The Nonlinear Balance Equation: a Survey of Numerical Methods

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1 Introduction

The full set of atmospheric equations are used to independently predict mass and momentum fields. There is a diagnostic relationship between these two quantities in some cases. The *geostrophic wind* is one such example, and is fairly accurate for synoptic scale motion away from the equator. The earliest numerical weather models were created for large scale mid-latitude flow, and so the simplifying assumptions incorporated into these models caused little problem. As computer power increased and domains began to include the tropics, more general balance conditions were needed. The *Momentum Equations* (ME) and the *Nonlinear Balance Equation* (NLBE) are models thought to accurately describe the highly curved flows seen in tropical cyclones. There has been much research concerning numerical solutions to these equations, but a unique numerical solution is often difficult to achieve. This paper discusses the theory behind some of these earlier methods, as well as their limitations. It was hoped that a numerical scheme could be implemented that would allow an accurate and efficient computation of a wind velocity field, induced by a prescribed pressure field.

The Advanced Microwave Sounding Unit (AMSU), orbiting on the satellite NOAA 15, is able to produce microwave radiation data that in turn is used to create high resolution temperature profiles on multiple levels of the atmosphere. Current temperature plots are produced with a resolution of as fine as 40 km^2, a great improvement of earlier (MSU) sensors. This highly refined data set is thought to allow accurate wind fields to be found in areas that are inaccessible to other sensing methods. Particularly, it was hoped that hurricane wind velocities could be determined purely from the remote sensing temperature data. Currently, accurate estimates are available only through aircraft and surface observations.

The AMSU data, while it provides a measure of the radiated microwaves on several frequency bands, does not provide a direct information about the wind field. It can, hoever, provide a measure of the pressure field. This is accomplished via the hydrostatic equation, and the temperature field can be converted into a geopotential height field. This height field is defined on a latitude-longitude grid in a domain of interest. For this domain, it is assumed that the surface pressure on the boundary of the domain is known, perhaps from actual ground data. This geopotential height Φ) is easily related to the standard geometric height coordinate - it is the geometric height normalized by gravity.

Under various assumptions, the pressure field may be related to the wind field via the *momentum equations* and the *nonlinear balance equation*. These equations have been used to determine pressure fields using a known wind field, but the solution to the inverse problem is a bit harder. Work began on this problem in the the mid 1950's, but a general numerical method has remained elusive.

The momentum equations are a quasi-linear system, with the non-linearity arising from a product of the wind components and their first derivatives. The NLBE is a single equation involving nonlinearities in the highest orders. While the creation of a fully non-linear system from a quasi-linear appears a needless complication, the NLBE is the basis for most numerical methods. This formulation has the benefit that it enforces a non-divergent, and therefore physical, solution. Also, it allows the problem to be viewed as Poisson's equation - a form with many pre-existing numerical approaches. Physically, however, this system should allow the wind field ($\mathbf{u}$) to adjust with the pressure field (Φ). With a prescribed pressure field we do not allow an adjustment process to take place between pressure and temperature fields, and are therefore attempting to solve reach a physical solution through a sequence of iterative solutions to a non-physical problem.

2 Preliminaries

Considering forces that effect the horizontal wind, a reasonably accurate model can be built using only the coriolis force ($f\mathbf{u}$) and the pressure gradient force ($\nabla\Phi$). Generally these forces appear to be in approximate balance. The *geostrophic balance* system of equations is given by

$$fv = \frac{\partial \Phi}{\partial x} \tag{1}$$

$$-fu = \frac{\partial \Phi}{\partial y} \tag{2}$$

with the horizontal wind $\mathbf{u}$ given by

$$\mathbf{u} := \begin{bmatrix} u \\ v \end{bmatrix}.$$

Adding $\frac{\partial}{\partial x}$ of (1) and $\frac{\partial}{\partial y}$ of (2), we get

$$f\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} \tag{3}$$

with the coriolis parameter, f, defined as $f := 2\Omega \sin\varphi$, with Ω the earth's rotation rate, and φ the latitude. While this system is useful for pointwise estimation, it does not include any type of centrifugal force and therefore is used to approximate only relatively straight flows. For the highly rotational flows seen in cyclones another model is needed.

A more accurate representation is given by the *f-plane momentum* equations. This system adds a centrifugal component to the geostrophic relationship. The coriolis parameter, f, is taken to be constant. In Cartesian coordinates the system is

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv + \frac{\partial \Phi}{\partial x} = 0 \tag{4}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu + \frac{\partial \Phi}{\partial y} = 0 \tag{5}$$

where the horizontal wind $\mathbf{u}$ is composed of the u velocity in the east-west (x) direction and the v velocity in the north-south (y) direction.

