* = d_0 + \frac{I(g - g_2)}{II(\hat{\phi})} \frac{1 + \varepsilon_1}{1 + \varepsilon_2} \approx d_0 + \frac{I(g - g_2)}{II(\hat{\phi})} (1 + \varepsilon_1 + \varepsilon_2),$$

and

$$\left| \hat{d}_1 - \hat{d}_1^* \right| \leq \left| \frac{I(g - g_2)}{II(\hat{\phi})} \right| (\varepsilon_1 + \varepsilon_2) = \left| \hat{d}_1 - d_0 \right| (\varepsilon_1 + \varepsilon_2).$$

The numerical integration errors are estimated by terms of the form,

$$|I - I^*| \leq C(\Delta t)^2 \quad \text{for } \Delta t = \text{inner mesh size}$$

and

$$|II - II^*| \leq C(\Delta x \Delta t) = C(\Delta t)^2$$

Use of higher order integration schemes is limited by the fact that reducing the mesh size of the outer or  $J$ -mesh in order to achieve identification accuracy absorbs  $I$ -mesh node points into the  $J$ -mesh leaving only enough points in the inner mesh to perform low order numerical integrations.

It follows from (3.5) and (3.6) that

$$\begin{aligned}I &= \int_0^{T_1} (g(t) - g_2(t)) A(T_1 - t) dt \approx T_1^{5/2}, \\ II &= \int_0^{T_1} \int_0^{x_0(t)} \lambda_1(u_2) \partial_x u_2 \partial_x \phi dx dt \approx T_1^{5/2}\end{aligned}$$

Then, since  $T_1 = k\Delta t$ , we find

$$\begin{aligned}\left| \hat{d}_1 - \hat{d}_1^* \right| &\leq \left| \frac{I(g - p_M)}{II(\hat{\phi})} \right| (\varepsilon_1 + \varepsilon_2) \\ &\leq \left| \hat{d}_1 - d_0 \right| \frac{C_1(\Delta t)^2}{C_2(k\Delta t)^{5/2}} \leq C(\Delta t)^{-1/2}.\end{aligned}$$

More generally, we have

**Lemma 3.4.** Under the conditions of (3.3), let  $\hat{d}_k^*$  reflect the error induced in  $\hat{d}_k$  by numerically approximating the integrals needed for (3.3). Then, as the (inner and outer) mesh size tends to zero,

$$\left| \hat{d}_k - \hat{d}_k^* \right| \leq C(\Delta t)^{-1/2}$$

This estimate suggests that as the outer mesh is refined in order to improve the accuracy of the identification of the nodal values of  $D(u_1)$ , more and more node points of the inner mesh are absorbed into the outer mesh, resulting in numerical integration

errors,  $|I - I^*|$  and  $|II - II^*|$ , that are of order  $\Delta t^2$ . At the same time, the approximate Gateaux derivative  $II(\hat{\phi})$  tends to zero like  $\Delta t^{5/2}$  so the effect of approximating the integrals becomes magnified at  $\Delta t$  tends to zero. Evidently, at some point the values of the integrals used to compute  $d_k$  become of the same order of magnitude as the numerical integration errors and the computation then no longer contains information. Further decreasing the mesh size then only increases the error.

Finally, we can combine lemmas 3.1, 3.3 and 3.4 to write

$$\begin{aligned} \left| D(\mu_k) - \hat{d}_k^* \right| &= \left| D(\mu_k) - d_k + d_k - \hat{d}_k + \hat{d}_k - \hat{d}_k^* \right| \\ &\leq |D(\mu_k) - d_k| + \left| d_k - \hat{d}_k \right| + \left| \hat{d}_k - \hat{d}_k^* \right| \end{aligned}$$

and,

$$\left| D(\mu_k) - \hat{d}_k^* \right| \leq C_1 \Delta t + C_2 (\Delta t)^{-1/2}. \quad (3.7)$$

Evidently the error in identifying  $d_k$  does not tend to zero as  $\Delta t$  tends to zero but is minimized by an optimal  $\Delta t$  different from zero.

