Cauchy-Kowalevski and Polynomial ODE

Roger Thelwell & Paul Warne

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Why?

- Picard-Lindelöf : fundamental local existence for IVODE
- Cauchy-Kowalevksi : fundamenal existence uniqueness of IBVPDE.

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- Cauchy-Kowalevksi : fundamenal existence uniqueness of IBVPDE.

But Cauchy arguments can be used in ODE setting, too. If the ODE is polynomial, the reasoning leads to an easy iterative constuction and a clear *a priori* error bound.

A brief history

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A brief history

Cauchy and Weierstrass **independently** developed the methods of majorants. Idea is to bound a series by one which is known to converge. If the bounding function is analytic, then so is the function represented by the original series.

Cauchy approach to ODE

The ODE:

$$d_t u(t) = f(u(t)) := \frac{1}{u} \exp(-16\, u^2), \quad \text{with } u(0) = 1.$$

What now?

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The ODE:

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What now?

$$\begin{aligned} d_t^2 u(t) &= d_u f(u) d_t u \\ &= -\frac{\mathrm{e}^{-32\, u^2} \left(32\, u^2 + 1\right)}{u^3} \\ d_t^3 u(t) &= d_u^2 f(u)[d_t u]^2 + d_u f(u) d_t^2 u \\ &= \frac{\mathrm{e}^{-48\, u^2} \left(2048\, u^4 + 96\, u^2 + 3\right)}{u^5} \\ \text{and} \qquad d_t^n u(t) &= p_n(f(u), d_u f(u), d_u^2 f(u), \ldots, d_u^{n-1} f(u)), \end{aligned} \tag{1}$$

Cauchy approach to ODE

Now just build a series representation of u.

$$u(t) = \sum_{k=0}^{\infty} \frac{1}{k!} d_t^k u(0)\, t^k. \tag{2}$$

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What's wrong? It looks so easy....

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It gets complicated really fast!

Cauchy approach to ODE

Consider even the forth derivative...

$$d_t^4 u(t) = d_u^3 f(u)[d_t u]^3 + 3 d_u^2 f(u) d_t^2 u d_t u + d_u f(u) d_t^3 u \quad (3)$$

$$= -\frac{e^{-64\,u^2}\left(196608\,u^6 + 11264\,u^4 + 576\,u^2 + 15\right)}{u^7} \quad (4)$$

In general, this is daunting.

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really, though, “we” can produce a closed form...

$$\frac{\sum_{\_k1=0}^{n} \binom{n}{\_k1} 2^{\text{-}k1}\, G_{2,3}^{1,2}\left(16 \,\middle|\, {}^{0,1/2}_{0,1/2+1/2\,\_k1,1/2\,\_k1}\right) poch\left(-n + \_k1,\, n - \_k1\right)}{n!}$$

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Consider:

$$C_\infty := \max_k \{|C_k|\} < \infty, \quad \text{where} \quad C_n = \frac{1}{n!} d_u^n f(1) r^n,$$

which provides the bound

$$\max_k \left| \frac{1}{k!} d_u^k f(1) \right| \leq C_\infty r^{-k}$$

on the Taylor coefficients of $f(u)$ about $u(0) = 1$

Cauchy approach to ODE

Now for the comparison IVODE:

$$d_t v(t) = g(v(t)) \text{ with } v(0) = 1. \tag{5}$$

where

$$g(v) := \sum_{k=0}^{\infty} C_\infty r^{-k}(v-1)^k = C_\infty \frac{r}{r-(v-1)} \quad \text{when} \quad |v-1| < r,$$

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Same form, same universal polynomial as

$$d_t u(t) = f(u(t)) \text{ with } u(0) = 1.$$

Cauchy approach to ODE

it follows that

$$\begin{aligned} |d_t^n u(0)| &= |p_n(f(1), \dots, d_u^{n-1} f(1))| \\ &\leq p_n(|f(1)|, \dots, |d_u^{n-1} f(1)|) \\ &\leq p_n(g(1), \dots, d_u^{n-1} g(1)) \\ &= d_t^n v(0), \end{aligned}$$

demonstrating that $u(t)$ is majorized by $v(t)$ in a neighborhood of $t = 0$.

Cauchy approach to ODE

Now since

$$|u(t)| = \left| \sum_{k=0}^{\infty} \frac{1}{k!} d_t^k u(0)\, t^k \right| \leq \sum_{k=0}^{\infty} \frac{1}{k!} d_t^k v(0)\, |t|^k \leq v(|t|),$$

and

$$v(t) = 1 + r - r\sqrt{1 - 2C_\infty t/r}, \tag{6}$$

we know that $u(t)$ must also be locally analytic about $t = 0$.