By adding $\frac{\partial}{\partial x}(4)$ and $\frac{\partial}{\partial y}(5)$ the *divergence equation* results

$$\frac{D}{Dt}\delta + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + 2\left(\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right) - f\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + \nabla^2\Phi = 0 \tag{6}$$

where $\delta := \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$ is divergence.

Define the horizontal wind $\mathbf{u}$ in terms of streamfunction ψ and velocity potential χ, such that

$$u := -\frac{\partial \psi}{\partial y} + \frac{\partial \chi}{\partial x} \quad \text{and} \quad v := \frac{\partial \psi}{\partial x} + \frac{\partial \chi}{\partial y}. \tag{7}$$

Write the divergence equation (6) in terms of (7). Assuming a non-divergent flow, ($\nabla \cdot \mathbf{u} = 0$), and discarding the time dependency, the approximation that results is called the *nonlinear balance equation*

$$2\left[\frac{\partial^2\psi}{\partial x^2}\frac{\partial^2\psi}{\partial y^2} - \left(\frac{\partial^2\psi}{\partial x \partial y}\right)^2\right] + f\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right) = \nabla^2\Phi. \tag{8}$$

If the divergence is retained, a more complicated system results. Equations (4) and (5) can be reformed as the system

$$\begin{aligned}
\frac{\partial}{\partial t}\left(\nabla^2\chi\right) + u\frac{\partial}{\partial x}\left(\nabla^2\chi\right) + v\frac{\partial}{\partial y}\left(\nabla^2\chi\right) + \frac{\partial(u,v)}{\partial(x,y)} - f\left(\nabla^2\psi\right) + \nabla^2\Phi &= 0 \\
\frac{\partial}{\partial t}\left(\nabla^2\psi\right) + u\frac{\partial}{\partial x}\left(\nabla^2\psi\right) + v\frac{\partial}{\partial y}\left(\nabla^2\psi\right) + \nabla^2\chi\nabla^2\psi + f\left(\nabla^2\chi\right) &= 0.
\end{aligned}$$

The NLBE (8) can also be rewritten as a Jacobian matrix. It is then

$$\begin{vmatrix} \hat{\psi}_{xx} & \hat{\psi}_{xy} \\ \hat{\psi}_{xy} & \hat{\psi}_{yy} \end{vmatrix} = \nabla^2\Phi/2 + (f/2)^2 \tag{9}$$

With this apparent, an integral variational equation is given by Courant and Hilbert (1962) as

$$J[\psi] = \iint_\Omega (\hat{\psi}_x^2\hat{\psi}_{yy} - 2\hat{\psi}_x\hat{\psi}_y\hat{\psi}_{xy} + \hat{\psi}_y^2(\hat{\psi}_{xx} - 6p\hat{\psi})\partial\Omega$$

with $\hat{\psi}_{xx} = \psi_{xx} + f/2$, $\hat{\psi}_{yy} = \psi_{yy} + f/2$ and $\hat{\psi}_{xy} = \psi_{xy}$. Also, $p(x,y) = \hat{\psi}_{xx}\hat{\psi}_{yy} - \hat{\psi}_{xy}^2$.

3 Historical Backround

A numerical solution to the nonlinear balance equation (NLBE) has been an area of research for years. Many computational methods have been proposed to obtain solutions to the mixed hyperbolic-elliptic domains that are often encountered in actual atmospheric data. Unfortunately, not as much attention has been paid to achieving a physical solution *without* modifying the physical data. Most methods seem to either manipulate the geopotential field in order to produce a convergent solution, or add enough divergence to the flow to allow a solution. Low order spectral methods have been applied to the shallow water equations by Tribbia (1982) that seem to suggest that the iterative methods typically fail in the regions where the *ellipticity condition* is violated. Tribbia suggests that this might be a consequence of approximations in the quasi-geostrophic theory, but proposes that the true cause might be more fundamental to the nature of the problem.

On a f-plane, where the coriolis parameter is assumed constant, the NLBE can be written

$$2\left(\psi_{xx}\psi_{yy} - \psi_{xy}^2\right) + f\nabla^2\psi - \nabla^2\Phi = F. \tag{10}$$

F contains the divergent part of the horizontal wind $\mathbf{u}$ as well as its time derivative. Generally, F is thought be negligible and is therefore approximated by zero. Equation (10) is then treated as Poisson's equation, with solutions for Φ easily obtained from a given streamfunction ψ.

