Stability via computed Power Series

JMM 2022 in Seattle

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July 1, 2022

The power of Series

JMM 2022 in ~~Seattle~~ online

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Abstract

We exploit the power of an iterative symbolic computation to efficiently compute the Lyapunov spectrum of highly nonlinear systems of ODEs. The technique constructs a truncated Taylor series representation (a jet of degree k) at a generic point of the separation of the flow. An approximate spectrum is then recovered and analyzed. Several examples will be discussed. (circa Sept 2, 2021)

Abstract

We exploit the power of an iterative symbolic computation to explore ODEs. We will talk about solutions, stability, error bounds, convergence, and control. The methods and theory are accessible to undergraduate students at the end of a first course in ODE, and the techniques are useful extensions that can be applied to power series solutions, and applicable to a wide range complex questions. (July 1, 2022)

Outline

$y' = \alpha y^2$

We'll begin with a toy problem...

$$y' = \alpha y^2 \qquad y(0) = y_0$$

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An analytic solution to this one is easy:

$$y(t) = -\frac{y_0}{\alpha y_0 t - 1}$$

$y' = \alpha y^2$: series solution?

What happens if we try (formal, for now) series?

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What happens if we try (formal, for now) series?
Let

$$y(t) = \sum_{k=0}^{\infty} y_k t^k$$

then

$$\sum_{k=0}^{\infty} (k+1) y_{k+1} t^k = \alpha \sum_{k=0}^{\infty} \left(\sum_{i,j \geq 0}^{i+j=k} y_i y_j \right) t^k$$

so we equate coefficients to get

$$y_{k+1} = \frac{\alpha}{k+1} \sum_{i+j=k} y_i y_j$$

$y' = \alpha y^2$: series solution?

Or, with MAPLE:

```
> ODE1 := diff(y(t),t) = alpha*y(t)^2;
> IC := y(0) = y0;
> y1 := dsolve({ODE1,IC},y(t))
```

and

```
> Y1 := dsolve({ODE1,IC},y(t),series);
      Y1 := 1+2*alpha*t*y0+3*alpha^2*y0^2*t^2+...
```

$y' = \alpha y^2$: sensitivity?

Consider the ODE system

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) \qquad \text{with } \mathbf{x}(0) = \mathbf{x}_0.$$

and let

$$\phi(t; \mathbf{x}_0)$$

represent the flow of this system through the initial point $\mathbf{x}_0$.

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Suppose we want to understand the divergence of two nearby trajectories:

$$\phi(t; \mathbf{y}_0) - \phi(t; \mathbf{x}_0) \approx D_{\mathbf{x}}\phi(t; \mathbf{x}_0)(\mathbf{y}_0 - \mathbf{x}_0),$$

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https://en.wikipedia.org/wiki/Lyapunov_exponent

Aside: The Lyapunov exponent

For any curve of initial conditions $\mathbf{x}_s$, define

$$\mathbf{v}(t) = \partial_{\mathbf{s}} \phi(t; \mathbf{x}_s)\Big|_{s=0},$$

then $\mathbf{v}(t)$ satisfies the first variation equation

$$\dot{\mathbf{v}} = D_{\mathbf{x}} \mathbf{F}(\phi(t; \mathbf{x})\Big|_{\mathbf{x}_0} \mathbf{v}_0 \quad \text{with} \quad \mathbf{v}_0 = \partial_{\mathbf{s}} \mathbf{x}_s.$$

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More generally, the full Lyapunov spectrum might be computed, but this is more difficult.
Can power series help?

$y' = \alpha(t)y$

With Maple

```
>> restart:
>> Order := 4:
>> alpha := t -> sum(a[k]*t^k,k=0..Order):
>> GROWTH := diff(y(t),t) = alpha(t)*y(t):
>> Yseries := dsolve({GROWTH,y(0)=y[0]},y(t),type='series');
```

$$y(t) = y_0 + a_0\, y_0 t + \left(1/2\, {a_0}^2\, y_0 + 1/2\, a_1\, y_0\right) t^2 + \left(1/6\, {a_0}^3\, y_0 + 1/2\, a_1\, a_0\, y_0 + 1/3\, a_2\, y_0\right) t^3 + O\left(t^4\right)$$

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which we can check

```
>> SOLN1 := y[0] * exp(int(alpha(tau),tau=0..t));
>> check := taylor(SOLN1,t=0) - Yseries;
```

Why?

