The (Hyper-Chaotic?) Sandwheel

Anthony Tongen, **Roger Thelwell** & David Becerra-Alonso

James Madison University, Virgina, USA
ETEA, Universidad de Códoba, ESP

CHAOS2011
Agios Nikolaos

Lorenz

$$\dot{x} = \sigma(y - x) \tag{1}$$

$$\dot{y} = rx - y - xz \tag{2}$$

$$\dot{z} = xy - bz \tag{3}$$

Lorenz

with $\sigma = 10, r = 28, \& \; b = 8/3$

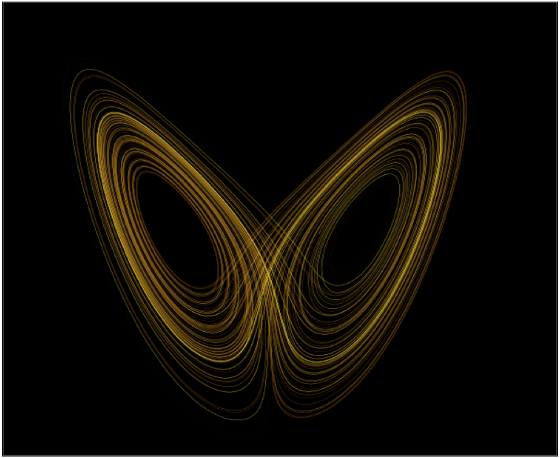

Figure: http://en.wikipedia.org

Malkus Waterwheel

http://people.web.psi.ch/gassmann/waterwheel/Wwheel1.HTML

JMU version

The model

Consider an arbitrary sector of the wheel $[\theta_1 \ , \ \theta_2]$. The mass can change in three ways:

1. Source $Q(\theta)$
2. Leakage $\ell(m)$, where

$$\ell(m) = \begin{cases} K_w m & \text{ww} \\ K_s H(m) & \text{sw} \end{cases} \tag{1}$$

   and $H(\cdot)$ is the Heaviside function.
3. Rotation

$$\omega \frac{\partial m}{\partial \theta},$$

   with angular velocity ω.

The model

So

$$\int_{\theta_1}^{\theta_2} \left[\frac{\partial m}{\partial t} - \left(Q(\theta) - \ell(m) - \omega \frac{\partial m}{\partial \theta} \right) \right] d\theta = 0, \tag{2}$$

or

$$\frac{\partial m}{\partial t} = Q(\theta) - \ell(m) - \omega \frac{\partial m}{al\theta}. \tag{3}$$

And the torque balance

$$I \frac{\partial \omega}{\partial t} = -\nu\omega + gr \int_0^{2\pi} m(\theta, t) \sin(\theta) d\theta, \tag{4}$$

closes the system, with rotational inertia I, frictional coefficient ν, gravitational force g, and wheel radius r.

Model - DFT

Typically, the DFT used to reduce to

$$\begin{aligned} \dot{a_1} &= \omega b_1 - K_w a_1 \\ \dot{b_1} &= -\omega a_1 - K_w b_1 + q_1, \\ \dot{\omega} &= \frac{-\nu\omega + \pi g r a_1}{I} \end{aligned} \tag{5}$$

Not so easy for sandwheel! In special case when $q_0 = K_s$, we have

$$\begin{aligned} \dot{a_1} &= \omega b_1 \\ \dot{b_1} &= -\omega a_1 + q_1, \\ \dot{\omega} &= \frac{-\nu\omega + \pi g r a_1}{I} \end{aligned} \tag{6}$$

Individual Mass tracking

Waterwheel

$$\frac{dm_i}{dt} = Q(\theta_i) - K_w m_i, \tag{7}$$

modeling the mass m_i in each cup, where

$$Q(\theta_i) = \begin{cases} Q, & \text{if } \cos\frac{\pi}{N} \leq \cos\left(\theta + \frac{2\pi i}{N}\right) \\ 0, & \text{otherwise.} \end{cases} \tag{8}$$

Individual Mass tracking

Waterwheel:

$$\frac{dm_i}{dt} = Q(\theta_i) - K_w m_i, \tag{7}$$

Sandwheel:

$$\frac{dm_i}{dt} = Q(\theta_i) - H(m_i)K_s \tag{8}$$

and torque balance (for both):

$$I\dot{\omega} = -\nu\omega + gr\sum_{i=0}^{N-1} m_i \sin(\theta_i). \tag{9}$$

I approaches steady state for the waterwheel, but not for the sandwheel.

Center of Mass - waterwheel

Introduce

$$y_{cm} = \frac{r}{M} \sum_{i=0}^{N-1} m_i \sin(\theta_i) \quad (10)$$

$$z_{cm} = \frac{r}{M} \sum_{i=0}^{N-1} m_i \cos(\theta_i) \quad (11)$$

Then, from the mass balance for the waterwheel

$$\frac{dm_i}{dt} = Q(\theta_i) - K_w m_i \quad (12)$$

we see

$$\dot{y}_{cm} = \omega z_{cm} - K_w y_{cm}, \quad (13)$$

and (for large N)

$$\dot{z}_{cm} = \frac{r q_0}{M_w} - \omega y_{cm} - K_w z_{cm}. \quad (14)$$

Center of Mass - sandwheel

For the sandwheel, we have

$$\sum_{i=0}^{N-1} \dot{m}_i \sin\theta_i = \sum_{i=0}^{N-1} Q(\theta_i)\sin(\theta_i) - \sum_{i=0}^{N-1} K_s H(m_i)\sin(\theta_i). \quad (15)$$

