Intro to Inverse Problems

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We'll talk about some very basic inverse problems. While the examples aren't very hard, they make us aware of some common difficulties. Our examples will come from ODE and PDE problems

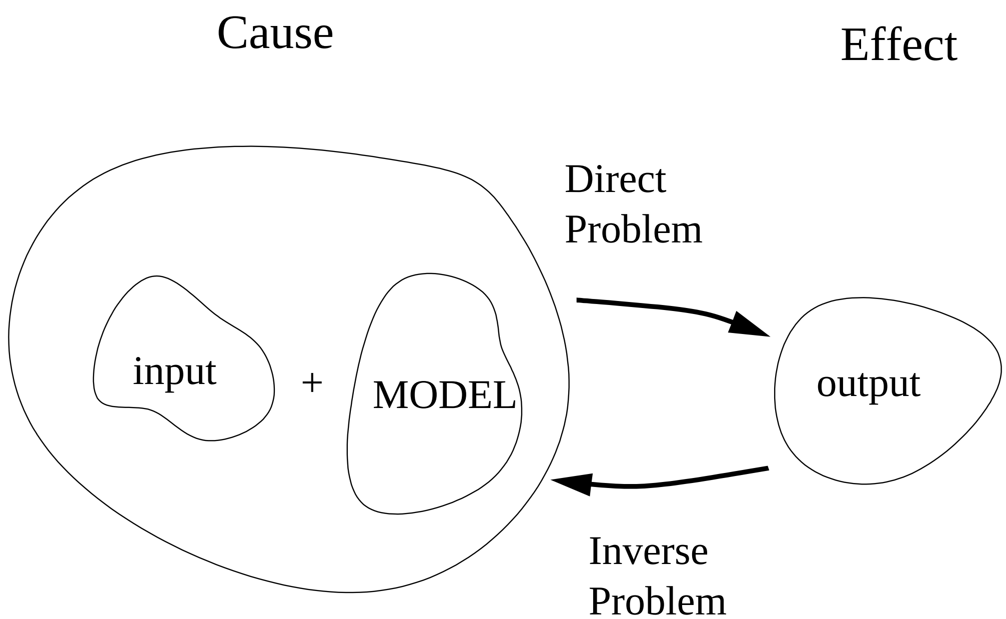
- Pressure Model

- Population Model

- Hyperbolic PDE

- Parabolic PDE

- Elliptic PDE



Inverse problems ask the following:

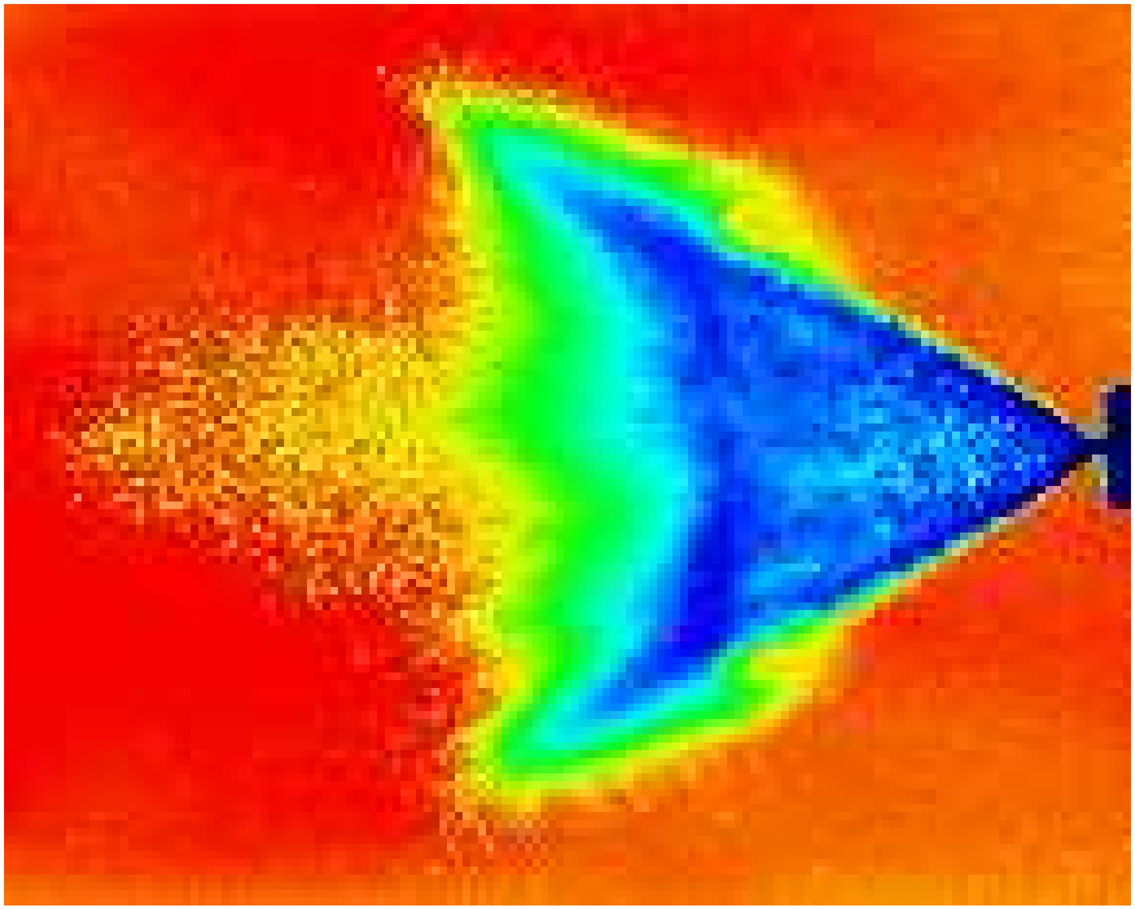
Can we, given some prescribed output, determine properties of the map M and or of the input x?

Inverse problems naturally occur in many fields:

- Geology (material properties, domain recovery),
- Ecology (parameter estimation,...),
- Medicine (parameter estimation, tomography)
- inverse scattering (ie acoustic obstacle analysis, material properties, non-destructive testing)
- Others ....

Magnetic Resonance Image (MRI)

Electric Capacitance Tomography (ECT)



Electrical Impedence Tomography, part I (EIT)