From

$$y\left(t\right) = y_0 + a_0\, y_0 t + \left(1/2\, {a_0}^2\, y_0 + 1/2\, a_1\, y_0\right) t^2 + \left(1/6\, {a_0}^3\, y_0 + 1/2\, a_1\, a_0\, y_0 + 1/3\, a_2\, y_0\right) t^3 + O\left(t^4\right)$$

we can find

$$\partial_{y_0} y(t) = 1 + a_0 t + \left(1/2\, {a_0}^2 + 1/2\, a_1\right) t^2 + \left(1/6\, {a_0}^3 + 1/2\, a_1\, a_0 + 1/3\, a_2\right) t^3 + O\left(t^4\right)$$

Sensitivity to initial conditions!

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Sensitivity to initial conditions! Which we can easily compute...

```
>> Yy0 := taylor(diff(Yseries,y_0),t=0);
```

The Lyapunov exponent

We have

$$\partial_{y_0} y(t) = \sum_{n=0}^{\infty} f_n(y_0, \ldots, y_{n-1}) t^n$$

and so $\lambda(t)$ is easy to compute:

```
>> simplify(taylor(ln(Yy0))/t);
```

$$\lambda(t) = a_0 + 1/2a_1 t + 1/3a_2 t^2 + O(t^3)$$

For our problem, a direct calculation verifies this:

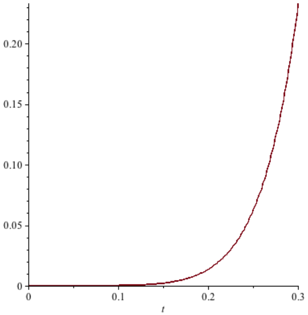
$$\frac{1}{t}\int_0^t \alpha(\tau)d\tau).$$

This time average is the mean coefficient on $[0, t]$

$y' = 2 * y^2, \quad y(0) = 1$: error?

What about the error?

```
> alpha := 2; y0 := 1; plot(abs(Y1-y1),t=0..0.5);
```



Can we quantify this?

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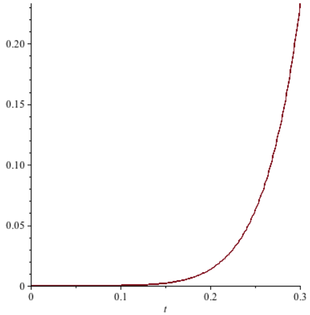


Can we quantify this?
Let's take a little diversion...

$y' = \alpha y^m$

The solution to the constant coefficient nonlinear IVODE

$$y' = \alpha y^m \qquad y(0) = y_0$$

is messy:

$$y(t) = \left((\alpha - \alpha m)t + y_0^{1-m}\right)^{-(m-1)^{-1}}.$$

But the ratio of $\frac{y}{y'}$ isn't!

$$\frac{y}{y'} = \frac{(\alpha - \alpha m)t + y_0^{1-m}}{\alpha}$$

and so

$$y'(t) = \underbrace{\frac{\alpha}{(\alpha - \alpha m)t + y_0^{1-m}}}_{K(t)} y(t),$$

a non-constant coefficient LINEAR ode.

$y' = \alpha y^m$: important aside

So

$$y'(t) = \underbrace{\frac{\alpha}{(\alpha - \alpha tm)t + y_0^{1-m}}}_{K(t)} y(t),$$

has solution

$$y(t) = y_0 exp\left(\int_0^t K(\tau)d\tau\right)$$

or, via series,

$$Y_{k+1} = \frac{\alpha(1 + (m-1)k)}{y_0^{1-m}(k+1)} Y_k$$

$y' = \alpha y^m$: important aside

From

$$Y_{k+1} = \frac{\alpha(1 + (m-1)k)}{y_0^{1-m}(k+1)} Y_k$$

and for $m \geq 2$,

$$Y_{k+1} \leq (m-1)|y_0|^{m-1} Y_k := C_\infty Y_k.$$

This leads directly to a geometric series bounding $y(t)$:

$$y(t) \leq \frac{|y_0|}{1 - C_\infty} = |y_0| \sum_{k=0} (C_\infty t)^k$$

Now for the bound...