The inverse problem, when ψ is sought from a given Φ field, has a much different form. It becomes a Monge-Ampere type equation. Oliker and Prusser (1982) proposed a numerical solution to a restricted class of problems of this type. While their method in computationally expensive, they claim that it produces a sequence that converges to a solution, irrespective of the initial iterate. This mathematical approach has not yet been applied by the atmospheric community, however. Instead, there has much study of the *ellipticity* and *realizability* conditions that transform this into a purely elliptic problem.

The NLBE in standard form is

$$A\psi_{xx} + 2B\psi_{xy} + C\psi_{yy} + \psi_{xx}\psi_{yy} - \psi_{xy}^2 = E \tag{11}$$

where

$$\begin{aligned} &A = C := f/2, \quad B = 0 \\ &E := (\nabla^2\Phi + F)/2. \end{aligned} \tag{12}$$

In general it is possible for nonlinear PDE's to have non-unique solutions. If, however, we restrict $A, B, C,$ and D to be continuous functions in the domain of interest, and in addition require that

$$AC - B^2 + E > 0 \tag{13}$$

the number of solutions is reduced to at most two (Courant and Hilbert 1962). This restriction (13) allows E to eliminated, and (11) to be factored into the form

$$(\psi_{xx} + C)(\psi_{yy} + A) - (\psi_{xy} - B)^2 > 0. \tag{14}$$

Equation (10) then can be written

$$(f/2 + \psi_{xx})(f/2 + \psi_{yy}) > \psi_{xy}^2 \geq 0. \tag{15}$$

By this inequality, $(f/2 + \psi_{xx})$ and $(f/2 + \psi_{yy})$ must both have the same sign. Absolute vorticity, $(f + \nabla^2\psi)$, will then either be positive or negative in the whole domain. In the Northern Hemisphere, it seems logical to choose the solution that corresponds to positive absolute vorticity. In the Southern Hemisphere, the opposite conditions would be imposed. It should also be noted that by an assumption of continuity on ψ across the equator, ψ_{xx} and ψ_{yy} should vanish there as well. The *realizability condition* can now be stated. It is

$$\nabla^2\Phi + \frac{f^2}{2} + F > 0. \tag{16}$$

Assuming that $F := 0$, the realizability condition reduces to the *ellipticity condition*

$$\nabla^2\Phi + \frac{f^2}{2} > 0. \tag{17}$$

An alternate formulation of the NLBE by Petterson (1953) is also interesting.

$$(\nabla^2\psi + f)^2 = 2Q + A_d^2 + B_d^2, \tag{18}$$

where

$$\begin{aligned} A_d &= -2\psi_{xy} \\ B_d &= \psi_{xx} - \psi_{yy} \\ Q &= \nabla^2\Phi + \frac{f^2}{2} + F. \end{aligned}$$

Petterson's form allows a direct solution for $\nabla^2\psi$

$$\nabla^2\psi = -f \pm \sqrt{(2Q + A_d^2 + B_d^2)}. \tag{19}$$

In order for this expression to have a real solution the radicand must be non-negative. This requires that $A_d^2 + B_d^2$ be larger than $-2Q$. In fact, the earliest numerical attempts (Miyakoda (1956), Schuman (1957)) both implemented this form of the NLBE.

The NLBE can also be recast in polar coordinates, which is perhaps a better system for examining rotational flow. The NLBE can then be written

$$1/r\frac{\partial}{\partial r}(r\psi_r)) + 1/(r^2)\psi_{\phi\phi} + 2/f\left[\psi_{rr}\left(1/(r^2)\psi_{\phi\phi} + 1/r\psi_r\right) - \left(1/r\psi_{r\phi} - 1/(r^2)\psi_\phi\right)^2\right] = \zeta_g \tag{20}$$

with

$$\zeta_g = 1/f\left[1/r\frac{\partial}{\partial\phi}(r\Phi_r) + 1/(r^2)\Phi_{\phi\phi}\right]$$

In the axisymmetric case, with $v = \psi_r$, equation (20) collapses to

$$\frac{\partial}{\partial r}\left[\underbrace{r\left(fv + \frac{v^2}{r} - \Phi_r\right)}_{\alpha}\right] = 0, \tag{21}$$

The *gradient wind* equation results once we notice that α is a constant with respect to r, and is zero at $r = 0$.