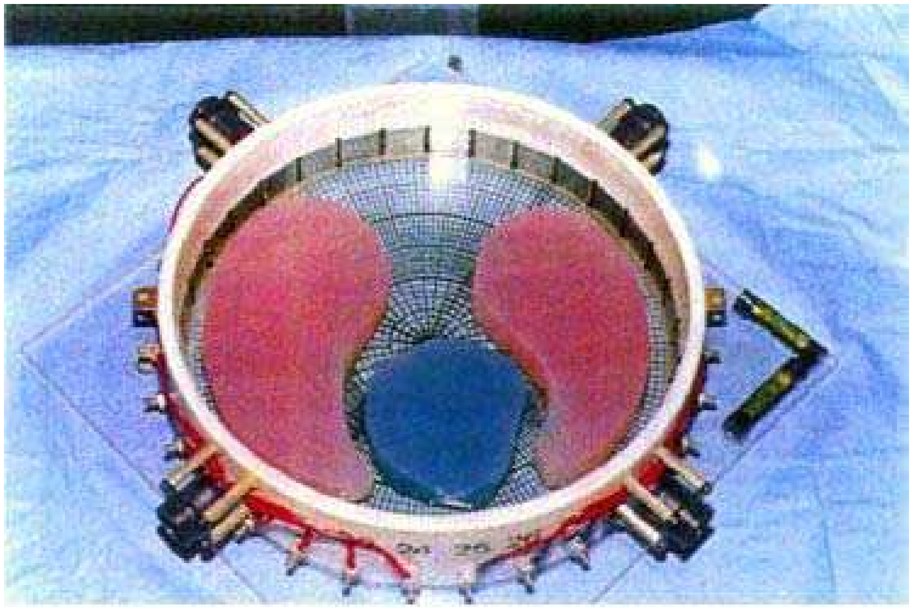
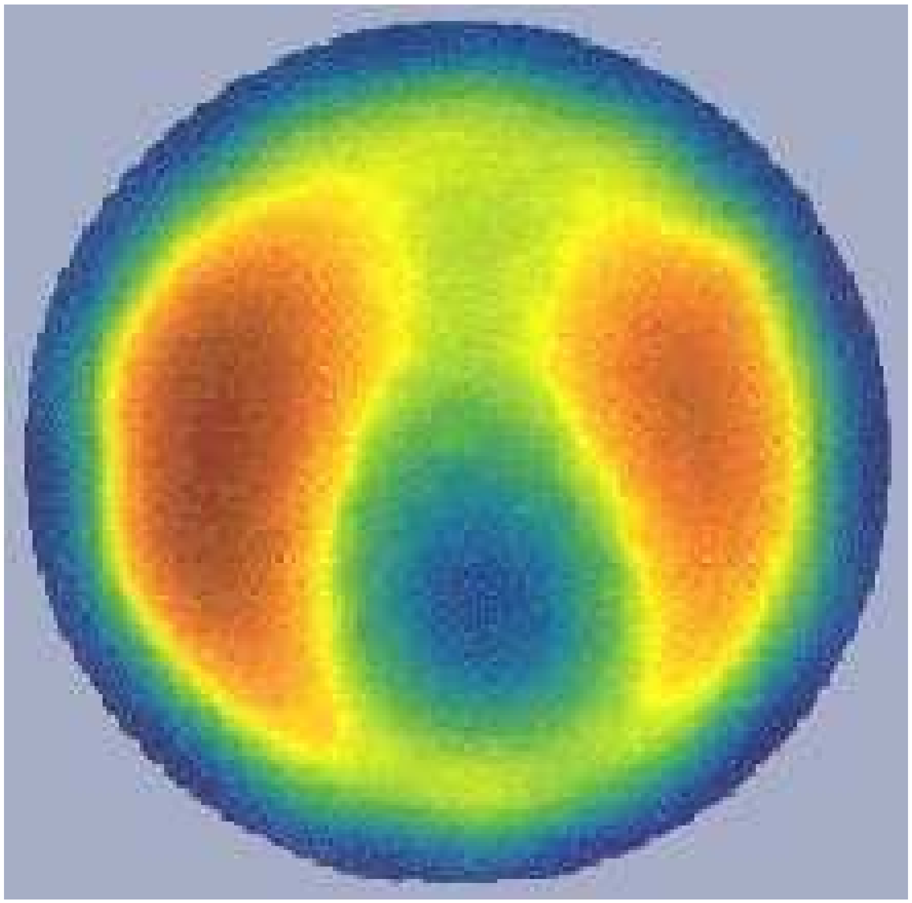
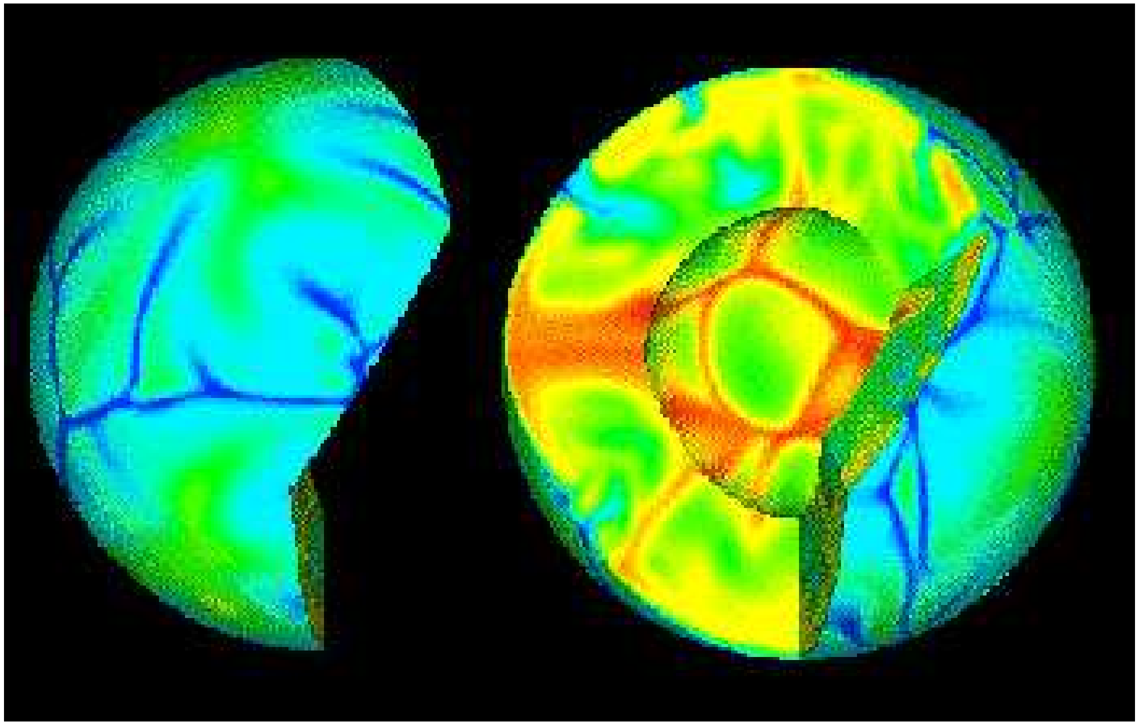


Figure A1. The two-dimensional phantom thorax with pink agar lungs, blue agar heart and black skin in saline. The electrodes are stainless steel, 2.54 x 2.54 cm. The resistivity of the heart is 150 ohm-cm, and that of the lungs is 1000 ohm-cm.

Electrical Impedence Tomography, part II (EIT)