#### 4. Numerical Experiments

In the numerical experiments we describe here, we chose  $f(t) = At$  for some positive constant  $A$  and defined the node points  $\mu_k$  for the outer mesh by  $\mu_k = AT_k$ ,  $k = 0, 1, \dots, M$ . Here, for each  $k$ ,  $T_k = t_j$  for some  $j > k$  where  $0 = t_0 < t_1 < \dots < t_N = T$  denotes the (inner) partition of  $[0, T]$ . The unknown nodal values for the coefficient  $D(u)$  are given by  $d_k = D(\mu_k)$ , and we assume that  $d_0$  is known. Since the initial state for the direct problem,  $u(x, 0)$ , is constant and  $f(t)$  is monotone increasing, the domain  $Q_T$  consists of a sequence of non-overlapping strips,  $S_k$ , with only one nodal value active on each strip. The algorithm to identify  $D$  from the  $(f, g)$  - data then proceeds as follows.

##### The Algorithm

To begin, we apply the  $g$ -integral identity (2.7) on  $Q_1$ . Since the solution of the direct problem satisfies lemma 2.1(a), we have  $\mu_0 \leq u_1(x, t) \leq \mu_1$  for  $(x, t) \in Q_1$ . Then only the known nodal value  $d_0$  and the unknown nodal value  $d_1$  are active on this strip. We are going to compute the unknown nodal values iteratively and we denote the  $i$ -th iteration for  $d_k$  by  $d_k^{(i)}$ . We set  $d_1^{(0)} = d_0$ .

We apply the integral identity (2.7) on  $Q_1$  with,

$$\begin{aligned} D_1 &= P_1 \left[ d_0, d_1^{(1)} \right] & \text{and} & & D_2 &= P_1 \left[ d_0, d_1^{(0)} \right] \\ u_2(x, t) &= u(x, t; D_2, At) & \text{and} & & g_2(t) &= \Phi[f, D_2] \\ \hat{\phi}(x, t) &= \phi(x, t; D_2, A(T_1 - t)) \end{aligned}$$

$$\text{We compute} \quad A_{11} = \int_0^{T_1} \int_0^{x_0(t)} \lambda_1(u_2) \partial_x u_2 \partial_x \hat{\phi} dx dt,$$

$$b_1 = \int_0^{T_1} (g(t) - g_2(t)) A(T_1 - t) dt$$

and solve

$$A_{11}(d_1^{(1)} - d_0) = b_1.$$

Note that  $A_{11}$  and  $b_1$  are computed from  $u_2$ ,  $\hat{\phi}$ ,  $g_2$  which are all based on the known coefficient  $D_2$ .

To continue, we apply the g-integral identity (2.7) first on  $Q_1$ , where only  $d_0, d_1$  are active, and then we apply the g-integral identity (2.7) again, but now on  $Q_2$  where  $d_0, d_1, d_2$  are active. That is,

$$\begin{aligned} \text{on } Q_1 \quad & D_1 = P_1 \left[ d_0, d_1^{(2)} \right] \quad d_1^{(2)} \text{ is unknown,} \\ \text{and} \quad & D_2 = P_1 \left[ d_0, d_1^{(1)} \right] \quad d_1^{(1)} \text{ is known,} \end{aligned}$$

and we compute  $A_{11}$  and  $b_1$  as before.

Note that  $u_2$ ,  $\hat{\phi}$ ,  $g_2$  are based on the updated coefficient  $D_2$  so that, in general,  $d_1^{(2)}$  will not be the same as  $d_1^{(1)}$ .

