Research Proposal

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Given a velocity field, it's fairly easy to produce a matching pressure field. The inverse problem is more difficult. It is this inverse problem that I'd like to work on. I'd like to to recover accurate velocity measures from a pressure snapshot, or series of snapshots.

Can a (unique) velocity field be recovered that matches a known pressure gradient field?

Specifically, I'd like to find a means to recover hurricane wind velocities from remotely sensed temperature data. Currently, accurate estimates are available only through aircraft and surface observations. As a sailor, I have a great interest in this topic, as do my friends still in the Caribbean. Theoretically, this project might provide a deeper understanding of cyclone dynamics, particularly in the spatial region containing the maximal pressure gradient. This understanding could lead to robust numerical methods of Potential Vorticity (PV) inversion.

I worked for six months on this problem as my masters project for my MS in math with Drs. Wayne Schubert and Mark DeMaria, both associated with the Atmospheric Science Department at Colorado State University. I'll be graduating with a PhD in applied math in the spring of 2004. I've been intrigued by this question for several years, but haven't had the opportunity to apply several new ideas to develop a more complete mathematical understanding of the balance equation.

1 Introduction

The full set of atmospheric equations are used to independently predict mass and momentum fields. There is a diagnostic relationship between these two quantities in some cases. The *geostrophic wind* is one such example, and is fairly accurate for synoptic scale motion away from the equator. The earliest numerical weather models were created for large scale mid-latitude flow, and so the simplifying assumptions incorporated into these models caused little problem. As computer power increased and domains began to include the tropics, more general balance conditions were needed. The *momentum equations* (ME) and the *nonlinear balance equation* (NLB) are models thought to accurately describe cyclonic flow.

On board several NOAA satellites, the Advanced Microwave Sounding Unit (AMSU) is able to produce microwave radiation data that in turn is used to create high resolution temperature profiles on multiple levels of the atmosphere. Current temperature plots are produced with a resolution of as fine as 40 km^2, a great improvement of earlier (MSU) sensors. This highly refined data set might allow accurate wind fields to be found in areas that are inaccessible to other sensing methods.

2 Mathematical formulations

Consider the quasi-static primitive equations on an f-plane. The momentum equations are

$$\partial_t u + u\partial_x u + v\partial_y u - fv + \partial_x \Phi = 0 \tag{1}$$

$$\partial_t v + u\partial_x v + v\partial_y v + fu + \partial_y \Phi = 0. \tag{2}$$

Computing the curl of this system produces the divergence equation,

$$\frac{D}{Dt}\delta + \delta^2 - 2\frac{\partial(u,v)}{\partial(x,y)} - f\zeta + \nabla^2\Phi = 0 \tag{3}$$

where $\delta := \partial_x u + \partial_y v$ is divergence, and $\zeta := \partial_x v - \partial_y u$ is vorticity.

Define the horizontal wind in terms of the streamfunction ψ and velocity potential χ, such that

$$u := -\partial_y \psi + \partial_x \chi \quad \text{and} \quad v := \partial_x \psi + \partial_y \chi. \tag{4}$$

Recasting the divergence equation in a streamfunction form and putting all terms involving divergence in on the right produces the nonlinear balance equation

$$2(\partial_{xx}\psi\partial_{yy}\psi - (\partial_{xy}\psi)^2) + f\nabla^2\psi - \nabla^2\Phi = F.$$

It can be shown there are at most two solutions to nonlinear balance equation boundary value problem if

$$\nabla^2\Phi + F > -\frac{f^2}{2}.$$

One solution corresponds to positive absolute vorticity and the second to negative absolute vorticity. The *realizability condition* can now be stated. It is

$$\nabla^2\Phi + \frac{f^2}{2} + F > 0. \tag{5}$$

Neglecting divergent terms by assuming that $F := 0$, the realizability condition reduces to the *ellipticity condition*

$$\nabla^2\Phi + \frac{f^2}{2} > 0. \tag{6}$$

3 Motivation

If the *realizability* and/or *ellipticity* conditions are met, then there are several numerical schemes that can recover velocity from pressure gradient information. As the resolution of remote sensing devices has improved, however, there have been an increasing number of examples where these conditions have been violated. Notably, this occurs in strong cyclonic flows in regions of large pressure gradient. While data smoothing 'fixes' these regions, a real question remains - Does the solution to the smoothed problem reasonably approximate the true solution?