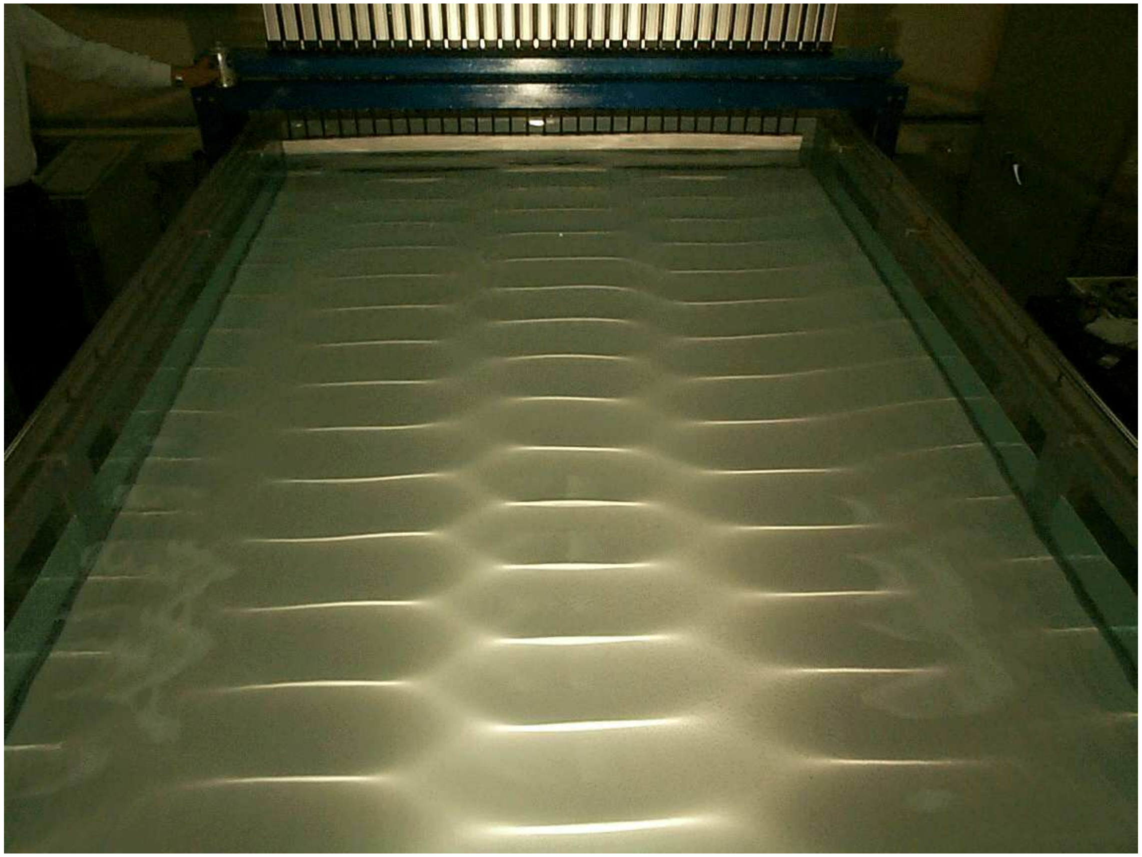


Tomography

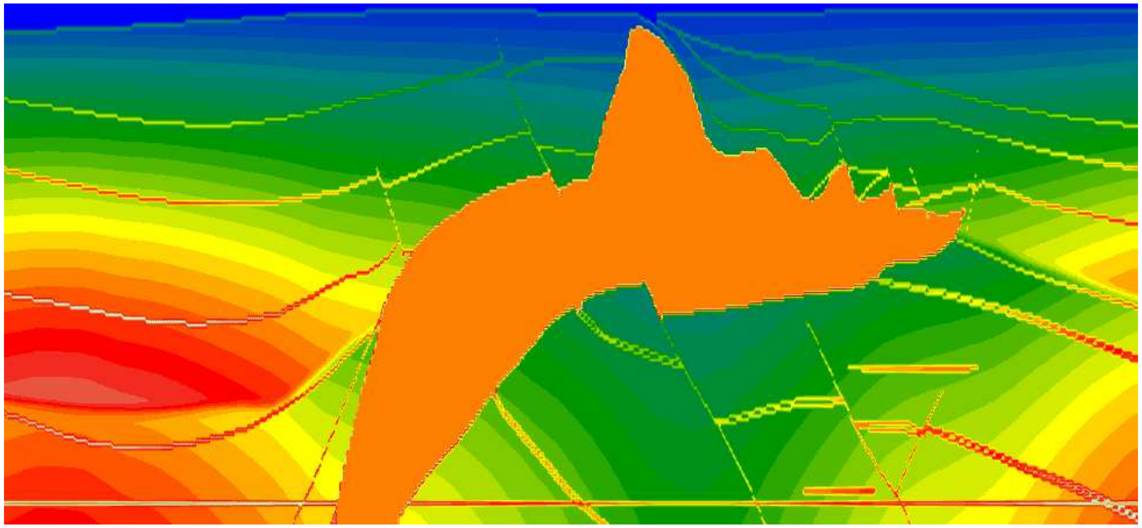


Domain Recovery

Ray Tracing



Seismic Tomography Image



Most inverse problems share the feature being not well-posed. Well-posedness is a concept developed by Hadamard (in the early 1900's).

A well-posed problem in one in which:

there **exists** *a* **unique** *solution that* **depends continuously** *on the data.*

We'll consider several examples that might be classified as follows:

- Missing parameter -

- Missing domain -

- Missing initial/boundary conditions -

A Pressure Example

Suppose that the rate of change of pressure with respect to depth in a column of fluid is constant, and denote the pressure at the surface by β. Writing this as an initial value problem , we have

$$\frac{dP}{dz} = \alpha, \qquad P(0) = \beta.$$

In the direct problem, we consider α and β known, and ask to find $P(z)$. Then

$$P(z) = \alpha z + \beta.$$

In the inverse problem, α and β are unknown, and we are given the data points $\{(z_1, P_1), ..., (z_n, P_n)\}$.

We could determine the coefficients by solving the system

$$\begin{bmatrix} z_1 & 1 \\ \vdots & \vdots \\ z_n & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} P_1 \\ \vdots \\ P_n \end{bmatrix}$$

This inverse solution requires a little more work...

The accuracy of the inverse solution depends on several components.

- Ill-posedness physically inherent in system (Condition number)
- Stability of inversion scheme
- The data

These are in general difficult questions to answer.

A Problem in Population

Consider the exponential growth equation

$$\frac{d}{dt}u(t) = r(t)u(t)$$

Given initial population u_0 and growth rate $r(t)$, find $u(t)$. The solution:

$$u(t) = u_0 \exp\left(\int_0^t r(s)\ ds\right)$$

This process is relatively stable.

Inverse Problem -

Can we find $r(t)$ given some measured $u(t)$?

$$r(t) = \frac{1}{u}\frac{du}{dt} = \frac{d}{dt}(\ln u)$$

If there is error, the derivative might blow up. Small perturbations in measured output can result in large perturbations of the growth function. It's an *ill-posed* problem.

Hyperbolic Problems

The prototype hyperbolic equation is the wave equation. The Cauchy problem for the general wave equation is

$$\begin{aligned}
\partial_{tt}u(\mathbf{x},t) &= \nabla\cdot(\sigma^2\nabla u(\mathbf{x},t)) \\
u(\mathbf{x},0) &= f(\mathbf{x}) \\
\partial_t u(\mathbf{x},0) &= g(\mathbf{x})
\end{aligned}$$

In 1d, for fixed wave speed, an exact solution is given by D'Alembert's formula

$$
\begin{aligned}
u(x,t) = \frac{1}{2}\left(f(x-\sigma t) + f(x+\sigma t)\right) \ldots \\
+\frac{1}{2\sigma}\int_{x-\sigma t}^{x+\sigma t} g(s)\ ds
\end{aligned}
$$

assuming $f \in C^2(\mathbb{R})$ and $g \in C^1(\mathbb{R})$.

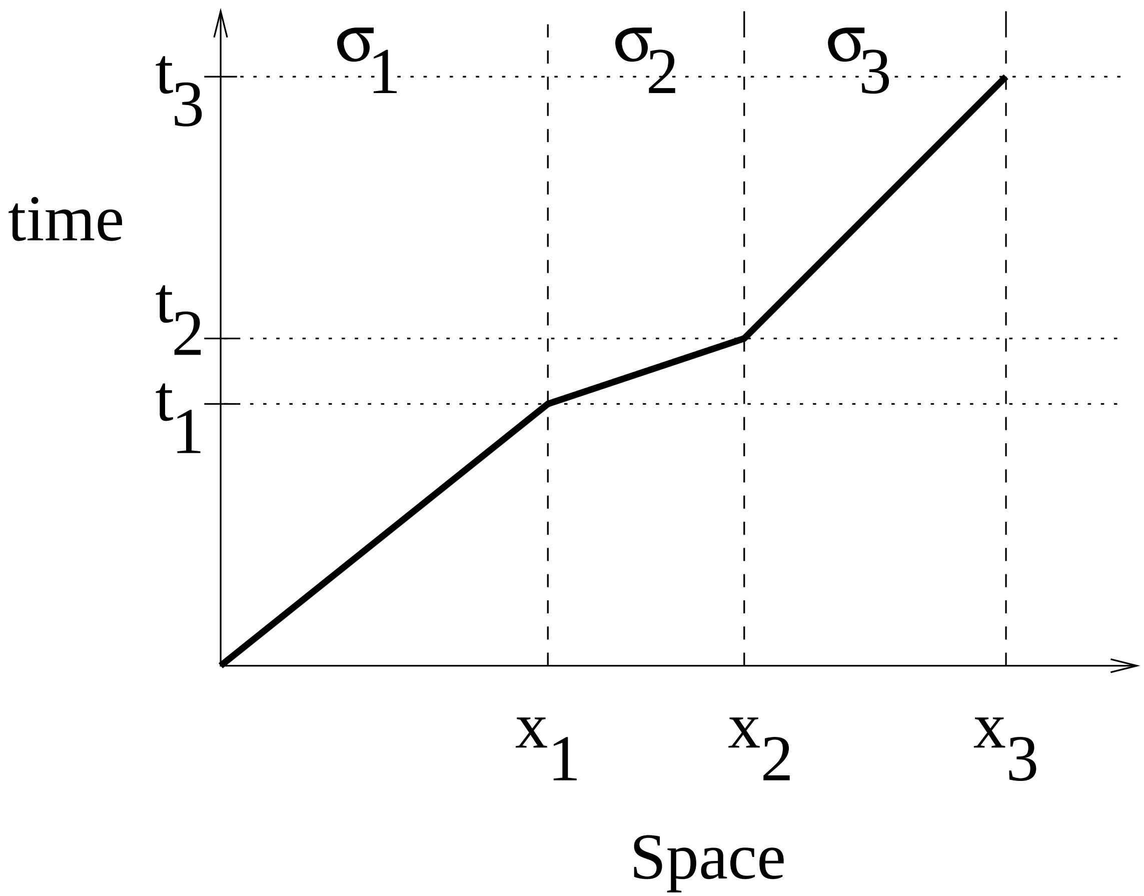
Resulting waves will have smooth trailing edges.

