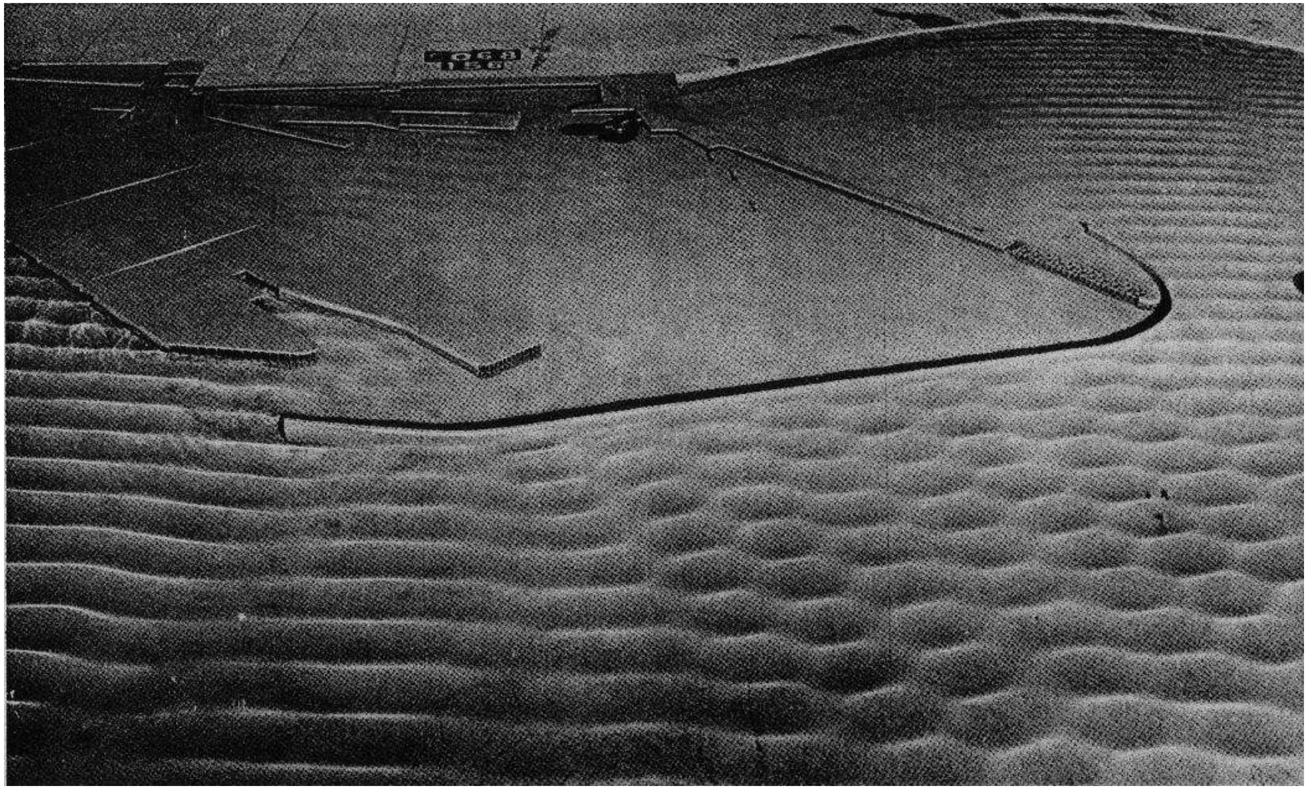
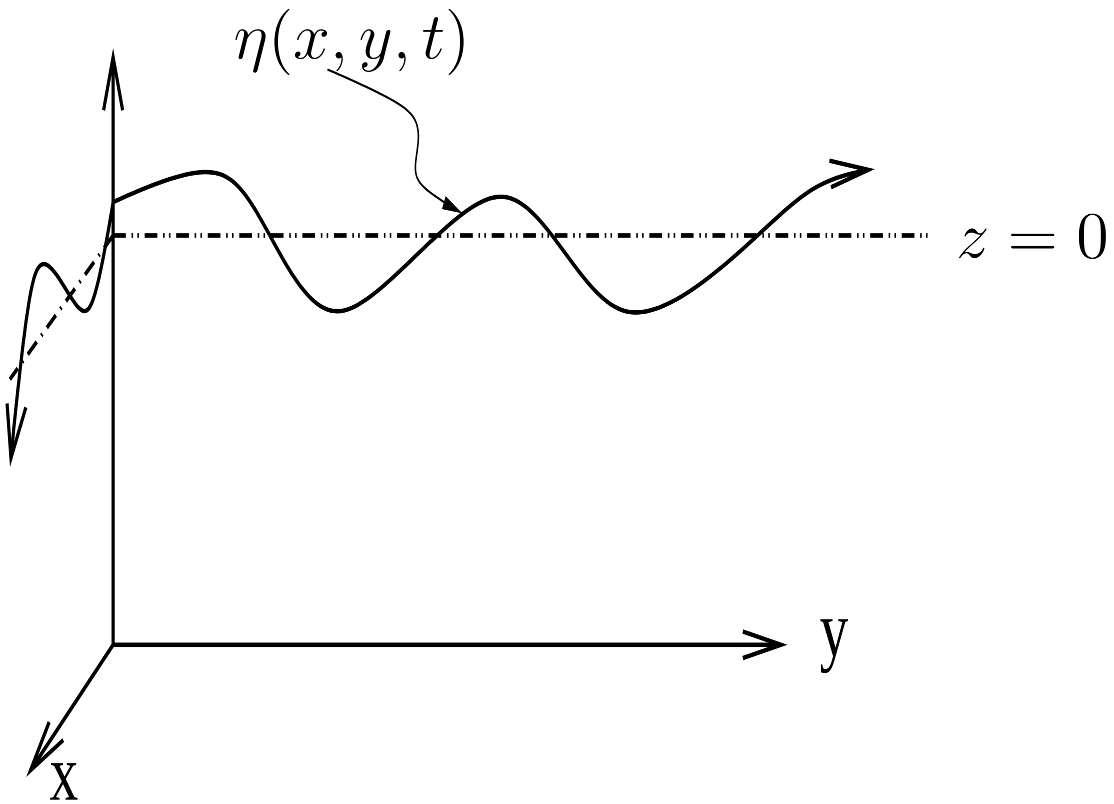
2D wave patterns