Tribbia (1982) explores the failure of nonlinear normal-mode initialization. The model discussed describes the flow of a rotating fluid in a right-circular cylinder. The upper surface is free and the fluid is assumed to have uniform density. The shallow water equations are used, and a low order spectral model is used to linearize the system. The solution to the iterative scheme is seen to be exactly the discrete gradient wind balance. It appears from Tribbia's analysis that there exists a true threshold for a real balancing wind to exist. The pressure gradient is only able to balance a certain portion of the centripetal and Coriolis forces. Once this maximum is reached, the mathematical model breaks down, and this is reflected in non-convergence of the scheme. He makes the following three conclusions:

1. There are limits on geostrophic vorticity (and therefore rotational mode amplitude). Unconstrained balancing breaks down at this point.

2. This limit corresponds to realizable wind field (the ellipticity condition in the NLB).

3. Current iteration methods work in cases where a real balance wind field exists.

This work would suggest that the search for stable numerical methods for the normal mode initialization (and, similarly, for the nonlinear balance equation) should remain restricted to *realizable* domains. Regions of non-ellipticity correspond to regions of non-convergence. It appears from these low order spectral methods that the transition from elliptic to hyperbolic regions violate the physical constraints of physical balance in the shallow water system, and also produces a numerical instability in the nonlinear normal mode balance scheme. The existence of non-elliptic ground data would suggest that these models require additional features in order to more accurately represent the true physics of flow in regions of non-ellipticity.

The current PV inversion schemes based on the streamfunction formulation of the divergence equation typically require balanced fields, no PV anomalies, for convergence. By the numerical addition of PV, the problem becomes increasing elliptic and therefore easier to solve.

4 Research Proposal

Can remote sensing provide enough information to recover accurate wind velocity estimates? I believe that this answer is **YES**.

I suspect that earlier methods based on the nonlinear balance equation will prove to be infeasible. The momentum equations, since they are quasi-linear, might provide a better foundation for future research. I also suspect that 'non-ellipticity' classification is a feature of the 2d model that might fail to persist in a full 3d time dependent flow. I suspect that incorporating vertical flow might help explain the difficulty of 'non-elliptic' ('hyperbolic' in the time dependent form) regions. When the flow becomes hyperbolic, that is when the pressure gradient can no longer contain the rotational flow, traveling waves are formed. Ignoring the vertical scale flow means that these waves must leave the computation region. But perhaps the z component of the pressure gradient is less than the horizontal component, and therefore would allow the flow to 'escape' vertically. In mesocale modeling, the slow mode is often considered to be the important physical flow, while the fast mode is treated minimally. Perhaps the fast mode doesn't 'propagate to infinity' but instead is captured by the system.

A 3d cylindrical formulation might allow these conjectures to be analyzed. I'd like to work with three sets of equations:

- Navier-Stokes
- Momentum
- Nonlinear Balance

I propose to work on several aspects of this problem, and their application to PV inversion schemes.

1. Survey of current cyclonic flow models. Do these produce 'non-elliptic' regions?
2. Validate the Navier-Stokes, momentum and nonlinear balance systems using available ground data.
3. If necessary, use scale analysis to test that the momentum equations and nonlinear balance are feasible approximations. Explore both 2d and 3d formulations; cartesian, polar and cylindrical.
4. Collect and analyze physical 'non-elliptic' data (ie Paegle 83). Is this a 2d phenomenon, or 3d? Compare to simulation data. Consider both a radially symmetric flow, which would allow a dimension reduction, as well as non symmetric flows.
5. Linearize both the momentum equations and the nonlinear balance equation around a known velocity field (both 'elliptic' and 'non-elliptic', and solve the resulting linear convection-diffusion system analytically. Do these differ? If so, how?
6. Explore numerical techniques for the linear and nonlinear recovery in 2 and 3 dimensions. Output Least Squares (OLS) techniques (coupled with regularization) should be feasible, but robustness and accuracy is a concern. Also, minimization doesn't provide access to the true dynamics. An OLS solution to a smoothed system should be a good initialization to other types of numerical methods, ie Finite Difference evolution schemes. For this phase, use simulation data initially, then move to AMSU data.
7. Apply numerical methods to PV inversion schemes in regions containing PV anomalies, including highly convective regions.
8. A bifurcation occurs in the nonlinear balance equation. In the literature, this the point where 'ellipticity' or 'realizability' condition is violated, and amounts to a region where the prescribed pressure gradient (the bifurcation parameter in this case) can no longer balance the velocity field. Do the momentum equations exhibit a bifurcation as well? Numerical continuation techniques can be used to explore this parameter space.
9. (possible) Consider existence and uniqueness for 2d and 3d flow regime for with a prescribed forcing (pressure gradient). A survey of work on Taylor-Couette flow might provide insight.

References

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