# Coefficient Recovery in the Richards Equation

## JMM 2018 in San Diego

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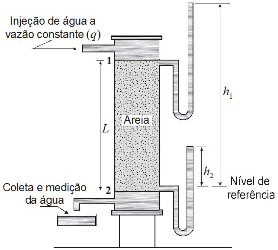
# Outline

# history: Darcy




(A. Jose Rosa, Wikimedia Commons)

The father of groundwater science was M. Henri Darcy. Through experiment, he developed a constitutive relationship for saturated flow through an aquifer.

$$q = -K(\hat{h})\frac{(h_1 - h_0)}{(z_1 - z_0)}$$

The flux $q$ in a soil column is proportional to the difference in head pressures at two points in the column divided by the vertical difference between these two points.

# history: Darcy & Richards

Combining Darcy's law with the conservation of mass for water content $\Theta(h)$ given by ($z = 0$ at top)

$$\partial_t \Theta + \partial_z q = 0$$

yields the equation attributed to Lorenzo Richards in 1931:

$$\partial_t \Theta(h) - \partial_z \left( K(h) \left( \partial_z h - 1 \right) \right) = 0$$

or, letting $C(h) = \partial_h \Theta$,

$$C(h) \partial_t h - \partial_z \left( K(h) \left( \partial_z h - 1 \right) \right) = 0$$

(1d vertical head-based or capacity/conductivity form)

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# The Inverse Problem

Goal: Recover approximations of $C(h)$ and $K(h)$ using easily availible experimental lab data.

A simple (direct) experiment, similar to Darcy's, could be modeled as :

$$\begin{aligned} C(h)\partial_t h - \partial_z\left[K(h)(\partial_z h - 1)\right] &= 0, && 0 < t < T, 0 < z < L \\ h(z,0) &= 0, && 0 < z < L \\ \partial_z h(0,t) - 1 = 0 \text{ and } h(L,t) &= s(t) && 0 < t < T \end{aligned}$$

This could represent a totally saturated column, driven by (a strictly decreasing) $s(t)$ at $z = L$ and no flow at $z = 0$.

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# theory: the integral ID

Common inversion techniques include OLS, EEM, and (more recently) Carleman estimates. The integral based algorithm presented will instead exploit and adjoint driven integral expression which directly relates changes in the unknown coefficients to corresponding changes in the measured output, and which provides clear insight about the inversion.

For suitable of coefficients $(C, K)$, the Richards equation has unique solution $h$ with observable quantities

- pressure head at inflow: $p(t) = h(0, t)$ and
- flux at outflow: $q(t) = K(h(L, T))(\partial_z h(L, t) - 1)$.

Similarly, $(\tilde{C}, \tilde{K})$ yields solution $\tilde{h}$ and output $(\tilde{p}, \tilde{q})$. In what follows, the first group will represent the true system and the second an approximation.

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# theory: DuChateau

Twenty years ago, DuChateau developed an integral identity well suited to the recovery of $C$ and $K$ from easily observable output data. The idea hinges on a duality paring and the solution $\phi$ to the adjoint problem

$$\begin{aligned} \alpha(z,t)\partial_t\phi + \beta\partial_{zz}\phi + \gamma\partial_z\phi &= 0 && 0 < t < T, 0 < z < L \\ \phi(z,T) &= 0 && 0 < z < L \\ \beta(0,t)\phi(0,t) = p^*(t), \qquad \phi(L,t) &= q^* && 0 < t < T \end{aligned}$$

where $\alpha = \Delta h \int_h^{\tilde{h}} C(s)\,ds, \beta = \Delta h \int_h^{\tilde{h}} K(s)\,ds$, and $\gamma = \Delta h \int_h^{\tilde{h}} K'(s)\,ds$ and $\Delta h = h_2 - h_1$.

# theory: Integral Identity

The full identity is

$$\int_0^\tau (\Delta q q^* + \Delta p p^*)\, dt$$
$$= \int_0^\tau \int_0^L \left[\Delta C \partial_t \tilde{h} \phi + \Delta K \left(\partial_z \tilde{h} - 1\right) \partial_z \phi\right] dz\ dt,$$

where $\Delta C = C - \tilde{C}, \Delta K = K - \tilde{K}, \Delta p = p - \tilde{p}$, $\Delta q = q - \tilde{q}$.