$y' = \alpha y^2$: back to error

From

$$y(t) \leq \frac{|y_0|}{1 - C_\infty} = |y_0| \sum_{k=0} (C_\infty t)^k$$

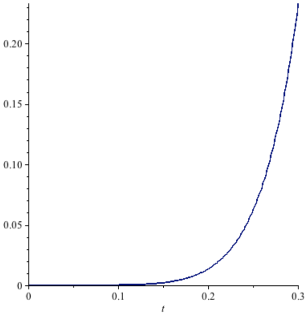
we see that the absolute error is

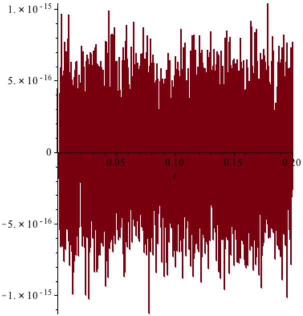
$$|Y1 - y1| \leq |y_0| \sum_{k=n+1}^{\infty} C_\infty |t|^k \leq \frac{|y_0| C_\infty^{n+1}}{1 - C_\infty |t|}$$

where $C_\infty = |y_0 \alpha|$.
An ERROR bound!

$y' = 2 * y^2, \quad y(0) = 1$: error plots

```
> ee :=  abs(Y1-y1);
> m := 2; Cinf := y0*alpha;
> EE := N -> abs(y0)*(Cinf*t)^(N+1)/( 1 - Cinf*abs(t));
> plot({ee,EE(5)},t=0..0.3);
> plot({ee-EE(5)},t=0..0.2);
```

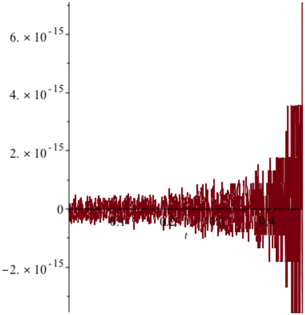




Great!

$y' = 2y^2, \quad y(0) = 1$: radius of convergence?

```
>plot({ee-EE(5)},t=0..0.48);
```



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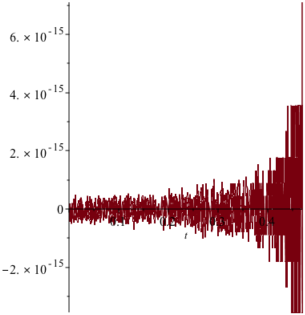

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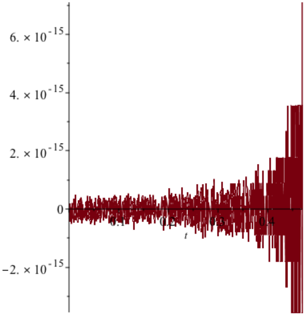

Hmmmm....

$$y(t) = -\frac{y_0}{\alpha y_0 t - 1}$$

$y' = \alpha y^2$: radius of convergence

But what if we only have this form?

```
> Y20 := dsolve({ODE1,IC},y(t),series);
        Y20 := 1+2*alpha*t*y0+3*alpha^2*y0^2*t^2+...
```

$y' = \alpha y^2$: radius of convergence

But what if we only have this form?

```
> Y20 := dsolve({ODE1,IC},y(t),series);
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```
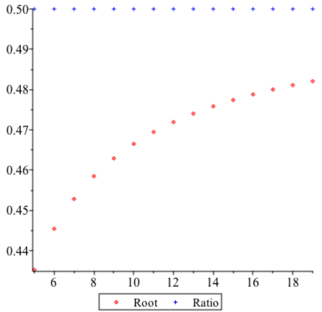

Approximate using the root or ratio test. Or calculate the Padé approximant.

$y' = 2y^2, y(0) = 1$: radius of convergence?