Tomography is a term which refers to the recovery of internal information from external measurement. Imagine an experiment in which an explosion is triggered. The vibrations are felt at distant points at future times. These acoustic vibrations can be modeled with the generalized 3d wave equation.

$$\begin{aligned} \partial_{tt}u(\mathrm{x},t) &= \nabla \cdot (\sigma^2 \nabla u(\mathrm{x},t)) \\ u(\mathrm{x},0) &= f(\mathrm{x}), \\ \partial_t u(\mathrm{x},0) &= g(\mathrm{x}), \end{aligned}$$

in which the speed σ is spatially dependent.

If we force $f(x) = \delta(x)$, the dirac delta function, we can draw the characteristics in space time:



Measuring the arrival time of the plane wave and calculating the distance of travel allows the construction of the linear system

$$\begin{bmatrix} x_1 & 0 & 0 \\ x_1 & x_2 - x_1 & 0 \\ x_1 & x_2 - x_1 & x_3 - x_2 \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{bmatrix} = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix}$$

which results in

$$\sigma_1 = t_1/x_1,$$

$$\sigma_2 = (t_2 - t_1)/(x_2 - x_1),$$

$$\sigma_3 = (t_3 - t_2)/(x_3 - x_2).$$

Domain Recovery

Acoustic scattering problem.

Recall that we can model the acoustic wave in a isotropic homogeneous medium in $\mathbb{R}^n, n = 2, 3$ using the wave equation

$$\partial_{tt} U - \sigma^2 \nabla U = 0.$$

Assuming a solution of the form $U(x,t) = u(x) e^{-i\omega t}$, the wave equation reduces to the Helmholtz equation

$$\nabla u - k^2 u = 0$$

where $k = \omega/\sigma$. The soft body direct scattering problem is to find $u = u^i + u^s$, where u^i is the incident wave and u^s the scattered wave, such that u satisfies the BVP

$$\begin{aligned} \nabla u(\mathbf{x}) - k^2 u(\mathbf{x}) = & \ 0 \quad \mathbf{x} \in \mathbb{R}^n - D \\ u(x) = & \ 0 \quad \mathbf{x} \in \partial \mathbf{D} \end{aligned}$$

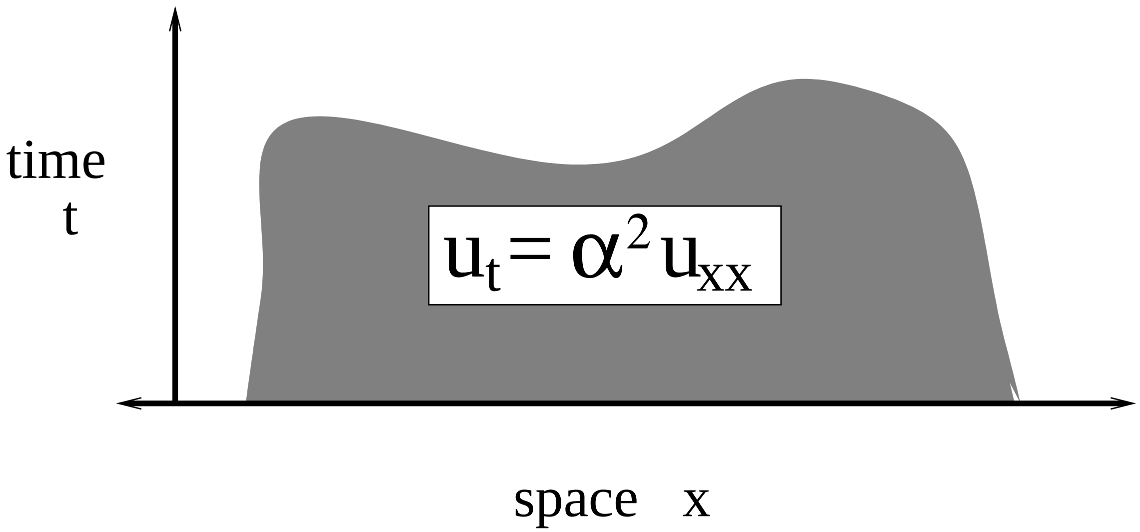
where D is the acoustic obstruction.

Inverse acoustic obstacle problem: *Use a given far field pattern* u_∞ *of a scattered wave* u^s *and one or more incoming plane waves* u^i *to determine the location and shape of the obstruction* D.

Parabolic Example

Consider the following initial boundary value problem in the quarter plane:

$$\begin{aligned} \partial_t u = &\ \alpha^2 \partial_{xx} u, && 0 < x < 1, 0 < t \\ u(0,t) = &\ g(t) && 0 < t \\ u(x,0) = &\ f(x) && 0 < x < \infty \end{aligned}$$



Standard Direct problem: Given α, $f(x)$ and/or $g(t)$ and u_0, find $u(x,t)$.

Since the HE is linear, we can split into two separate problems. Let $u = v + w$, where v satisfies the forward heat equation (FHE)

$$\begin{aligned} \partial_t v = & \ \alpha^2 \partial_{xx} v, && 0 < x, 0 < t \\ v(0,t) = & \ 0 && 0 < t \\ v(x,0) = & \ f(x) && 0 < x < \infty \end{aligned}$$

and w satisfies the sideways heat equation (SHE)

$$\begin{aligned} \partial_t w = & \ \alpha^2 \partial_{xx} w, && 0 < x, 0 < t \\ w(0,t) = & \ g(t) && 0 < t \\ w(x,0) = & \ 0 && 0 < x < \infty \end{aligned}$$

First, consider the FHE. A Fourier transform with respect to x yields

$$\begin{aligned} V'(t) = & -\alpha^2\xi^2 V(t), \quad x \in \mathbb{R}, t > 0 \\ V(0) = & \; F \end{aligned}$$

Inverting this (assuming $\alpha^2 t > 0$), we see that the solution is a convolution of the initial condition $f(x)$ and the fundamental solution $K(x,t)$, where

$$K(x,t) = \frac{1}{2\alpha\sqrt{\pi t}} \exp\left(\frac{-x^2}{4\alpha^2 t}\right)$$

then

$$v(x,t) = \int f(s)K(x-s,t)ds$$

Some Inverse Problems:

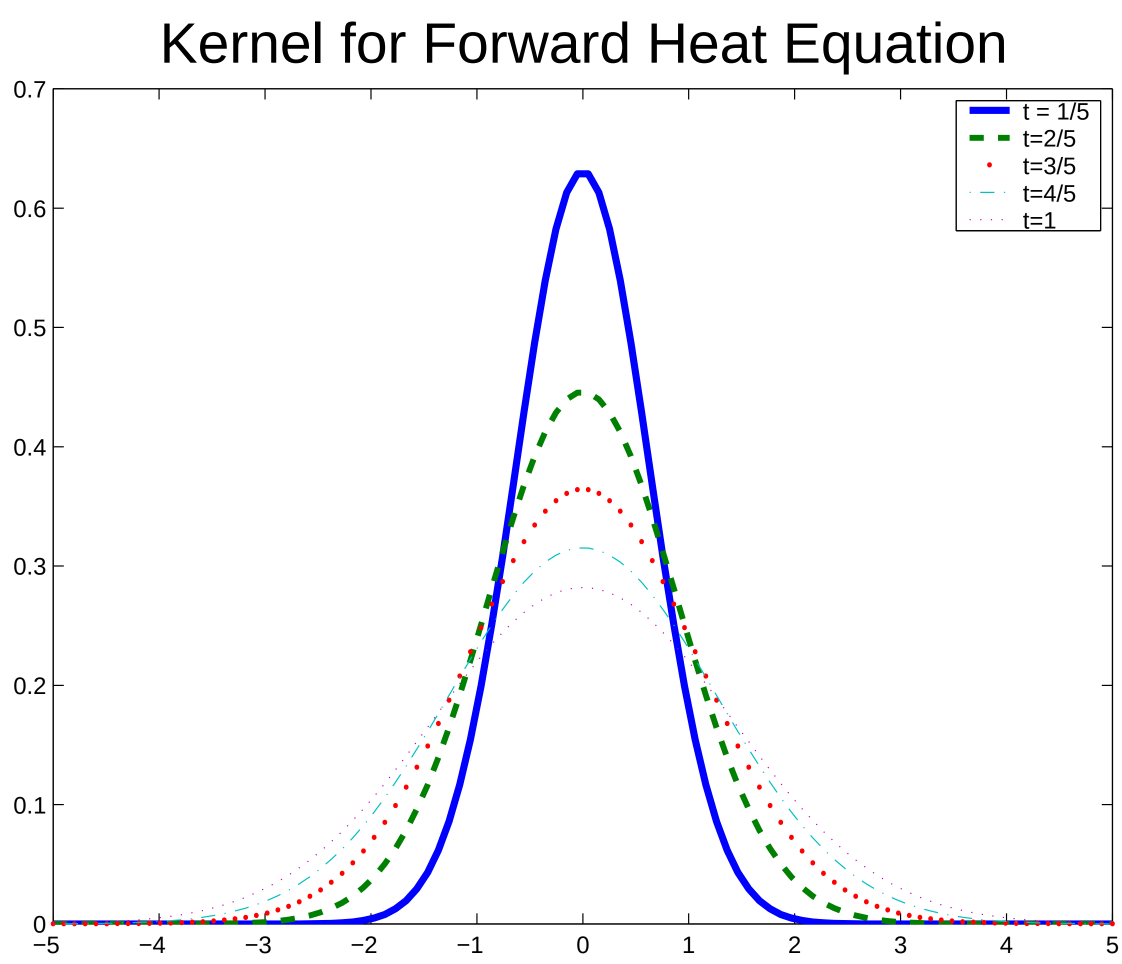
The Backward heat equation

$$\begin{aligned} \partial_\tau u(x,t) = & \ -\alpha^2 \partial_{xx} u, & x \in \mathbb{R}, \tau \in [0,T] \\ u(x,T) = & \ f(x) & x \in \mathbb{R} \end{aligned}$$

The (Inverse) Sideways Heat Equation

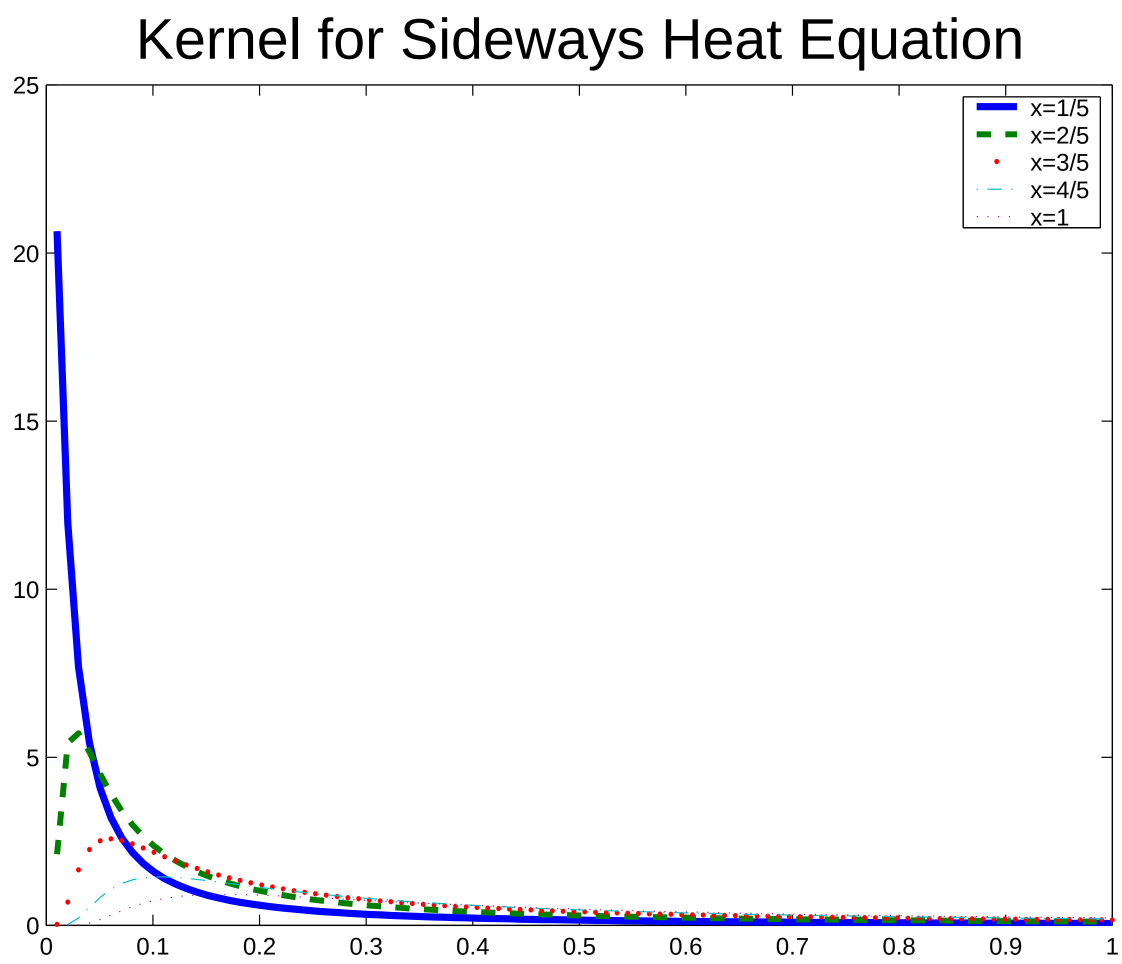
$$\begin{aligned} \partial_t w = & \ \alpha^2 \partial_{xx} w, & 0 < x, 0 < t \\ w(1,t) = & \ g(t) & 0 < t \\ w(x,0) = & \ 0 & \end{aligned}$$

Both considered ill-posed. A plot of the Forward Heat Equation Kernel: $\alpha^2 = 1$.