# Dual Data

Notice that the dual appears in a duality pairing with terms of the direct problem. Setting $q^* \equiv 0$ results in the $p$-identity

$$\int_0^\tau \Delta p p^* dt$$
$$= \int_0^\tau \int_0^L \left[ \Delta C \partial_t \tilde{h} \phi_p + \Delta K \left( \partial_z \tilde{h} - 1 \right) \partial_z \phi_p \right] dz \; dt,$$

and $p^* \equiv 0$ in the $q$-identity

$$\int_0^\tau \Delta q q^* dt$$
$$= \int_0^\tau \int_0^L \left[ \Delta C \partial_t \tilde{h} \phi_q + \Delta K \left( \partial_z \tilde{h} - 1 \right) \partial_z \phi_q \right] dz \; dt,$$

where $\phi_p, \phi_q$ are solutions for the appropriate dual data.

# theory: Linear system

A monotonicity result allows a continuous piecewise linear approximation of $\tilde{C}$ and $\tilde{K}$ to be recovered a linear system valid over an arbitary region $\Omega = (0, \tau) \times (0, L)$

$$M \begin{bmatrix} \tilde{C}_i - \tilde{C}_{i-1} \\ \tilde{K}_i - \tilde{K}_{i-1} \end{bmatrix} = \begin{bmatrix} d_{i-1} \\ e_{i-1} \end{bmatrix}$$

with

$$M_{11} = \iint_\Omega \Lambda_1(\tilde{h})(\partial_z \tilde{h} - 1)\partial_z \phi_p \qquad M_{12} = \iint_\Omega \Lambda_1(\tilde{h})\partial_t \tilde{h} \phi_p$$

$$M_{21} = \iint_\Omega \Lambda_1(\tilde{h})(\partial_z \tilde{h} - 1)\partial_z \phi_q \qquad M_{22} = \iint_\Omega \Lambda_1(\tilde{h})\partial_t \tilde{h} \phi_q$$

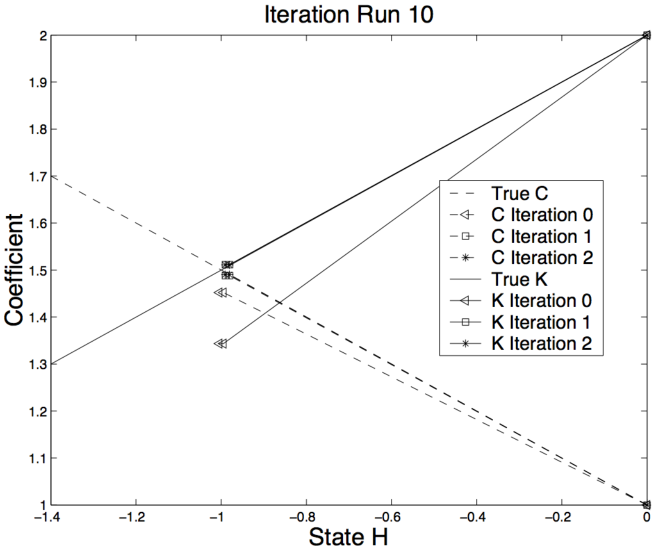
$$d_1 = \int_0^\tau \Delta q q^* \text{ and } \qquad e_1 = \int_0^\tau \Delta p p^*.$$

where $\tilde{C} \approx \sum_i C_i \Lambda_i$ and $\tilde{K} \approx \sum_i K_i \Lambda_i$.

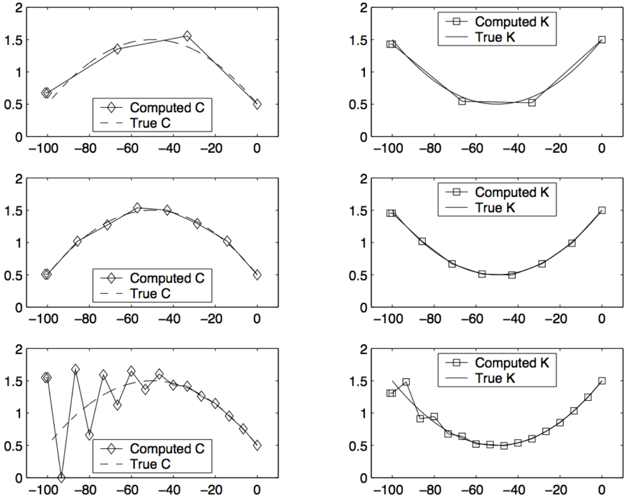
# numerics: snippet

```
[T,P,Q,C0,K0] = load Data.file
while (max(t) < Tmax)
   for strip = level-1:level
      tspan = ????;
      [tspan,h,h_z,h_t,p,q] = solve_forward(C,K,...); % FORWAR
      dp = P-p; dq = Q-q;  % ERROR IN OUTPUT
      [phip,phip_z] = solve_dualp(...) % DUAL p
      [phiq,phiq_z] = solve_dualq(...) % DUAL q
      % CONSTRUCT M and b
      [M,b] = make_M(h,h_z,h_t,phip,phip_z,phiq,phiq_z,dp,dq,t
      [deltaC,deltaK] = M\b;            % COMPUTE COEFFS
      C = C + deltaC; K = K + deltaK;  % UPDATE COEFFS
      % Iterate?
      level = level + 1; % MOVE TO NEXT COEFFICIENT INTERVAL
   end
end
```