Patterns

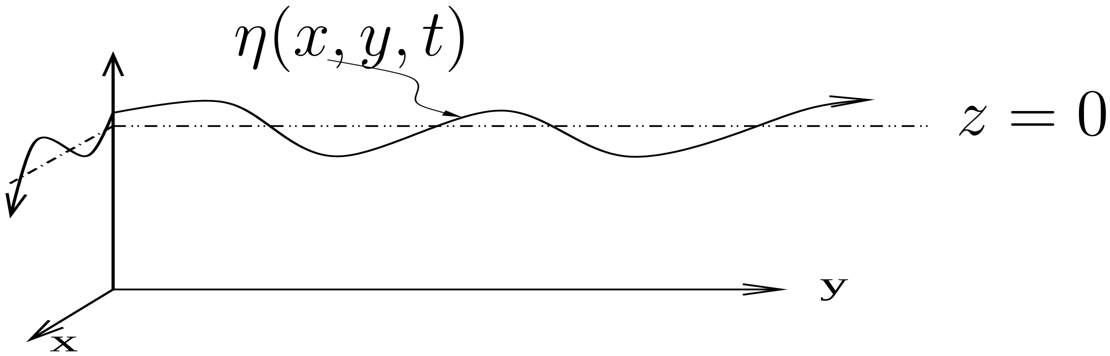


Bernard's webpage: CERC harbor model (Source unknown)

