Solutions, Stability, Bounds, and Control?

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Outline

$y' = \alpha y^2$

We'll explore a toy problem...

$$y' = \alpha y^2 \qquad y(0) = y_0$$

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An analytic solution to this one is easy:

$$y(t) = -\frac{y_0}{\alpha y_0 t - 1}$$

$y' = \alpha y^2$

What happens if we try (formal, for now) series?

$y' = \alpha y^2$

What happens if we try (formal, for now) series?
Let

$$y(t) = \sum_{k=0}^{\infty} y_k t^k$$

then

$$\sum_{k=0}^{\infty} (k+1) y_{k+1} t^k = \alpha \sum_{k=0}^{\infty} \left(\sum_{i,j \geq 0}^{i+j=k} y_i y_j \right) t^k$$

so we equate coefficients to get

$$y_{k+1} = \frac{\alpha}{k+1} \sum_{i+j=k} y_i y_j$$

$y' = \alpha y^2$

Or, with MAPLE:

```
> ODE1 := diff(y(t),t) = alpha*y(t)^2;
> IC := y(0) = y0;
> y1 := dsolve({ODE1,IC},y(t))
```

and

```
> Y1 := dsolve({ODE1,IC},y(t),series);
       Y1 := 1+2*alpha*t*y0+3*alpha^2*y0^2*t^2+...
```

$y' = \alpha y^2$

Given either solution, we see

$$\partial_\alpha y1 = \frac{y_0^2 t}{(\alpha y_0 t - 1)^2}$$

or

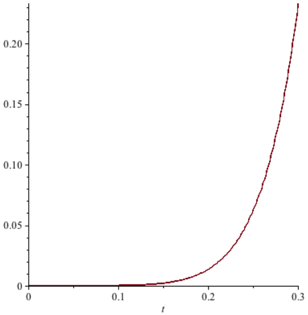
$$\partial_{y_0} Y1 = 1 + 2\alpha t y_0 + 3\alpha^2 y_0^2 t^2 + \ldots$$

Computing sensitivity to perturbation in initial condition and/or parameter(s) is easy!

$y' = \alpha y^2$

What about the error?

```
> alpha := 1; y0 := 2; plot(abs(Y1-y1),t=0..0.5);
```



Can we quantify this?

$y' = \alpha y^2$

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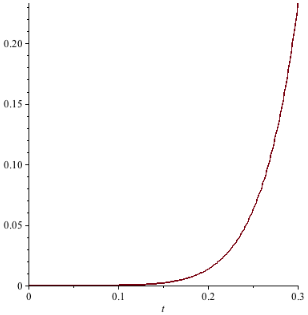


Can we quantify this?
Let's take a little diversion...

$y' = \alpha y^m$

The solution to the constant coefficient nonlinear IVODE

$$y' = \alpha y^m \qquad y(0) = y_0$$

is messy:

$$y(t) = \left((\alpha - \alpha m)t + y_0^{1-m}\right)^{-(m-1)^{-1}}.$$

But the ratio of $\frac{y}{y'}$ isn't!

$$\frac{y}{y'} = \frac{(\alpha - \alpha m)t + y_0^{1-m}}{\alpha}$$

and so

$$y'(t) = \underbrace{\frac{\alpha}{(\alpha - \alpha m)t + y_0^{1-m}}}_{K(t)} y(t),$$

a non-constant coefficient LINEAR ode.

$y' = \alpha y^m$: important aside

So

$$y'(t) = \underbrace{\frac{\alpha}{(\alpha - \alpha tm)t + y_0^{1-m}}}_{K(t)} y(t),$$

has solution

$$y(t) = y_0 exp\left(\int_0^t K(\tau)d\tau\right)$$

or, via series,

$$Y_{k+1} = \frac{\alpha(1 + (m-1)k)}{y_0^{1-m}(k+1)} Y_k$$

$y' = \alpha y^m$: important aside

From

$$Y_{k+1} = \frac{\alpha(1 + (m-1)k)}{y_0^{1-m}(k+1)} Y_k$$

and for $m \geq 2$,

$$Y_{k+1} \leq (m-1)|y_0|^{m-1} Y_k := C_\infty Y_k.$$

This leads directly to a geometric series bounding $y(t)$:

$$y(t) \leq \frac{|y_0|}{1 - C_\infty} = |y_0| \sum_{k=0} (C_\infty t)^k$$

Now for the bound...