```
> Order := 20: alpha := 2; y0 := 1;
> Y20 := rhs(dsolve({ODE1,IC},y(t),series)):
> Ycoeff := [seq(coeff(convert(Y20,polynom),t,i),i=1..20)];
> RatioT := i -> abs(a[i+1]/a[i]): RootT := i -> abs(a[i])^(1/
> RootEst := [seq(1/RootT(i),i=1..Order)]: RatioEst := [seq(1/
> plot ....
```



$y' = \alpha y^2$: for control?

Suppose we want to control

$$y' = \alpha y^2 \qquad y(0) = y(0)$$

so that $y(T) = \beta$.
If we apply frictional damping to the system

$$y' - \alpha y^2 = u,$$

where $u = kty'$, can we drive the system to the desired state?

$y' = \alpha y^2$: for control?

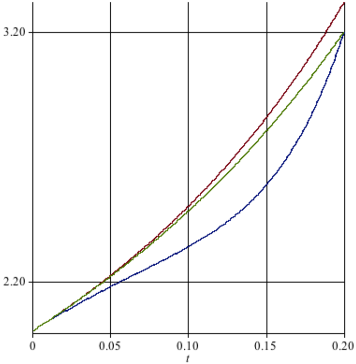
Let's try to drive the system so that $Y(0.2) = 3.2$.

```
> ODE := diff(y(t),t) = alpha*y(t)^2 + k*t*diff(y(t),t);
> IC := y(0) = 2;
> Y := convert(rhs(dsolve({ODE1,IC},y(t),series)),polynom);
> kvals := solve(subs(t=0.2,Y)=3.2);
       -2.051118481+8.300750459*I, -.6504685102 , ...
> subs({k=kvals[2],t=0.2},Y);
        3.200000000
```

$y' = \alpha y^2 + kty'$

It looks like we can.

```
> kvals := solve(subs(t=0.2,Y)=3.2);
       -2.051118481+8.300750459*I, -.6504685102 , ...
>plot({Yk(0),Yk(kvals[2]),Yk(kvals[3])},t=0..0.2);
```



Repeated application allows trajectory control, and our error bound still applies to the forced system!

Torsional Deformation of Blatz-Ko material?

Torsional deformation of a compressible elastic solid cylinder can be modelled as:

$$y'' \left[1 + \frac{\sqrt{2}}{y'^2} \left(\frac{x}{yy'} \right)^{\sqrt{2}-1} \right] = \left[\sqrt{2} \left(\frac{x}{yy'} \right)^{\sqrt{2}} + 1 \right] \left[\frac{y}{x^2} - \frac{y'}{x} \right] + \frac{\pi^2}{16} y$$

A solution to arbitrary order can also be computed via power series.

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A solution to arbitrary order can also be computed via power series. Just introduce auxiliary variables to generate a polynomial system. All the same ideas apply!

Six degree-of-freedom flight mechanics

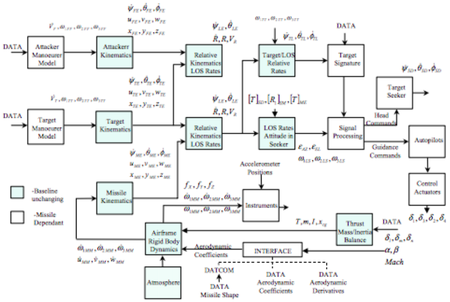

Figure 1. Block diagram of generic baseline model.

from: `http://plantarchy.us/katko/projects/dope/DSTO-TR-0931-PR.pdf`

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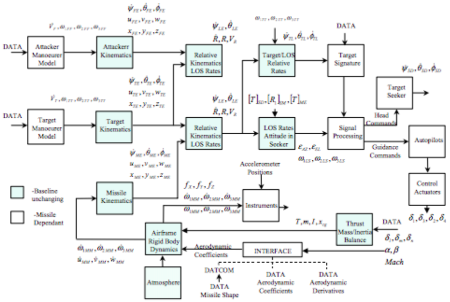

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Conclusions

We considered a toy problem already cast as a polynomial ode. Extension and application of these methods will rely on the use of auxiliary variables to build a system of polynomial IVODEs. Once the system is polynomial, series methods allow remarkably direct analysis.

- Analytic approximation of solution
- Stability and sensitivity
- Easy error BOUND
- Radius of Convergence estimate
- Simple control?

These techniques apply to a broad range of highly nonlinear ODE.

?

Thanks!

Questions?

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http://educ.jmu.edu/~sochacjs/PSM.html