$$\left(fv + \frac{v^2}{r}\right) = \Phi_r \tag{22}$$

Kasahara (1982) discusses the gradient wind equation in an attempt to explore the possible behavior of the full NLBE. He begins by defining U as the absolute velocity of a vortex

$$U := v + \frac{fr}{2}. \tag{23}$$

The absolute vorticity, ζ_a of U, is found by taking $1/r\frac{d}{dr}(rU)$,

$$\zeta_a = \frac{r}{4U}\left[2M + f^2\left(\frac{4\Phi_r}{f^2 r} + 1\right)\right] \geq 0. \tag{24}$$

where

$$M = \nabla^2\Phi + \frac{f^2}{2} \tag{25}$$

is exactly the *ellipticity condition*. Eliminating $\nabla^2\Phi$ results in

$$\frac{d}{dr}U^2 = Mr. \tag{26}$$

This can be solved for U^2, by integrating with respect to r, and noting that $U(0) = 0$,

$$U^2 = \int_0^\alpha M\alpha \; d\alpha > 0. \tag{27}$$

This integral form (27) involves the ellipticity condition M less restrictively than the condition required by eq. (24). By (27), the integral over the entire *domain* to be forced positive, while (24) is a *local* expression, requiring ellipticity in a pointwise sense.

The ellipticity and realizability conditions (16 and 17) work well in most cases. As the resolution of remote sensing devices has improved, however, there have been an increasing number of examples where these condtions have been violated. In some local domains divergence may be larger than the vorticity. In cases of strong heating the dominant forces appear to include divergent flow accelerations and divergence in the pressure field (Paegle 1983).

3.1 Conclusions

The early attempts often transformed the NLBE into a Poisson's equation. Relaxation, FFT's, and other methods were used to solve these formulations. The lagging of the nonlinear contributions did not seem to effect the solution, since a steady state solution was being looked for. These methods seemed to have convergence problems, in large part due to 'non-elliptic' data sets. For years, these numerical methods were examined and re-examined in the hope that a more robust numerical method could be discovered. Only in the 1980's did reseachers begin to look at the non-convergence as consequence of a fundamental component of the model (Tribbia 1982). An interesting approach might be further consideration of a global integral 'ellipticity' condition, as Kasahara's work on the gradient wind equation seems to suggest.

4 Implementable Approaches

4.1 Notation

Vectors should always be interpreted as column vectors, and will be denoted as $\mathbf{x}$. In the iterative scheme u^0, v^0, and ψ^0 will represent the initial iterate. Similarly, the ν^{th} iterate will be denoted u^ν, v^ν, and ψ^ν, respectively.

The *tensor product* of two matrices A and B, written $A \otimes B$,is defined to be

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ \vdots & & & \\ a_{n1}B & a_{n2}B & \cdots & a_{nn}B \end{bmatrix}$$

Also define $\mathbf{f} \circ \mathbf{g}$, $\mathbf{f}, \mathbf{g} \in \mathbb{R}^n$, where

$$\mathbf{f} \circ \mathbf{g} := \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix} \circ \begin{bmatrix} g_1 \\ \vdots \\ g_n \end{bmatrix} = \begin{bmatrix} f_1 g_1 \\ \vdots \\ f_n g_n \end{bmatrix} \in \mathbb{R}^n$$

Similarly, $A \circ \mathbf{g}, \quad A \in \mathbb{R}^{n \times n}, \mathbf{f} \in \mathbb{R}^n$, where

$$A \circ \mathbf{f} := \begin{bmatrix} \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_n \end{bmatrix} \circ \begin{bmatrix} g_1 \\ \vdots \\ g_n \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1 g_1 \\ \vdots \\ \mathbf{a}_n g_n \end{bmatrix} \in \mathbb{R}^{n \times n}.$$

4.2 Geometric approach

The non-linear balance equation is one of a family of problems of the Monge-Ampere type. In some cases, a good guess is required for solutions methods to converge. Oliker and Prussner (1982) have developed an algorithm which they claim produces a monotonic sequence that converges to a solution, irrespective of the initial iterate. The equation they study is

$$M(z) := z_{xx} z_{yy} - (z_{xy})^2 = g \tag{28}$$

for a bounded convex domain Ω, nonnegative g and Dirichlet data on Γ.