Cauchy approach to ODE

This argument relies on C_∞, a constant which in practice is often difficult to ascertain.

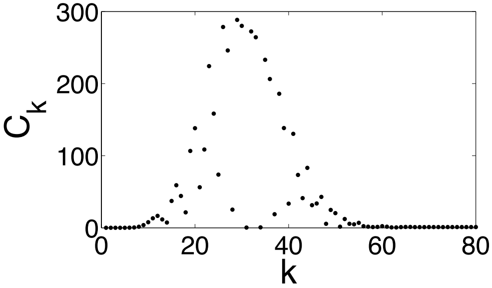

Figure: C_k coefficient list

An argument for polynomials

Recall:

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And build a system of polynomial ODE:

$$\begin{aligned} d_t u &= x & u(0) &= 1 \\ d_t x &= (-32xu - xy)\, d_t u = -32x^2 u - x^2 y & x(0) &= \exp(-16) \\ d_t y &= -\frac{1}{u^2} d_t u = -y^2 x & y(0) &= 1. \end{aligned}$$

An argument for polynomials

The *new* comparison IVODE

$$d_t z = \mathcal{C} z^m \qquad z(0) = c, \tag{7}$$

which has analytic solution

$$z\left(t\right) = \left(\mathcal{C}t - \mathcal{C}tm + c^{1-m}\right)^{-(m-1)^{-1}}.$$

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We claim that $z(t)$ majorizes $u(t), x(t)$ and $y(t)$, if $\mathcal{C} = 33, m = 3$ and $c = 1$.

An argument for polynomials

Consider just

$$d_t x = -32x^2 u - x^2 y \quad x(0) = \exp(-16)$$

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and

$$\begin{aligned} z_{n+1} &= \frac{1}{n+1} \cdot 33 \sum_{k=0}^{n} \left(\sum_{i=0}^{k} z_i z_{k-i} \right) z_{n-k} \\ &\geq \frac{1}{n+1} \cdot \left| -32 \sum_{k=0}^{n} \left(\sum_{i=0}^{k} x_i x_{k-i} \right) u_{n-k} - \sum_{k=0}^{n} \left(\sum_{i=0}^{k} x_i x_{k-i} \right) y_{n-k} \right| \qquad (8) \\ &= |x_{n+1}| \end{aligned}$$

An argument for polynomials

IMPORTANT

$$x_{n+1} = \frac{1}{n+1} \cdot \left[-32 \sum_{k=0}^{n} \left(\sum_{i=0}^{k} x_i x_{k-i} \right) u_{n-k} - \sum_{k=0}^{n} \left(\sum_{i=0}^{k} x_i x_{k-i} \right) y_{n-k} \right]$$

An error bound for polynomials

From

$$d_t z = \mathcal{C} z^m \qquad z(0) = c, \tag{9}$$

we can extract the recurrence relation:

$$z_{n+1} = \frac{(1+(m-1)n)c^{m-1}\mathcal{C}}{n+1} z_n \qquad z_0 = c, \text{ for } n \geq 1. \tag{10}$$

An error bound for polynomials

For $m \geq 2$,

$$\frac{(1 + (m-1)n)c^{m-1}\mathcal{C}}{n+1} \leq (m-1)c^{m-1}\mathcal{C} := \mathcal{C}_\infty. \tag{11}$$

Combining (10) and (11) yields $z_{n+1} \leq \mathcal{C}_\infty z_n$. If

$$w_{n+1} = \mathcal{C}_\infty w_n, \qquad \text{with } w_0 = c, \tag{12}$$

An argument for polynomials

From this recurrence,

$$w(t) = \frac{c}{1 - \mathcal{C}_\infty t} = c \sum_{k=0} (\mathcal{C}_\infty t)^k, \quad \text{when} \quad |t| < \frac{1}{\mathcal{C}_\infty}.$$

The function w may be interpreted as a solution to the IVODE

$$d_t w(t) = \mathcal{C}_\infty w, \qquad w(0) = c \tag{13}$$

where $\mathcal{C}_\infty$ bounds the coefficient growth of terms of z, playing much the same role as C_∞.

An argument for polynomials

Now for a simple bound on the remainder term $\mathcal{R}_n$!

$$\mathcal{R}_n(t) := \left| u(t) - \sum_{k=0}^{n} u_k t^k \right| \leq c \sum_{k=n+1}^{\infty} \mathcal{C}_\infty |t|^k$$
$$\leq c|\mathcal{C}_\infty t|^{n+1} \frac{1}{1 - |\mathcal{C}_\infty t|}. \quad (14)$$

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- Majorization is explicit, and leads to ....
- A clear *a priori* error bound!