Models



Models



Euler: $\eta(x,y,t)$ and $\vec{v}(x,y,z,t)$ unknown

$$\vec{v}_t + (\vec{v}\cdot\nabla)\vec{v} + \nabla p = -g\vec{k}$$

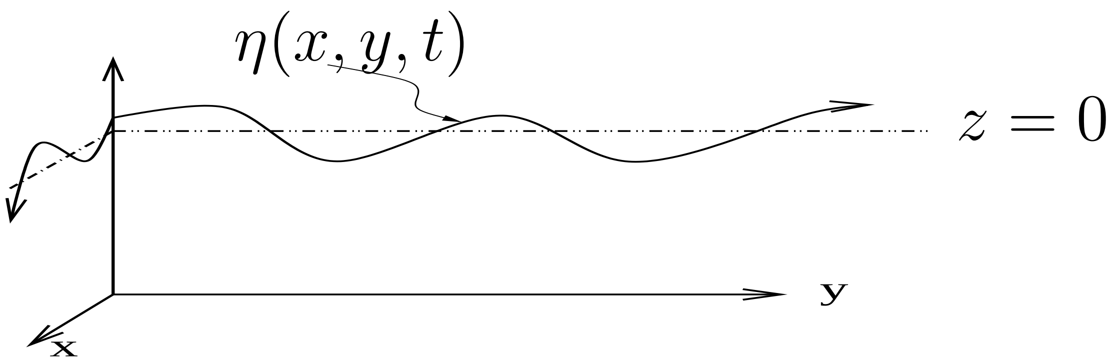
and $\nabla\cdot\vec{v} = 0$

$\vec{v}\cdot\vec{n} = 0$ at $z = 0$

$\frac{D}{Dt}(\eta - z) = 0$ and $p = P_0$ at $z = \eta$

Navier Stokes + viscous free

Models



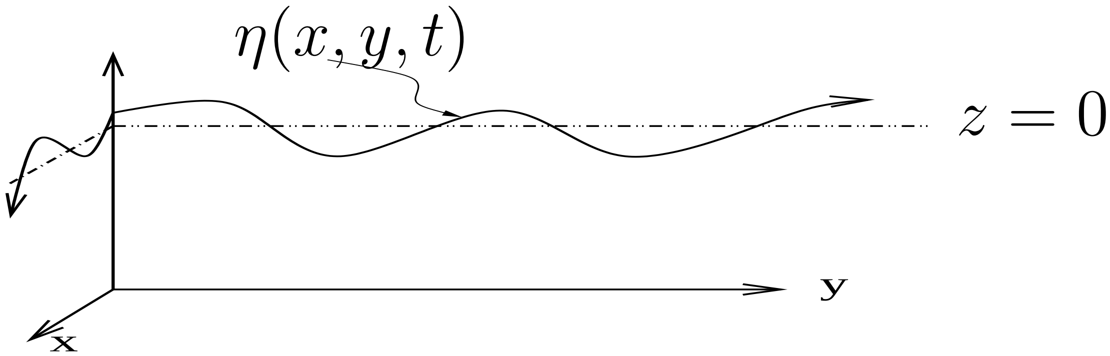
Boussinesq: $\eta(x,y,t)$ and $\vec{v}(x,y,z,t)$ unknown

$$\eta_t + \nabla \cdot \vec{v} + \nabla \cdot \eta\vec{v} - \frac{1}{6}\nabla^2\eta_t = 0$$

$$\vec{v}_t + \nabla\eta + \frac{1}{2}\nabla|\vec{v}|^2 - \frac{1}{6}\nabla^2\vec{v}_t = 0.$$

Euler + almost constant density
shallow water + symmetric + arbitrary solutions shape

Models

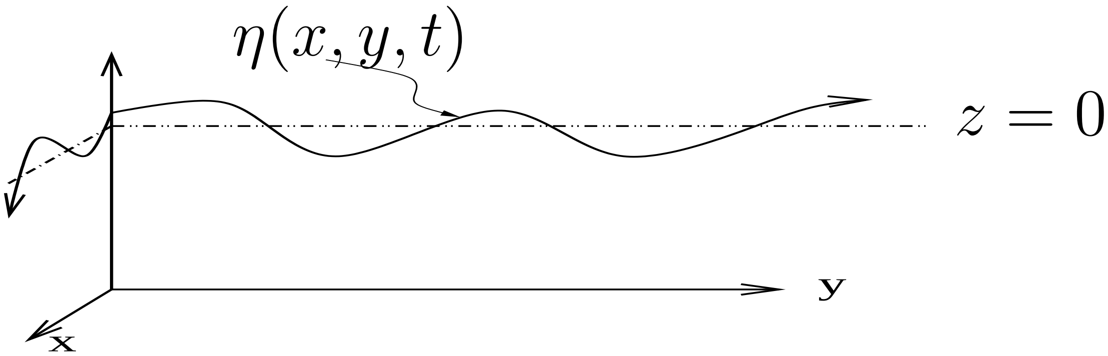


KP: $\eta(x, y, t)$ unknown

$$\left(\eta_t + \eta_x + \frac{2}{3}\eta\eta_x \pm \frac{1}{6}\eta_{xxt}\right)_x + \frac{1}{2}\eta_{yy} = 0$$

Perturbation of carrier wave
Small amplitude + shallow water + unidirectional
Integrable!