Let $z \in \mathcal{C}^2(\Omega)$ be a solution to eq. (28). Then $M(z)$ may be thought of at the determinate of the Jacobian matrix

$$M(z) = det\left(\begin{bmatrix} z_{xx} & z_{xy} \\ z_{yx} & z_{yy} \end{bmatrix}\right)$$

and the function z must be either convex or concave. Without loss of generality, we consider the case when z is concave. It is possible to create a map $\gamma : (x, y) \rightarrow (p, q) = (z_x(x, y), z_y(x, y))$ through a Legendre transform. By the $\mathcal{C}^2$ nature of z, the map γ is a diffeomorphism (a differential homeomorphism) with Jacobian M. For any Borel subset $\omega \in \Omega$, we can form $\nu(\omega)$ where

$$\nu_z(\omega) = \iint_{\gamma(\omega)} dp\ dq.$$

By the properties of M and equation (28), it can be shown that $\nu_z(\omega) = \mu(\omega)$, where

$$\mu(z) = \iint_{\omega} g\ dx\ dy, \qquad \text{for all Borel subsets} \quad \mathcal{B}(\omega) \in \Omega.$$

This correspondence between ν_z and μ allows the set of arbitrary convex functions $z(x, y)$ to be included in the set of admissible functions. We begin by choosing arbitrary points on the surface S, and construct tangent planes T to this surface at these points. We are able to produce a new discrete map $\bar{\gamma}$ where

$\bar{\gamma}(x, y) \to (p, q) \in T$. By the previous argument, if the $\iint_\omega \bar{\gamma}(\omega) = \nu_z(\omega)$, then a *generalized* solution $\bar{z}$ has been found. We now able to approximate the measure of $\mu_z(\omega)$ as the sum of the measures ν_z of evaluated at the discrete points that we have chosen on the surface S. By incorporating more points chosen from S, a convergent sequence of piecewise linear functions $\bar{z}_i$ can be generated. In the limit, $\bar{z}_\infty$ is equal to z, the solution to the original equation (28)

4.2.1 Legendre Transformation

First consider a surface S in $\mathbb{R}^3$ such that

$$S = \{(x, y, u) : u = u(x, y)\} \quad \text{for } (x, y) \in \Omega \subset \mathbb{R}^2 \tag{29}$$

and the plane T in $\mathbb{R}^3$ such that

$$T = \{(x, y, u) : u - px - qy + \lambda = 0\} \tag{30}$$

The plane T is uniquely determined by (p, q, λ) (these can be thought of as the zeros of the plane). The plane T is tangent to S at (x_0, y_0, u_0) if

$$(u - u_0) - \partial_x u(x - x_0) - \partial_y u(y - y_0) = 0 \tag{31}$$

That is, T is uniquely determined by the system

$$p = \partial_x u(x_0, y_0) \tag{32}$$
$$q = \partial_y u(x_0, y_0) \tag{33}$$
$$\lambda = x_0 p + y_0 q - u_0. \tag{34}$$

Now S can be described by $S(\lambda(p, q))$, instead of $S(u(x, y))$, by solving (34) for (x_0, y_0) in term of (p, q). Substitution into (34) gives

$$\lambda = x_0(p, q)p + y_0(p, q)q - u(x_0(p, q), y_0(p, q)) \tag{35}$$

Differentiating (35) gives

$$\partial_p \lambda = x_0(p, q) + p\left[\frac{\partial}{\partial p} x_0(p, q)\right] - q\left[\frac{\partial}{\partial p} y_0(p, q)\right] - \frac{\partial u}{\partial x}\left[\frac{\partial}{\partial p} x_0(p, q)\right] - \frac{\partial u}{\partial y}\left[\frac{\partial}{\partial p} y_0(p, q)\right] \tag{36}$$

But $\frac{\partial u}{\partial x} = p$ and $\frac{\partial u}{\partial y} = q$ by (32), so (36) reduces to to

$$\frac{\partial \lambda}{\partial p} = x \qquad (\text{similarly, } \frac{\partial \lambda}{\partial q} = y).$$

Equation (35) can now be written as

$$\lambda(p, q) = u(x, y) = xp + yq \tag{37}$$

where

$$p = u_x, \quad q = u_y \tag{38}$$

$$x = \lambda_p, \quad y = \lambda_q \tag{39}$$

The BVP is given by

$$\begin{array}{ll} z_{xx}z_{yy} - z_{xy}^2 = g(x,y) \geq 0 & (x,y) \in \Omega \\ z(x,y) = F(x,y) & (x,y) \in \Gamma \end{array} \tag{40}$$

where Ω is the domain and Γ is the boundary of Ω.
Now define

$$\mu(\omega) = \iint_{\omega} g \ dx \ dy \qquad \forall \ \omega \in \Omega, \omega \in \mathcal{B}(\Omega) \tag{41}$$

where $\mathcal{B}(\Omega)$ are the Borel subsets of Ω, and

$$\begin{array}{ll} \nu_\omega & = \iint_\omega (z_{xx}z_{yy} - z_{xy}^2) \ dx \ dy \\ & = \iint_{\gamma(\omega)} \ dp \ dq \end{array} \tag{42}$$

where $\gamma(\omega)$ is the image of $\omega \in \Omega$ under the Legendre Transform.