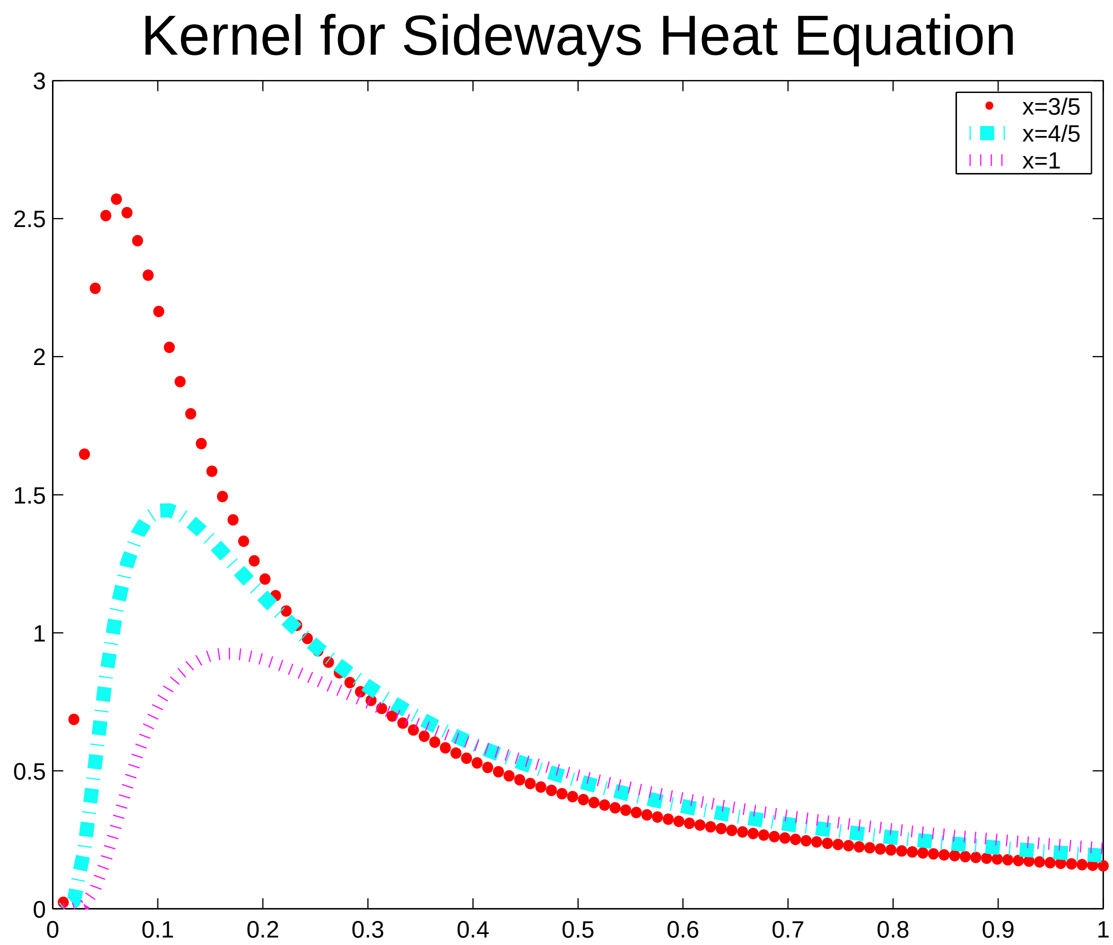


Kernel for Forward Heat Equation
t = 1/5
t=2/5
t=3/5
t=4/5
t=1
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
−5
−4
−3
−2
−1
0
1
2
3
4
5

The kernel for the Sideways Heat Equation: $\alpha^2 = 1$.



Consider the kernel of the SHE, limited to $x >= 3/5$.



An Elliptic Example

The prototypical forward elliptic problem is Laplace's equation, given by

$$\nabla \cdot (\sigma \nabla u(\mathbf{x})) = 0 \quad \mathbf{x} \in \Omega$$

over some domain Ω and various side conditions.

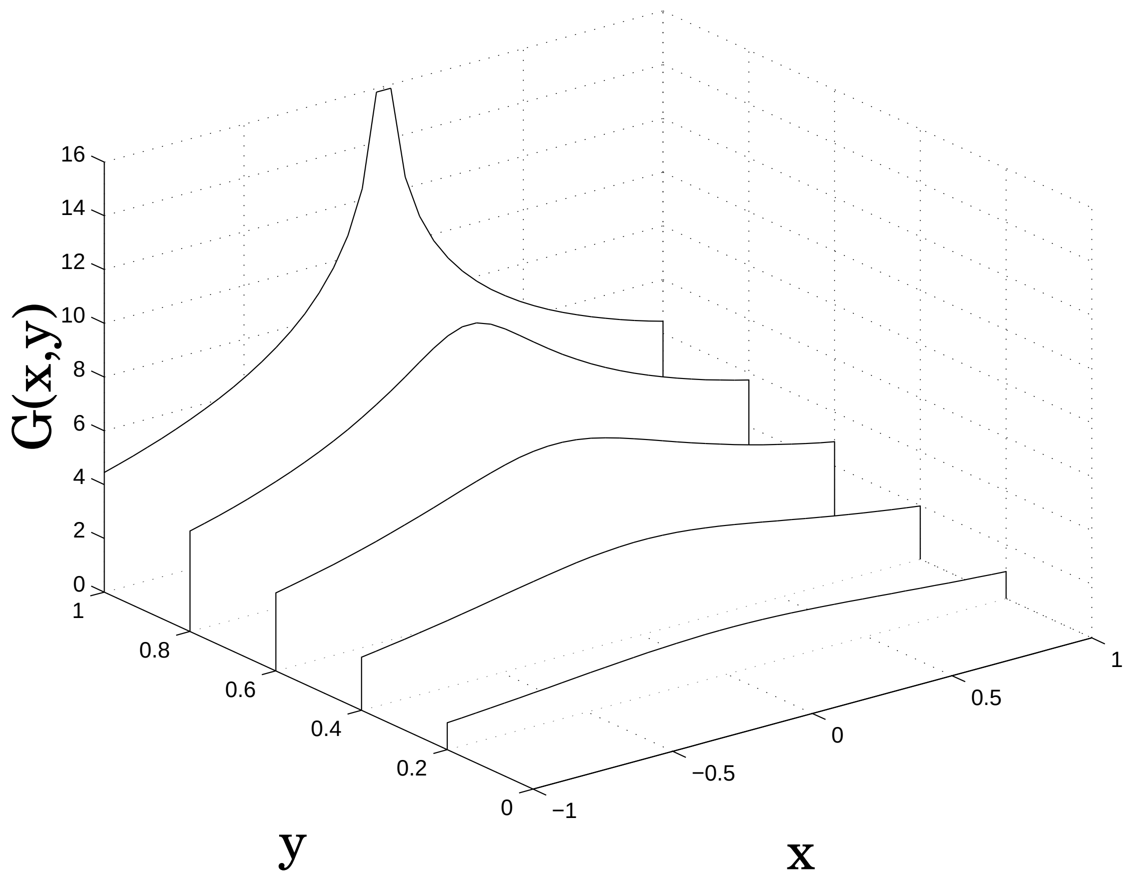
Consider the problem for which the parameter σ is unknown,

$$\begin{aligned}
\nabla \cdot (\sigma(x,y)\nabla u(x,y)) &= 0 && 0 < x,y < 1 \\
u(x,0) &= f(x) && x \in \Gamma_1 \\
\sigma(x,y)\partial_y u(x,0) &= j(x) && x \in \Gamma_2 \\
u(x,y) &= 0 && x,y \in \Gamma_3,
\end{aligned}$$

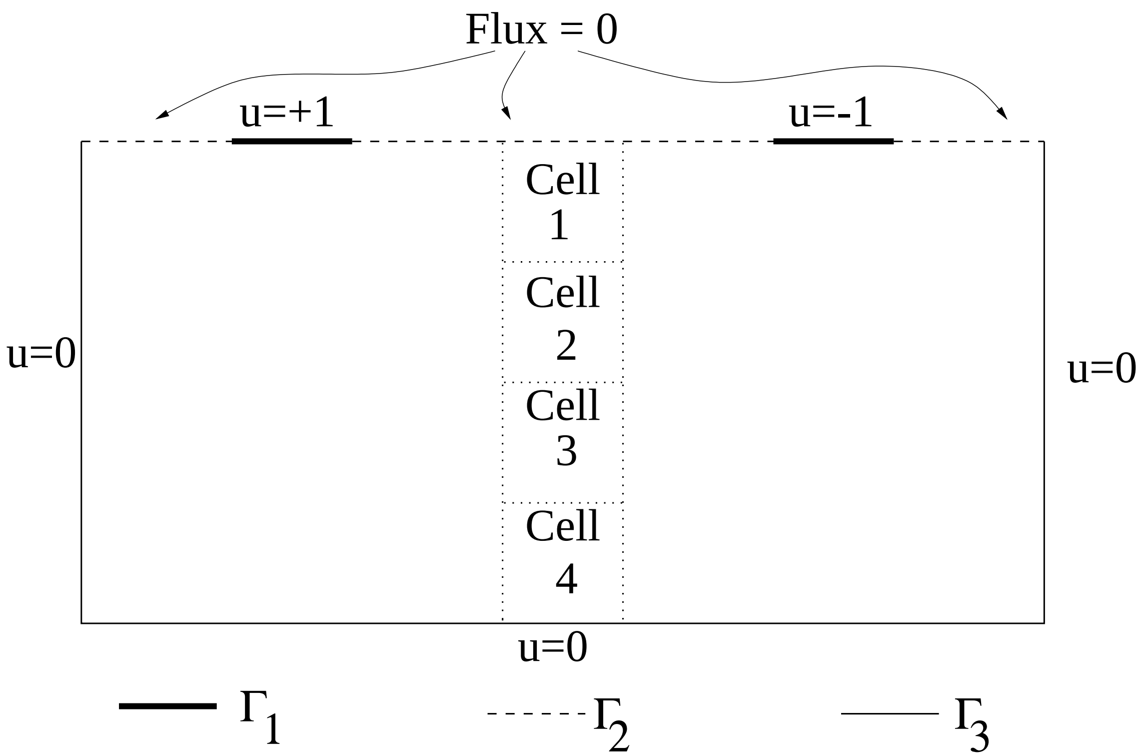
for which data measurements are made on the surface $(\Gamma_1 \cup \Gamma_2)$,

$$\begin{aligned}
u(x,0) &= F(x) && x \in \Gamma_2 \\
\sigma(x,y)\partial_y u(x,0) &= J(x) && x \in \Gamma_1
\end{aligned}$$