Models



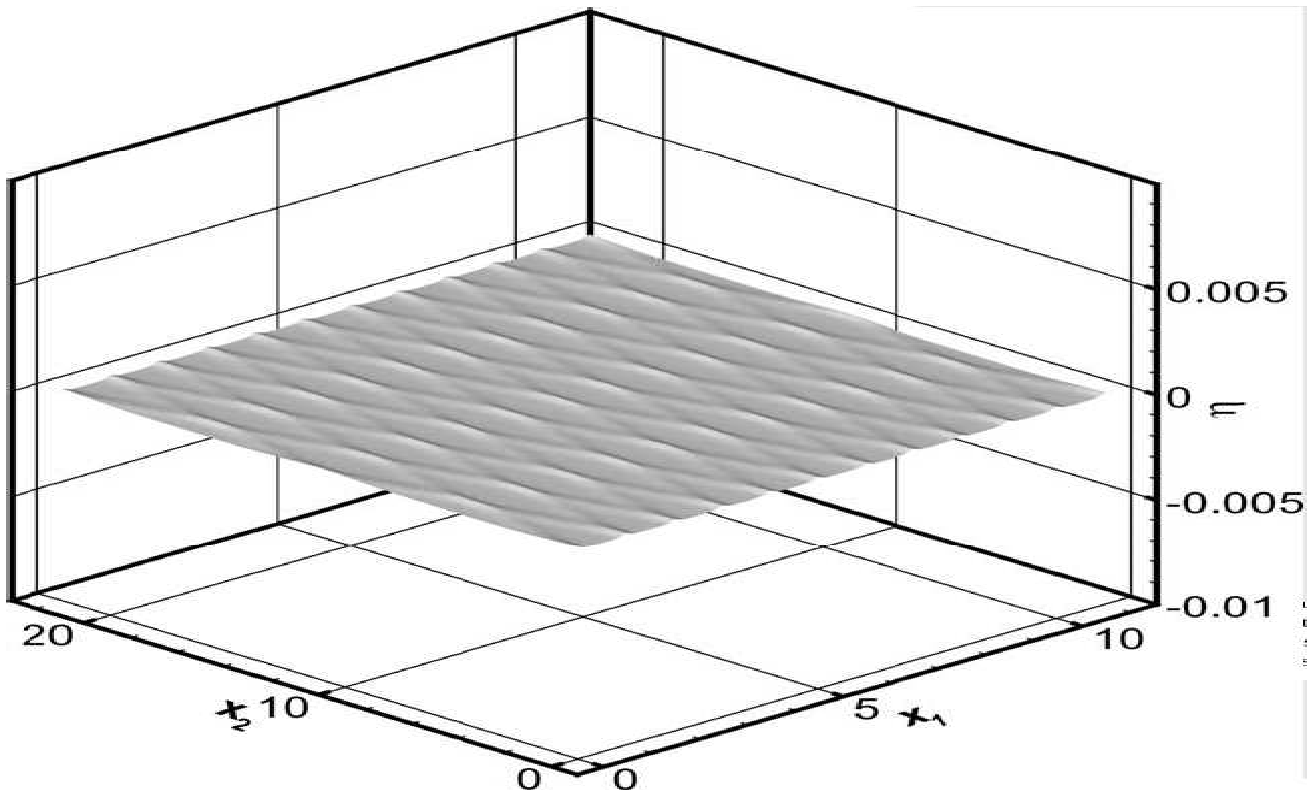
KdV: $\eta(x,t)$ unknown

$$\eta_t + \eta\eta_x + \eta_{xxx} = 0$$

Perturbation of carrier wave
2D + small amplitude + shallow water + unidirectional

Integrable!

Euler ‘solutions’



Craig & Nicholls: TWW in 2 and 3D

Boussineq ‘solutions’