The area of the discrete $\nu_z(\omega)$ with vertices $\{(x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3)\}$ may also be computed as

$$|\frac{1}{2} det \left(\begin{bmatrix} 1 & 1 & 1 \\ (x_2 - x_1) & (y_2 - y_1) & (z_2 - z_1) \\ (x_3 - x_1) & (y_3 - y_1) & (z_3 - z_1) \end{bmatrix} \right)|.$$

Then $z = z(x,y)$ is a solution to the BVP if

$$\mu(\omega) = \nu_z(\omega) \quad \forall \omega \in \mathcal{B}(\Omega) \tag{43}$$

This transformation is always possible if

$$p = u_x, \quad q = u_y \tag{44}$$

can be solved for $x = x(p,q), y = y(p,q)$. This is always possible when

$$J = u_{xx}u_{yy} - u_{xy}^2 \neq 0 \quad \text{ for } (x,y) \in \Omega. \tag{45}$$

This condition implies that S has a family of T planes, each tangent to S at a unique point.

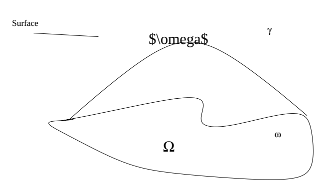


Figure 1: *S is the solution surface, Ω its projection. γ is the mapping between the two*

This creates a mapping from the solution surface S to the 'shadow' of the surface. By comparison of the weighted area of the shadow Ω to the area of the solution surface S, it becomes possible to construct a sequence of discrete approximations that converge to the general solution. It is interesting to note that for this method to work the function g must be non-negative. There are regions in the physical data that are negative. Examining more closely, it becomes apparent that the 'non-negative' restriction enforces the ellipticity condition (from 9). Perhaps a way to explore the NLBE is through a separation of the solution space to regions that met the non-negative restriction and regions where this is not satisfied.

4.3 Explicit Matrix Iterative method

4.3.1 Overview

The *nonlinear balance equation* is

$$2\left[\frac{\partial^2\psi}{\partial x^2}\frac{\partial^2\psi}{\partial y^2} - \left(\frac{\partial^2\psi}{\partial x\partial y}\right)^2\right] + f\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right) = \nabla^2\Phi \tag{46}$$

It is possible to write a linear approximation of the NLBE in the matrix form

$$L\psi = \nabla^2\Phi. \tag{47}$$

In order to do this, the linear operator L must contain information of the previous iteration. We attempt to linearize the system about the current iterate, and solve for the next. It would be better to write the linear approximation of the system as

$$L^\nu\psi^{\nu+1} = \nabla^2\Phi. \tag{48}$$

The iteration then becomes

$$\left\{(2\left[{\psi_{xx}}^\nu \circ \frac{\partial^2}{\partial y^2} - \left({\psi_{xy}}^\nu \circ \frac{\partial^2}{\partial x\partial y}\right)\right] + f\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\right\})\psi^{\nu+1/2} = \nabla^2\Phi \tag{49}$$

and

$$\left\{2\left[{\psi_{yy}}^{\nu+1/2} \circ \frac{\partial^2}{\partial x^2} - \left({\psi_{xy}}^{\nu+1/2} \circ \frac{\partial^2}{\partial x\partial y}\right)\right] + f\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\right\}\psi^{\nu+1} = \nabla^2\Phi. \tag{50}$$

It was hoped that alternating the choice of ψ_{xx} and ψ_{yy} would stabilize the iteration. Another possibility is the form

$$\left[\psi_{xx}^\nu \circ \frac{\partial^2}{\partial y^2} + \psi_{yy}^\nu \circ \frac{\partial^2}{\partial x^2} - 2\psi_{xy}^\nu \circ \frac{\partial^2}{\partial x\partial y} + f\frac{\partial^2}{\partial x^2} + f\frac{\partial^2}{\partial y^2}\right]\psi^{\nu+1} = \nabla^2\Phi. \tag{51}$$

In this way the linear operator L^ν is produced each iteration. The system can then be solved for $\psi^{\nu+1}$. This $\psi^{\nu+1}$ term is then used to build the $L^{\nu+1}$ operator.