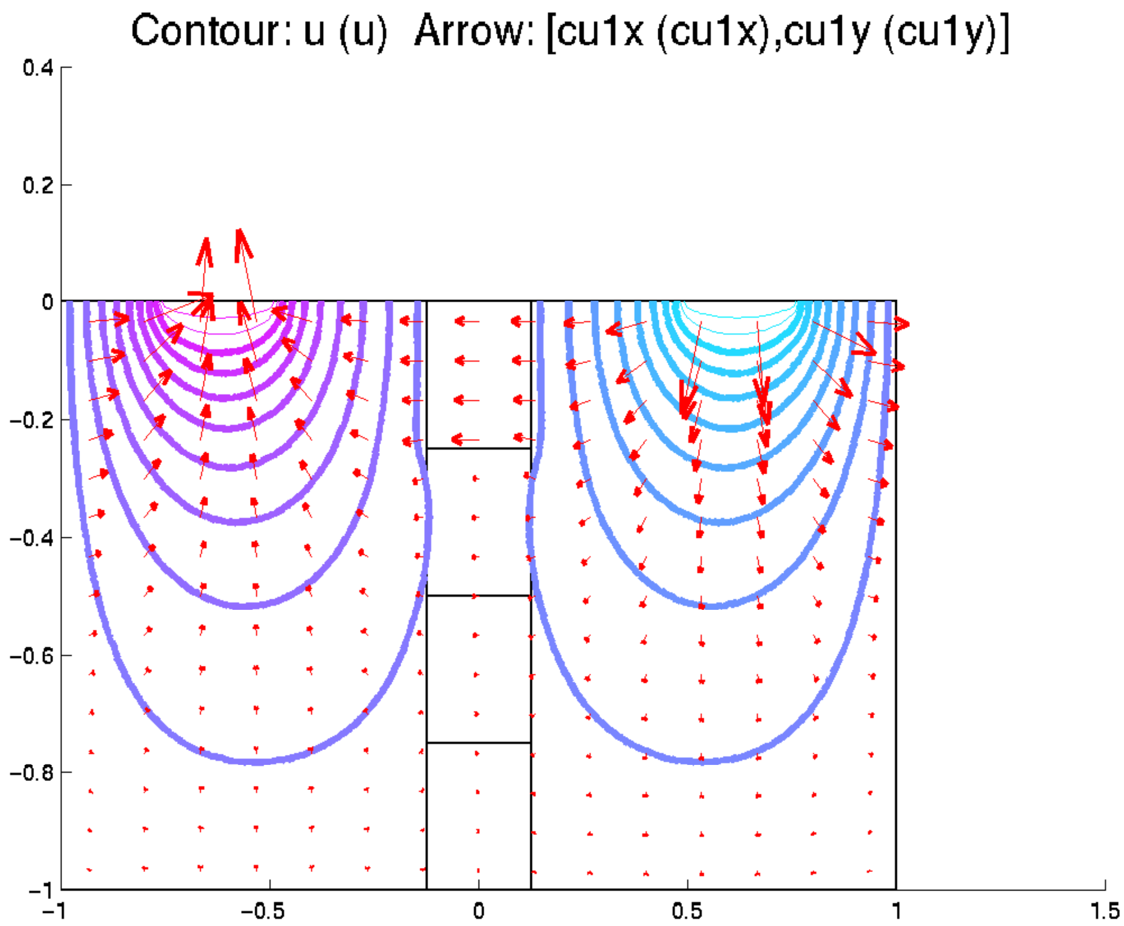
Fundamental solution for constant σ



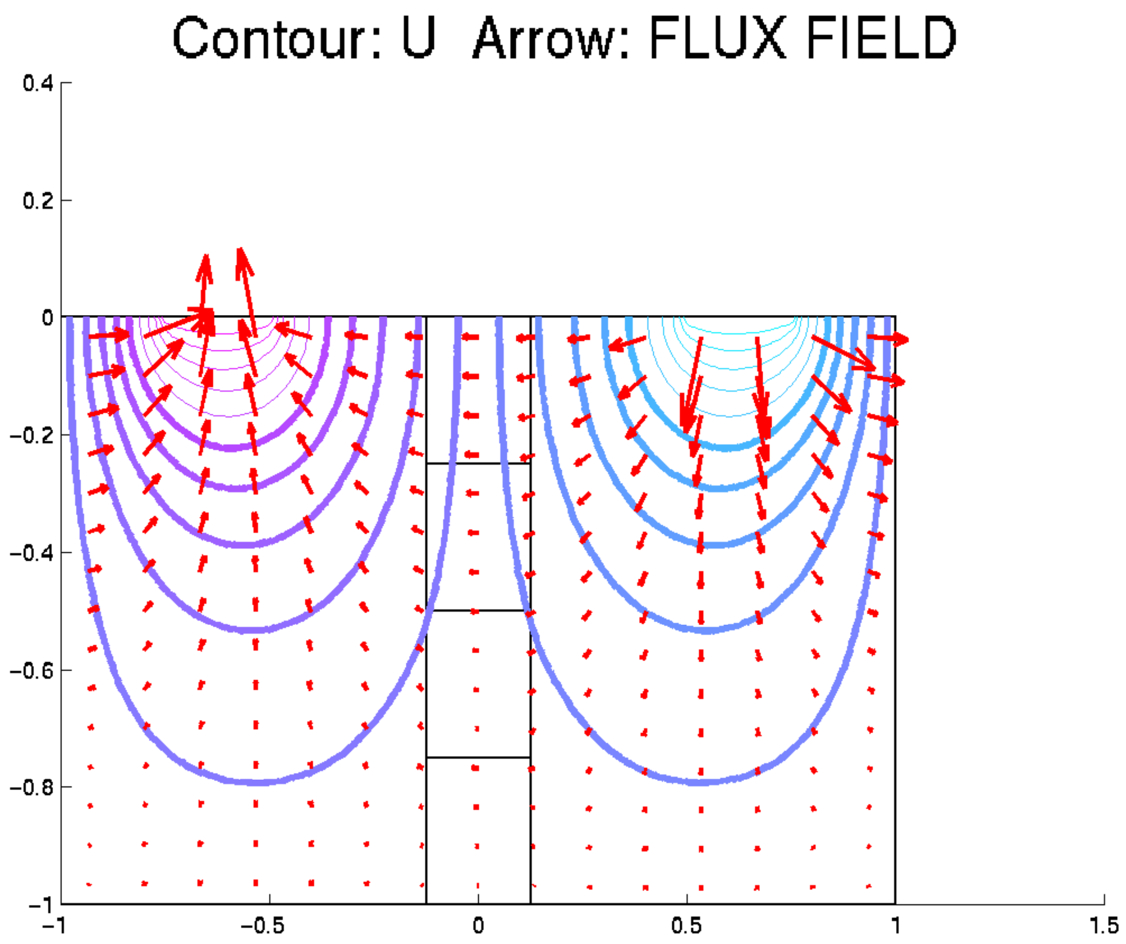
Here, a problem with spatially dependent conductivity has been simulated using Femlab.