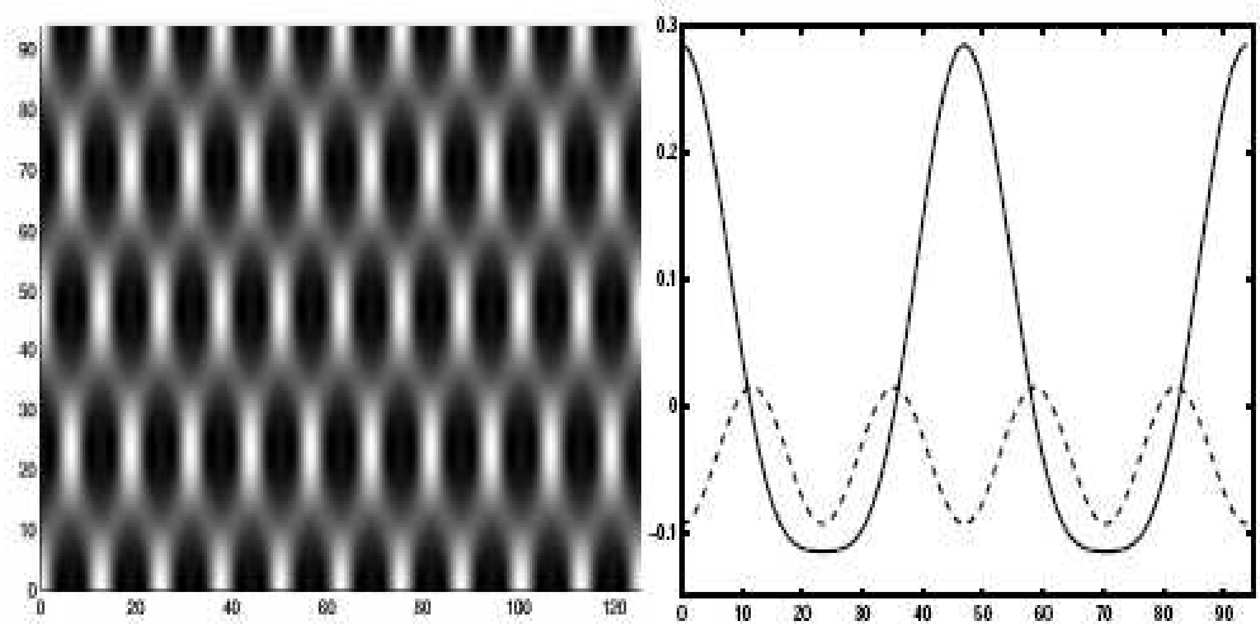
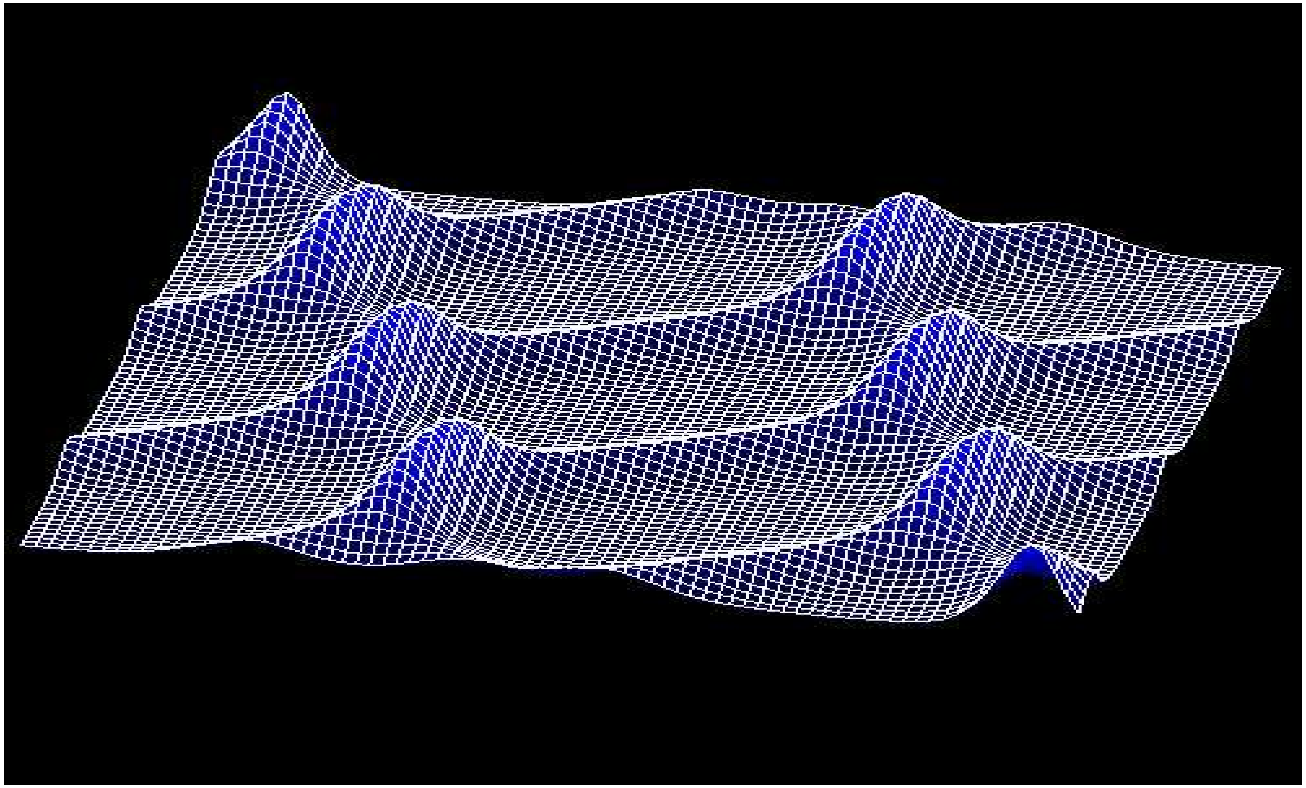