$$\begin{aligned} \text{On } Q_2 \quad & D_1 = P_2 \left[ d_0, d_1^{(2)}, d_2^{(1)} \right] \\ \text{and} \quad & D_2 = P_2 \left[ d_0, d_1^{(1)}, d_2^{(0)} \right] \quad \text{note : } d_2^{(0)} = d_1^{(1)} \\ \text{we compute} \quad & A_{2,1} = \int \int_{Q_{21}} \lambda_1(u_2) \partial_x u_2 \partial_x \hat{\phi} dx dt, \\ & Q_{21} = \{ \mu_0 \leq u_2(x, t) \leq \mu_1, 0 \leq t \leq T_2 \} \\ & A_{2,1} = \int_{T_1}^{T_2} \int_0^{x_1(t)} \lambda_2(u_2) \partial_x u_2 \partial_x \hat{\phi} dx dt \\ & b_2 = \int_0^{T_2} (g(t) - g_2(t)) A(T_2 - t) dt \\ \text{and we solve} \quad & \begin{bmatrix} A_{11} & 0 \\ A_{2,1} & A_{2,2} \end{bmatrix} \begin{bmatrix} d_1^{(2)} - d_1^{(1)} \\ d_2^{(1)} - d_2^{(0)} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}. \end{aligned}$$

We proceed in this way, where at the  $k$ -th stage we apply the integral identity  $k$  times, once on each of the strips  $Q_1$  to  $Q_k$ . Of course this produces  $k$  equations, one for each strip. On each strip,  $Q_j$  there are only  $j$  unknown active node values  $d_1^{(p)}, \dots, d_j^{(q)}$ , at various stages of iteration, hence the  $j$ -th equation contains only the first  $j$  unknowns. This leads to a  $k$  by  $k$  lower triangular system for the differences  $d_j^{(p)} - d_j^{(p-1)}$ . At the  $k$ -th stage of the algorithm we are solving for the first iterate for  $d_k$ , for the second iterate of  $d_{k-1}$ , etc. This algorithm, which we will call the iterative algorithm, differs from the non-iterative algorithm described in the preceding section. The non-iterative algorithm amounts to suppressing the iterative feature so that for each  $k$ , the nodal value  $d_k$  is obtained by solving just a single equation,  $A_{kk}(d_k - d_{k-1}) = b_k$ . Suppressing the iteration leads to cascading errors in the sequentially computed nodal values  $d_k$  as shown in Figure 2. The coefficient shown in this figure,

$$D(u) = 2 - \arctan [6(u - 1/2)], \quad 0 < u < 1, \quad (4.1)$$

was recovered in two ways. In the first, the non-iterative algorithm applied to the data  $\{f, g\}$  to produce the dashed line plot, while the iterative algorithm was applied to produce the solid line plot. The data was generated by solving the direct problem (2.1) using a functional form of the coefficient (4.1) on a mesh of 70 nodes with the Matlab solver `ode15s`. The flux,  $g(t)$ , was then computed using a difference formula. This flux data was submitted to the recovery algorithms, both which used a 40 node mesh and



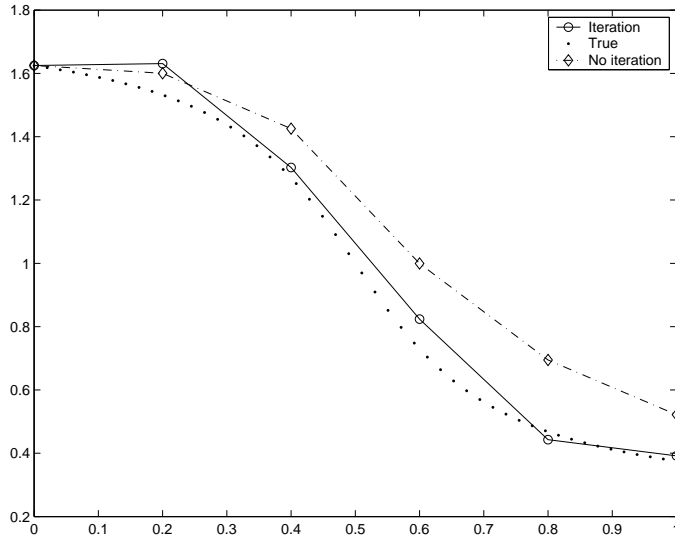


Figure 2:

`ode15s` to compute solutions to the direct and adjoint problem. It is clear from the figure that the errors in non-iterated nodal values for  $D(u)$  accumulate as the values are sequentially determined. We point out that determining  $d_k$  we are obliged to integrate over the approximately triangular region  $\{0 < x < x_k(t), T_{k-1} < t < T_k\}$ . However, the algorithm must numerically approximate  $x_0(t_j)$  on the inner mesh, and this leads to a systematic overestimation of the value of  $A_{kk}$  which, in turn leads to a correction term that is too small. The fact that  $D$  is a decreasing function of  $u$ , as given in equation (4.1), leads to a negative  $\Delta g(t)$  and a negative correction,  $b_k/A_{kk}$ . This is evident in the dashed-line plot of Figure 2. The fact that  $A_{kk}$  is too large causes the negative correction to be too small so that the graph of the computed polygonal function lies above the graph of the true coefficient. Since the integrals for  $A_{kk}$  and  $b_k$  involve only the interval  $[T_{k-1}, T_k]$ , each identified value,  $d_k$ , can do nothing to diminish errors in previously identified values, hence the identification error accumulates.

This suggests that iteration might prove useful. The solid line plot in Figure 2 shows the result of identifying the coefficient (4.1) but now iterating as follows. We use the identity (2.7) on  $Q_1$  together with the known value,  $d_0$ , to identify  $d_1^{(1)}$ . Here the known value,  $d_0$ , is used to compute  $u_2(x, t)$ ,  $g_2(t)$  and  $\hat{\phi}(x, t)$ . Next we use the identity (2.7) on  $Q_1$  and  $Q_2$  together with known values,  $d_0, d_1^{(1)}$  to identify  $d_1^{(2)}$  and  $d_2^{(1)}$ . In the next step, we use the identity (2.7) on  $Q_1, Q_2$  and  $Q_3$  together with known values,  $d_0, d_1^{(2)}$  and  $d_2^{(1)}$  to identify  $d_1^{(3)}, d_2^{(2)}$  and  $d_3^{(1)}$ . At each stage, the known nodal values are used to compute  $u_2(x, t)$ ,  $g_2(t)$  and  $\hat{\phi}(x, t)$ . Continuing in this way, we eventually obtain  $d_M^{(1)}, d_{M-1}^{(2)}, \dots, d_1^{(M)}$ . It is evident from the solid line plot in Figure 2 that as a result of the iteration, the errors no longer exhibit the cumulative character seen in the dashed line plot, where iteration was not applied.

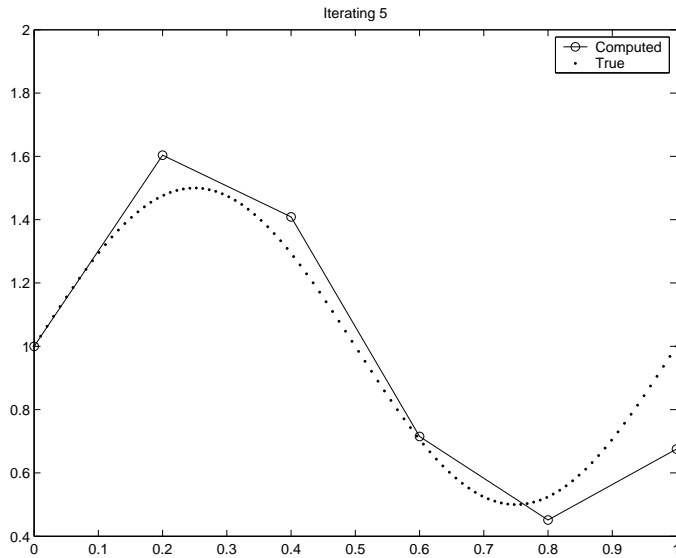


Figure 3:

Here the coefficient

$$D(u) = 1 + \frac{1}{2} \sin(2\pi u), \quad 0 < u < 1, \quad (4.2)$$

was used to generate flux data as in the previous example, although here the Matlab solver `ode23s` was used. This data was passed to the iterative recovery algorithm, the results of which are plotted in Figure 3

The qualitative agreement between the computed and true coefficient appears reasonable in this figure. Notice that the approximation initially lies above the plot of the true coefficient (4.2) in regions where  $D$  is increasing, which is in agreement with the analysis of the previous experiment. The value at the last nodal is not iterated in this scheme, and is visibly less accurate than the computed values on other nodes.