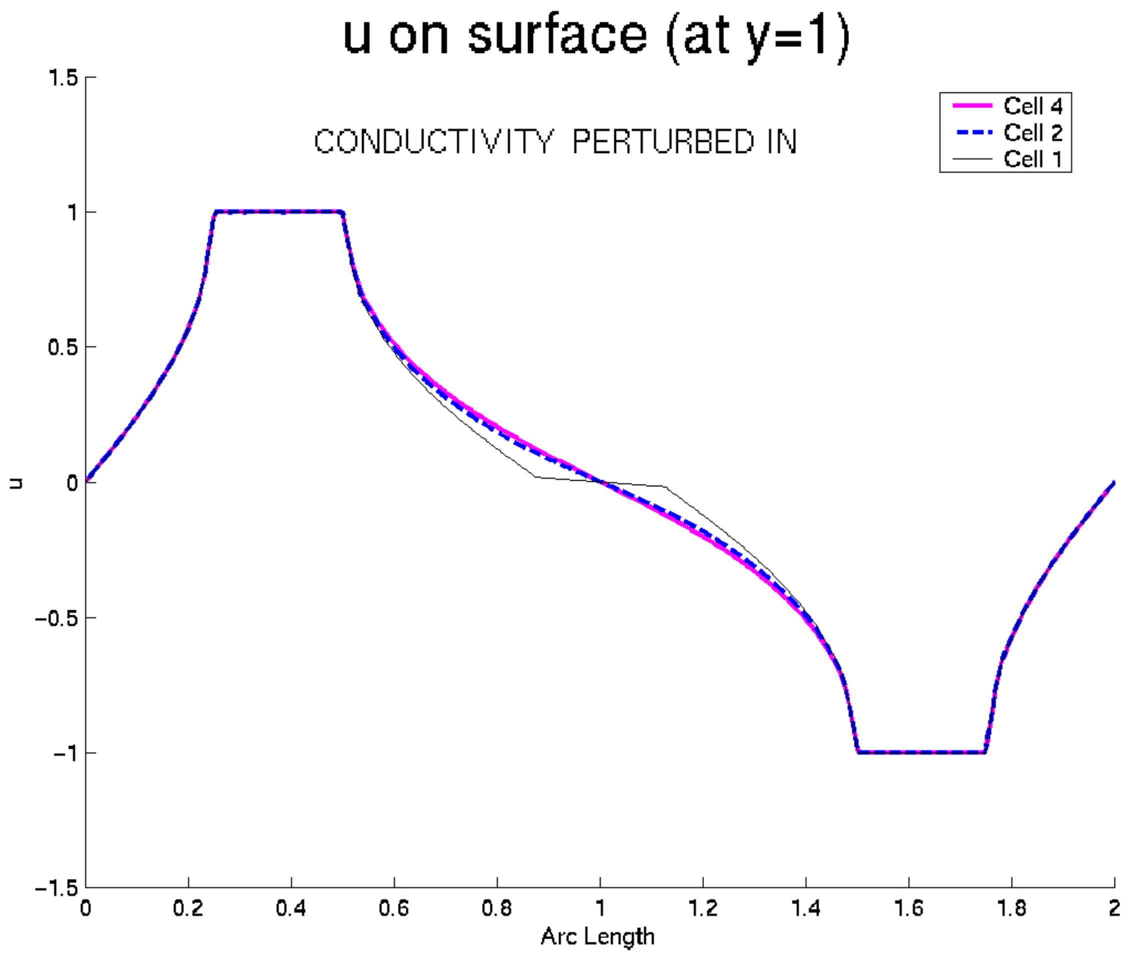
$\sigma = 10$ in top cell.



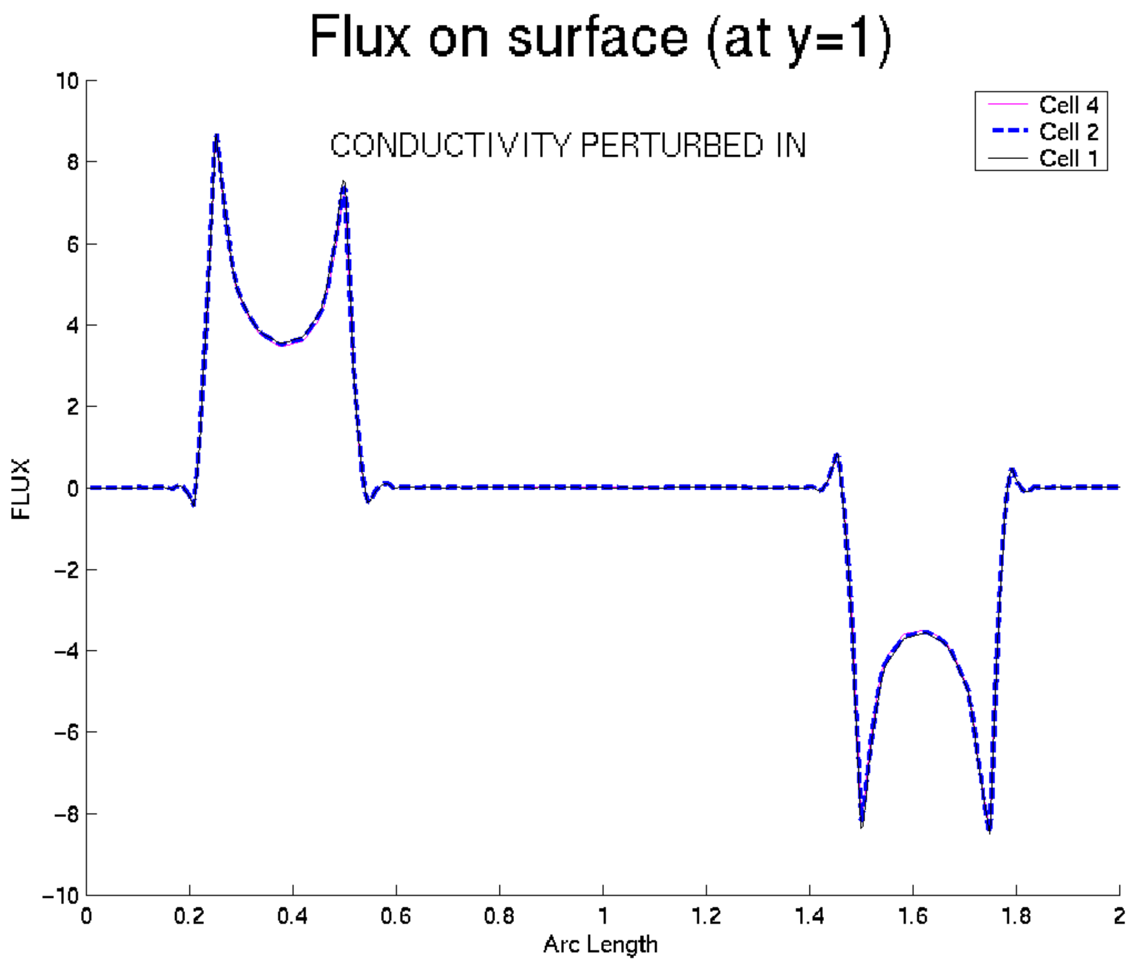
$\sigma = 10$ in bottom cell.



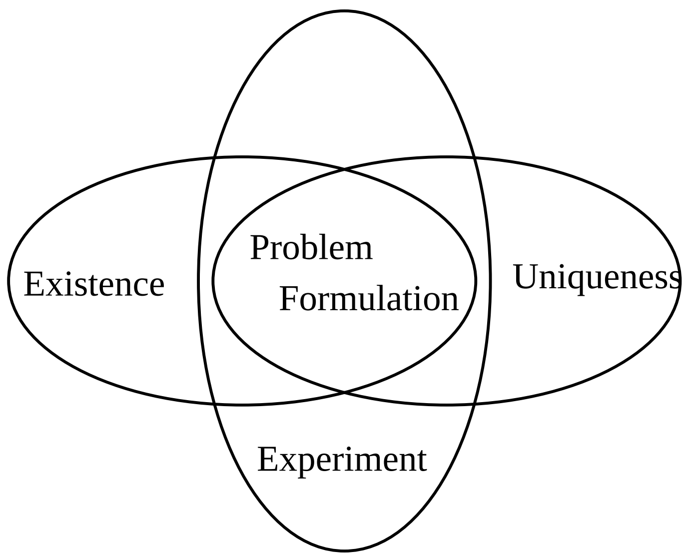
Plot of current as measured on the surface.



Plot of current flux as measured on the surface.



Inverse problems typically attempt to answer specific questions, but rarely prescribe the setting for the problem. The challenge is to find a formulation of the problem that balances experimental feasiblity with information. This balance is hard to achieve.



Analysis of the direct problem is often the first step in inverse research.

3D Ultrasound

1

John Rozier Cannon. *The One-Dimensional Heat Equation*. Addison - Wesley, Menlo Park, CA, 1984.

W Rundell D. Colton, R. Ewing, editor. *Inverse Problems in Partial Differential Equations*. SIAM, Philadelphia, PA, 1990.

Charles Groestch. *Inverse Problems in the Mathematical Sciences*. Vieweg, Braunschweig, Wiesbaden, 1993.