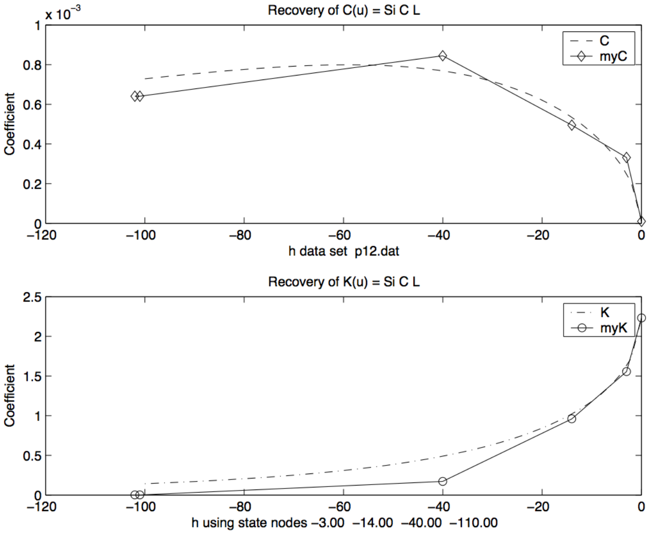
# numerics: Iteration?



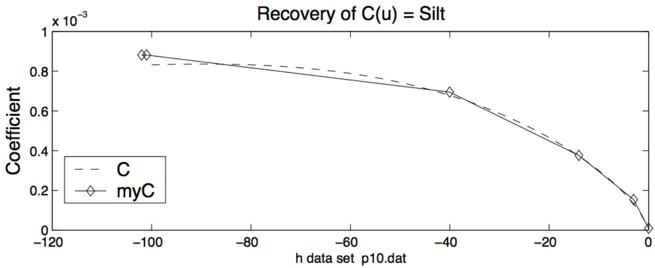
# numerics: Dimension

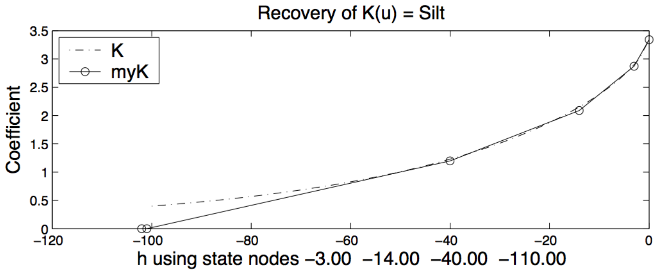


# numerics: Silty Clay Loam

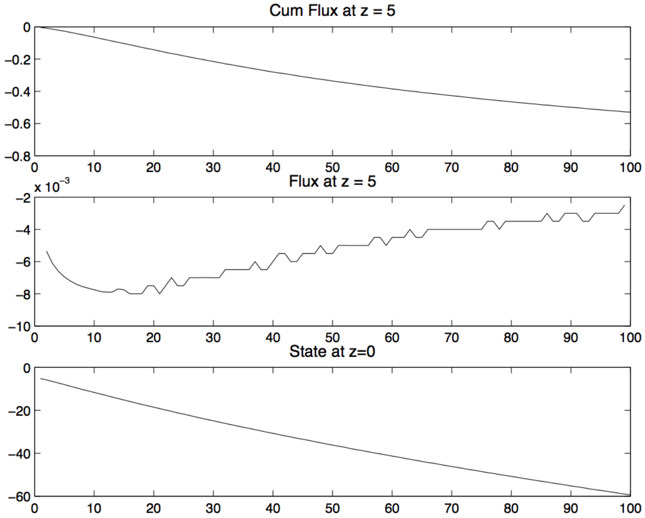


# numerics: Silt



# numerics: Cumulative flux? (SL)



# numerics: Cumulative flux formulation

Consider the output integral term in q-idenity:

$$\int_0^T \Delta q q^*(t)dt$$

If $Q(t) := \int_0^t q(s)dt$ and $\tilde{Q}(t) := \int_0^t \tilde{q}(s)dt$, then

$$b_2 = \int_0^T (Q(t) - \tilde{Q}(t))Q^{'*}(t)\ dt.$$

(after integrating by parts and noting that $Q(0) - \tilde{Q}(0) = 0$ and choosing $Q^*(T) = 0$.)

# Conclusions

- $M$ is the realization of the discrete (approximate) map.
- Easy to understand and interpret!
- Useful in recovery
- Useful in failure!
- Seems natural setting