$y' = \alpha y^2$: back to error

From

$$y(t) \leq \frac{|y_0|}{1 - C_\infty} = |y_0| \sum_{k=0} (C_\infty t)^k$$

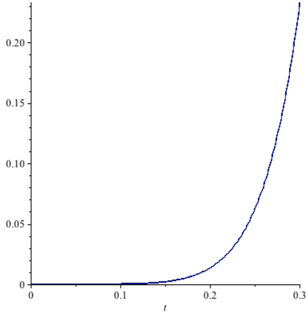
we see that the absolute error is

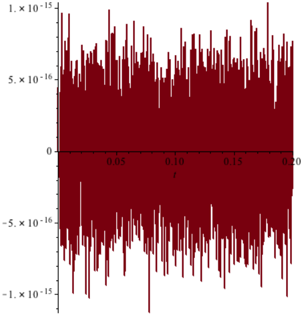
$$|Y1 - y1| \leq |y_0| \sum_{k=n+1}^{\infty} C_\infty |t|^k \leq \frac{|y_0| C_\infty^{n+1}}{1 - C_\infty |t|}$$

where $C_\infty = |y_0 \alpha|$.
An ERROR bound!

$y' = \alpha y^2$: error plots

```
> ee :=  abs(Y1-y1);
> m := 2; Cinf := y0*alpha;
> EE := N -> abs(y0)*(Cinf*t)^(N+1)/( 1 - Cinf*abs(t));
> plot({ee,EE(5)},t=0..0.3);
> plot({ee-EE(5)},t=0..0.2);
```





$y' = \alpha y^2$: for control?

Suppose we want to control

$$y' = \alpha y^2 \qquad y(0) = y(0)$$

so that $y(T) = \beta$.
If we apply frictional damping to the system

$$y' - \alpha y^2 = u,$$

where $u = kty'$, can we drive the system to the desired state?

$y' = \alpha y^2$: for control?

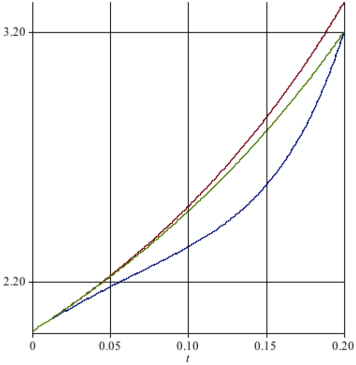
Let's try to drive the system so that $Y(0.2) = 3.2$.

```
> ODE := diff(y(t),t) = alpha*y(t)^2 + k*t*diff(y(t),t);
> IC := y(0) = 2;
> Y := convert(rhs(dsolve({ODE1,IC},y(t),series)),polynom);
> kvals := solve(subs(t=0.2,Y)=3.2);
       -2.051118481+8.300750459*I, -.6504685102 , ...
> subs({k=kvals[2],t=0.2},Y);
        3.200000000
```

$y' = \alpha y^2 + kty'$

It looks like we can.

```
> kvals := solve(subs(t=0.2,Y)=3.2);
       -2.051118481+8.300750459*I, -.6504685102 , ...
>plot({Yk(0),Yk(kvals[2]),Yk(kvals[3])},t=0..0.2);
```



Repeated application allows trajectory control, and our error bound still applies to the forced system!

Conclusions

We considered a toy problem already cast as a polynomial ode. Extension and application of these methods will rely on the use of auxiliary variables to build a system of polynomial IVODEs. Once the system is polynomial, series methods allow remarkably direct analysis.

- Analytic approximation of solution
- Stability and sensitivity
- Easy error BOUND
- Simple control?

These techniques should apply to a broad range of highly nonlinear ODE.

Thank you

Thanks!

Questions?

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Thelwell et al.: Cauchy Kowalevski and Polynomial ODE *EJDE*, 11, 1–8, 2012.

James Sochacki: Polynomial ordinary differential equations *Neural Parallel & Scientific Computations*, 18(3-4):441–450, 2010.

$y' = \sin(y)$???

What now?

$y' = \sin(y)$???

What now?
We introduce auxiliary variables to generate a polynomial system.

$$v_1 = \sin(y) \quad v_2 = \cos(y)$$

$$\begin{aligned} y' &= v_1 \\ v_1' &= v_2 v_1 \\ v_2' &= -v_1^2 \end{aligned}$$

All the same ideas apply!