In order to factor $\psi^{\nu+1}$, all operators need to act from the left side. To do this, ψ must be vectorized, which allows positions to be addressed through one row of the operator matrix. Considering the $n \times n$ grid, $\nabla^2\Phi$ and $\psi^{\nu+1}$ become n^2 long vectors. The operators, formed again using the typical difference forms, are n by n matrices and include the boundary conditions. Similarly, difference methods applied to $\Phi_{n\times n}$ (with BC's) produces $\nabla^2\Phi_{n\times n}$.

In implementation, $\psi^{\nu+1}$ can be solved directly. GMRES, a minimized residual solver that uses a tolerance, might be utilized on order to speed convergence. The direct method, while slower, did not create an additional source for error. The sparsity of L was utilized by MATLAB's built in sparse solving techniques.

Individual operators will be used to build the complete operator L. Assume a uniform grid with spacing h. D_n^1 will be used to denote the $\frac{\partial}{\partial i}$ operator, acting from the left with dirichlet boundary condtions.

$$D_n^1 = \frac{1}{((2h)^2} \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ -1 & \ddots & 1 & & \vdots \\ 0 & & & 0 & \\ \vdots & & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & -1 & 0 \end{bmatrix}$$

Let D_n^2 denote the $\frac{\partial^2}{\partial i^2}$ operator, acting from the left with dirichlet boundary condtions.

$$D_n^2 = \frac{1}{h^2} \begin{bmatrix} -2 & 1 & 0 & \cdots & 0 \\ 1 & \ddots & \ddots & & \vdots \\ 0 & \ddots & & \ddots & 0 \\ \vdots & & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & 1 & -2 \end{bmatrix}$$

Let C be a $(n \times n)$ matrix on a uniform grid, with grid spacing h. If we concatenate the matrix $C_{(n\times n)}$ into $\mathbf{c}_{n^2\times 1}$, we get

$$\mathbf{c} = \begin{bmatrix} c_{11} \\ \vdots \\ c_{n1} \\ \vdots \\ c_{1n} \\ \vdots \\ c_{nn} \end{bmatrix}$$

Now, assuming dirichlet BC's, $\mathbf{c}_{xx}$ can be formed. It is

$$\mathbf{c}_{xx} = \begin{bmatrix} D_n^1 & & 0 \\ & \ddots & \\ & & D_n^1 \end{bmatrix} \mathbf{c}$$

Using tensor notation, this becomes

$$\mathbf{c}_{xx} = (I_n \otimes D_n^2)\mathbf{c}$$

Similarly,

$$\begin{aligned} \mathbf{c}_{yy} &= (D_n^2 \otimes I_n)\mathbf{c} \\ \mathbf{c}_{xy} &= (D_n^1 \otimes D_n^1)\mathbf{c} \\ \nabla^2 \mathbf{c} &= (I_n \otimes D_n^2 + D_n^2 \otimes I_n)\mathbf{c} \\ \mathbf{c}_{xx}\mathbf{c}_{yy} &= \left[(I_n \otimes D_n^2)\mathbf{c} \circ (D_n^1 \otimes I_n)\mathbf{c}\right] \end{aligned}$$

Writing the iterative scheme (51) using this tensor notation produces

$$\left[\psi_{xx}^\nu \circ (D_n^2 \otimes I_n) + \psi_{yy}^\nu \circ (I_n \otimes D_n^2) - 2\psi_{xy}^\nu \circ (D_n^1 \otimes D_n^1) + f(D_n^2 \otimes I_n) + f(I_n \otimes D_n^2)\right] \psi^{\nu+1} = \nabla^2 \Phi \quad (52)$$

with

$$\begin{aligned} {\psi_{xx}}^\nu &= (I_n \otimes D_n^2)\psi^\nu \\ {\psi_{yy}}^\nu &= (D_n^2 \otimes I_n)\psi^\nu \\ {\psi_{xy}}^\nu &= (D_n^1 \otimes D_n^1)\psi^\nu \end{aligned}$$

Combining these, operator L is seen to be block tridiagonal. For matrices of this form, there are some fast computational methods.

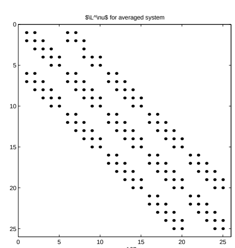


Figure 2: *Operator matrix L for a 6 by 6 grid*

4.4 Minimization methods

The NLBE (8) can also be reformulated as minimization problem

$$2(\psi_{xx}\psi_{yy} - (\psi_{xy})^2) + f(\psi_{xx} + \psi_{yy}) - \nabla^2 \Phi = R$$

where R is the quantity to be minimized.