M_s is time-dependent, so

$$\dot{y}_{cm} + \frac{\dot{M}_s}{M_s} y_{cm} = -\frac{r}{M_s}\sum_{i=0}^{N-1} K_s H(m_i)\sin(\theta_i) + \omega z_{cm}. \quad (16)$$

Similarly,

$$\dot{z}_{cm} + \frac{\dot{M}_s}{M_s} z_{cm} = -\frac{r}{M_s}\sum_{i=0}^{N-1} K_s H(m_i)\cos(\theta_i) - \omega y_{cm} + \frac{rq_0}{M_s}. \quad (17)$$

Center of Mass - sandwheel

One way to interpret this is

$$\dot{y}_{cm} = -\frac{\dot{M}_s}{M_s} y_{cm} - K_s \gamma(t) y_{cm} + \omega z_{cm} \quad \text{and} \tag{18}$$

$$\dot{z}_{cm} = -\frac{\dot{M}_s}{M_s} z_{cm} - K_s \gamma(t) z_{cm} - \omega y_{cm} + \frac{r q_0}{M_s}, \tag{19}$$

where $\gamma(t) = \sum_{i=1}^{N} \frac{H(m_i)}{m_i}$, the proportion of cups losing mass.
recall:

$$\dot{y}_{cm} = \omega z_{cm} - K_w y_{cm}, \tag{20}$$

$$\dot{z}_{cm} = \frac{r q_0}{M_w} - \omega y_{cm} - K_w z_{cm} \tag{21}$$

for the waterwheel.

Butterfly Sandwheel

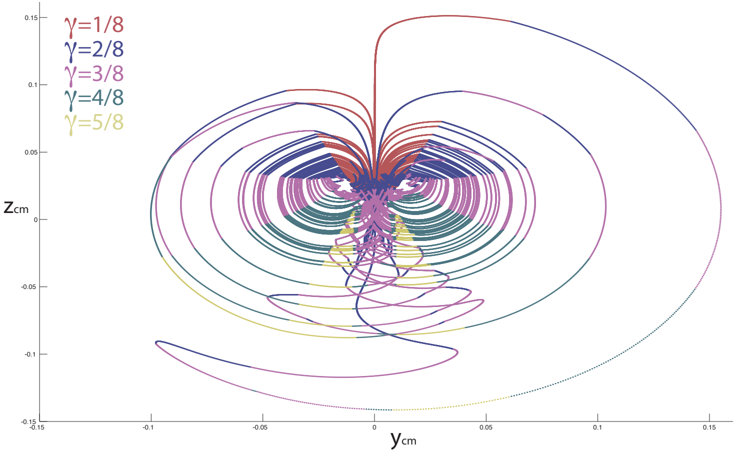

Figure: Coloured on γ

Bifurcation Waterwheel

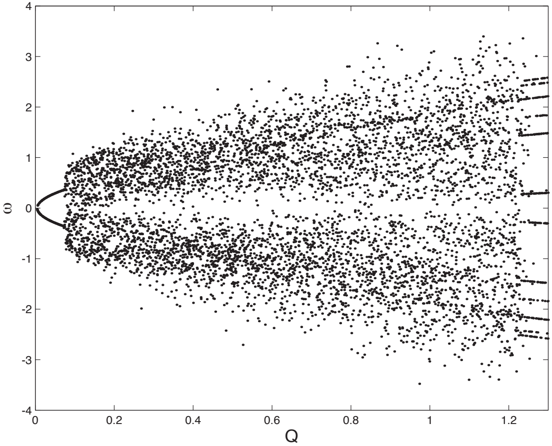

Figure: Waterwheel bifurcation

Bifurcation Sandwheel

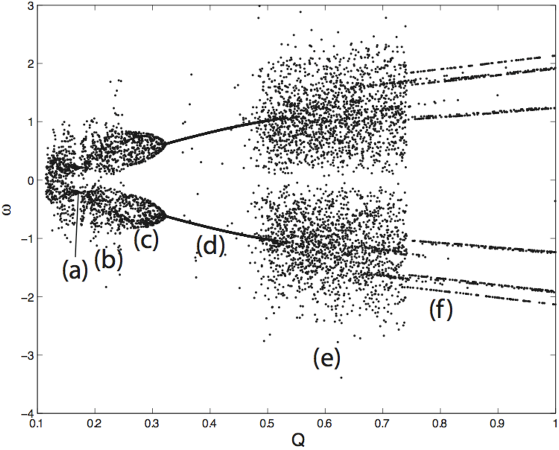

Figure: Sandwheel bifurcation

Bifurcation Sandwheel

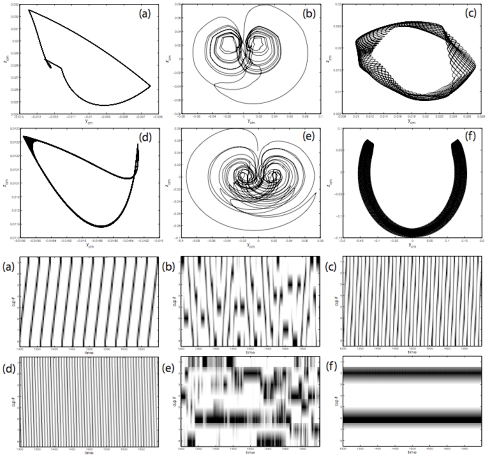

Figure: Sandwheel bifurcation

Conclusions

- Rich array of behaviour.
- Both exhibit 2 forms of periodicity and chaos.
- Quasi-periodicity on border of chaos.
- $Q > NK_s$ implies periodicity
- Heaviside seems to control chaos, driving system to quasiperiodicity and periodicity.