Figure 4 displays the effect of refining the outer mesh by increasing  $M$ , the number of nodes in order to identify the coefficient

$$D(u) = 1 + u, \quad 0 < u < 1,$$

The results for  $M = 2, 6$  and  $9$  are shown in addition to a plot of the  $L^2$  - error versus  $M$ . This last display shows the error decreasing with increasing  $M$  up to about  $M = 6$ , at which point the error begins again to increase. This result is in qualitative agreement with (3.7).

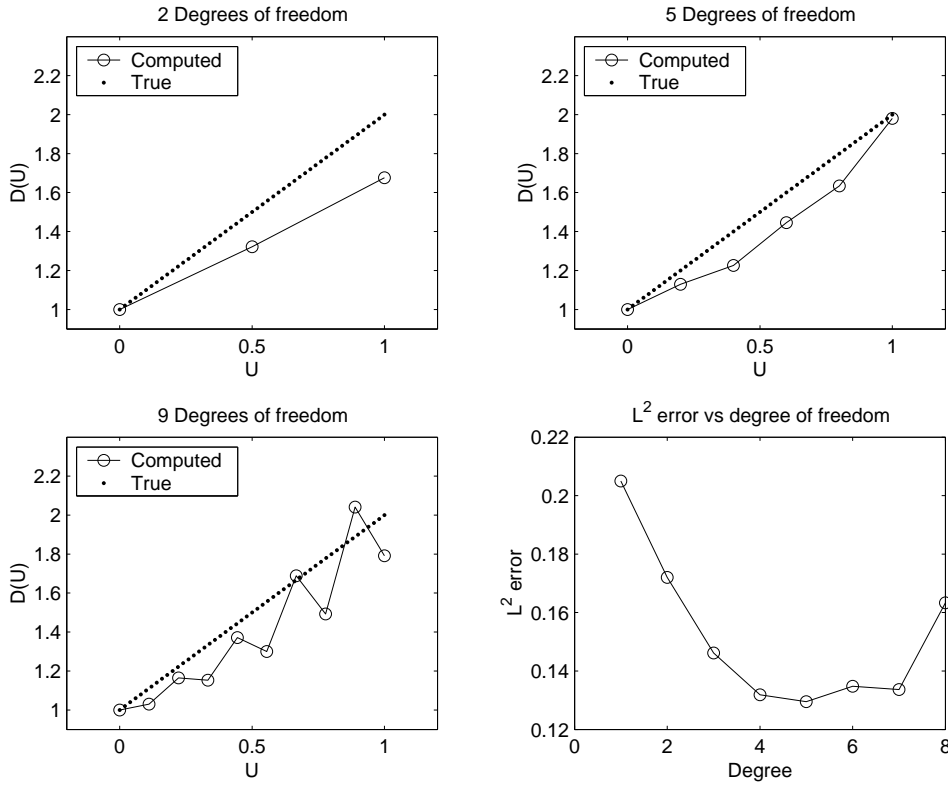


Figure 4:

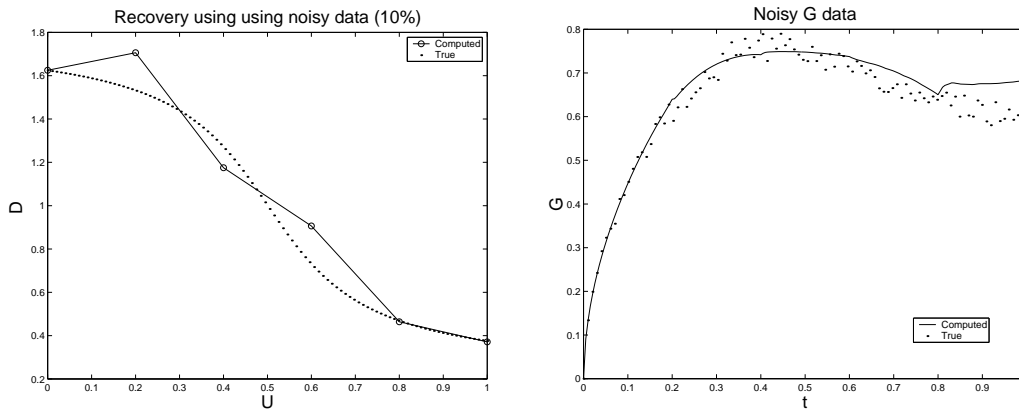


Figure 5:

Figure 6:

Figure 5 represents coefficient recovery in which the data contained induced error. A relative uniform random error of 10% was induced in the flux data, and the iterative algorithm was applied. The flux data used for recovery is plotted in Figure 6. The recovered coefficient, plotted in Figure 5 appears to capture the general structure of the true coefficient. No preprocessing was applied to this data, which was possible since the error had mean zero. The integration of the of  $g$  data in (3.3) allows much of this error to cancel.

## 5. Conclusions

The integral identities (2.7) and (2.9) are equations providing explicit representations for the input to output mappings associated with the inverse problem of identifying an unknown diffusion coefficient from overspecified data measured on the boundary. These equations provide a means for proving that the input output maps are continuous, injective and strictly monotone. Such information is not so readily obtained from an output least squares approach nor from equation error techniques. The equations may also be the basis for a numerical approximation procedure although here it might be more difficult to compete with sophisticated OLS implementations.

## 6. References

- [1 ] H. B Aneur, G. Chavent, J. Jaffre, Refinement and coarsening of parameterization for the estimation of hydraulic transmissivity, Proceedings of Inverse Problems in Engineering, 3rd Int'l Conf on Inverse Problems in Engineering, 1999, Port Ludlow Wa.
- [2 ] J. Cannon and P. DuChateau, Some asymptotic boundary behavior of solutions of nonlinear parabolic initial boundary value problems, JMAA vol 68, no. 2, (1997) pp536-547
- [3 ] J. Cannon and P. DuChateau, Design of an experiment for the determination of a coefficient in a nonlinear diffusion equation, Int. J. Eng. Sci., vol 25, No. 8, pp1067-78, 1987
- [4 ] G. Chavent, and P. Lemonnier, Identification de la non-linearite d'une equation parabolique quasilineaire, Appl. Math. Optim., 1(1974) pp121-162
- [5 ] P. DuChateau, An inverse problem for the hydraulic properties of porous media, SIAM J Math Anal 28(1997) pp611-632.
- [6 ] M. Hanke and O. Scherzer, Error analysis of an equation error method for the identification of the diffusion coefficient in a quasilinear parabolic differential equation, SIAM J. Appl. Math. 59 (1999), 1012-1027
- [7 ] D. Hinestroza and D. Murio, Identification of transmissivity coefficients by mollification techniques. Part I: 1-dimensional elliptic and parabolic problems, Comp Math Applic, 25(1993) pp 59-79
- [8 ] Y. Jarny, M. N. Ozisik and J P Bardon, A general optimization method using adjoint equation for solving multidimensional inverse heat conduction. Int. J. Heat Mass Transfer, Vol 34, No 11, 1991 pp2911-2918
- [9 ] P. Knabner and S. Bitterlich, An Efficient Method for Solving an Inverse Problem for the Richards Equation. Journal of Computational and Applied Mathematics, Vol. 147, Issue 1, 2002, pp153-173

- [10 ] R. Nabakov, An inverse problem for porous medium equation, in Parameter Identification and Inverse Problems in Hydrology, Geology and Ecology, J. Gottlieb and P DuChateau, eds, Kluwer, Dordrecht, Boston, London 1996 pp155-163.
- [11 ] Reeve, D E; Spivack, M; Recovery of a variable coefficient in a coastal evolution equation, Journal of Computational Physics, Vol. 151, pp 585-596, 1999.
- [12 ] G. R. Richter, Numerical identification of a spatially varying diffusion coefficient, Math Comp 36(1981) pp375-386.