This can be stated in an informal sense as finding the local minimizer x^* such that

$$f(x^*) \leq f(x) \quad \forall\ x \text{ near } x^*$$

Since the nonlinear balance equation occurs in a finite domain with a known boundary condition, this is a constrained minimization problem. Calling the region Ω and its boundary Γ, the problem can now be formalized

$$\min_{x \in \Omega} f(x) \qquad \text{where } f(x) = g(x) \quad \forall\ x \in \Gamma.$$

This problem may be approached in several ways - but the classical approach requires sufficient smoothness of function. With discontinuities and or irregularities, these methods might fail.

4.4.1 Gradient based schemes

Gradient based schemes are extensions of the simple newton method - where a linear approximation is made to the function, and the gradient is chosen as the new search direction. Gradient methods exist in many forms. All utilize the gradient, its action, or an approximation. Others include the hessian or the eigenvalues of the system. This information is used to modify the search direction. Gradient based schemes are typically the first methods attempted in non-linear problems. The quadratic convergence, and availability of many existing pre-written codes make this a good choice for the start of preliminary experiments. Since these methods are well understood, they will not be discussed here.

For the NLBE, data occurs which leads to a non-positive definite operator. These hyperbolic regions seem to create difficulty for iterative solvers. One solution to this was proposed by 'Arnason (1957). He manipulated the data in order to achieve an elliptic form. His technique involved the smoothing of data until the required ellipticity was reached.

4.5 Adaptive Operators

Writing the f-plane momentum equations in matrix/vector form, with $\mathbf{u} = \begin{pmatrix} u \\ v \end{pmatrix}$ the system becomes

$$\frac{\partial}{\partial t}\mathbf{u} + \begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix} \frac{\partial}{\partial x}\mathbf{u} + \begin{bmatrix} v & 0 \\ 0 & v \end{bmatrix} \frac{\partial}{\partial y}\mathbf{u} + \begin{bmatrix} 0 & -f \\ f & 0 \end{bmatrix} \mathbf{u} + \nabla\Phi = 0. \qquad (53)$$

The eigenvalues of the two matrices

$$\begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix}, \text{ and } \begin{bmatrix} v & 0 \\ 0 & v \end{bmatrix}$$

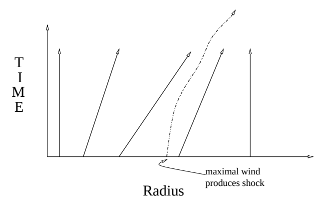
are easy to find - they are just u and v, respectively. In general, this system has the form $\mathbf{u}_t + \alpha\mathbf{u}_x + \beta\mathbf{u}_y + \gamma\mathbf{u} + G(x, y) = 0$.

Compare this to the similar equation of the form

$$U_t + aU_x = 0 \tag{54}$$

for which there are traveling wave solutions $U(x,t) = F(x - at)$. The slope of the characteristics of (54) are determined by the value of the coefficient a. It seems reasonable to expect for similar behavior to the system (53). This is a hyperbolic system, so the upstream scheme is used. 'Upstream' is determined by the done that is determined by the **u** values.

The characteristics might look like this.



There is a possible shock as these characteristics collide. This is a feature that elliptic finite difference would be unable to capture. The CFL was applied in a similar fashion

$$CFL = \frac{\Delta x}{||\mathbf{u}||_\infty}$$

5 Experiments

Of the schemes mentioned, the explicit, the adaptive, and the gradient methods were coded and tested in the MATLAB environment. A known streamfuction ,ψ, was chosen. This utilized parameters controlling the maximal wind velocity, the radius of maximal wind as well as mean wind over the entire domain. From the known streamfunction, Φ was computed. Φ (or its necessary derivatives) were used to provide the forcing in the schemes.

Each method seemed to converge in some local domain, but a global method has yet to be developed. Steepest descent methods seem to work well for axisymmetric flows - but fail when this is augmented with a mean flow across the domain. Similarly, the adaptive and explicit methods seems to handle the mean flow but have difficulty in the region just outside the region of maximal wind. Possibly these methods can be combined in such a way that a reasonable global solution may be found but more work is needed to achieve this.

It appears that the increase in data resolution is providing many more examples of non-elliptic regions in the data. Consequently, the smoothing of data is becoming more and more suspect. This conversion of a hyperbolic system into one that appears elliptic is quite troubling. Is there some underlying physical justification for this method? Or was this method only suited to coarse data sets, in regions that involve little divergent flow? Are the primitive momentum equations more applicable to numerical solutions for these types of flows? The scheme presented by Oliker and Prusser(1982) is quite interesting and might lead to a efficient and robust solver, one that does not require data manipulation. More work is needed to begin to answer these questions.

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