Figure 2: $\tau = 2 - \sqrt{3}$, $L_{x_1} = 12.6$, $L_{x_2} = 46.96$, $c_0 = 0.99$, $c \approx 1.04$

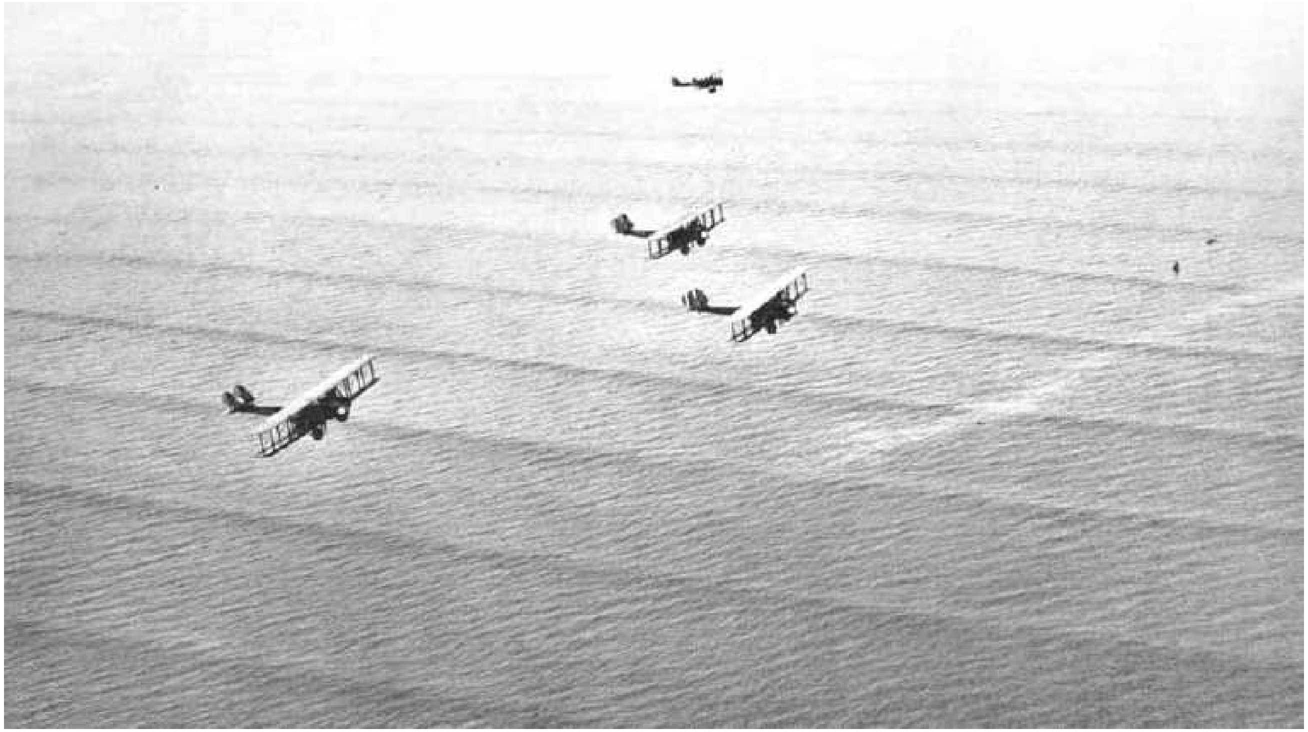
Chen & Iooss: Periodic Wave Patterns 2D Boussinesq

KP solutions



Bernard's KP webpage

KdV solutions



Bernard's KP webpage

Stability

Are these patterns stable?

Stability

Are these patterns stable?
We have access to:

- exact KdV solutions
- exact KP solutions
- numerical Boussinesq ‘solutions’
- numerical Euler ‘solutions’

Consider the Boussinesq problem ....

Why??

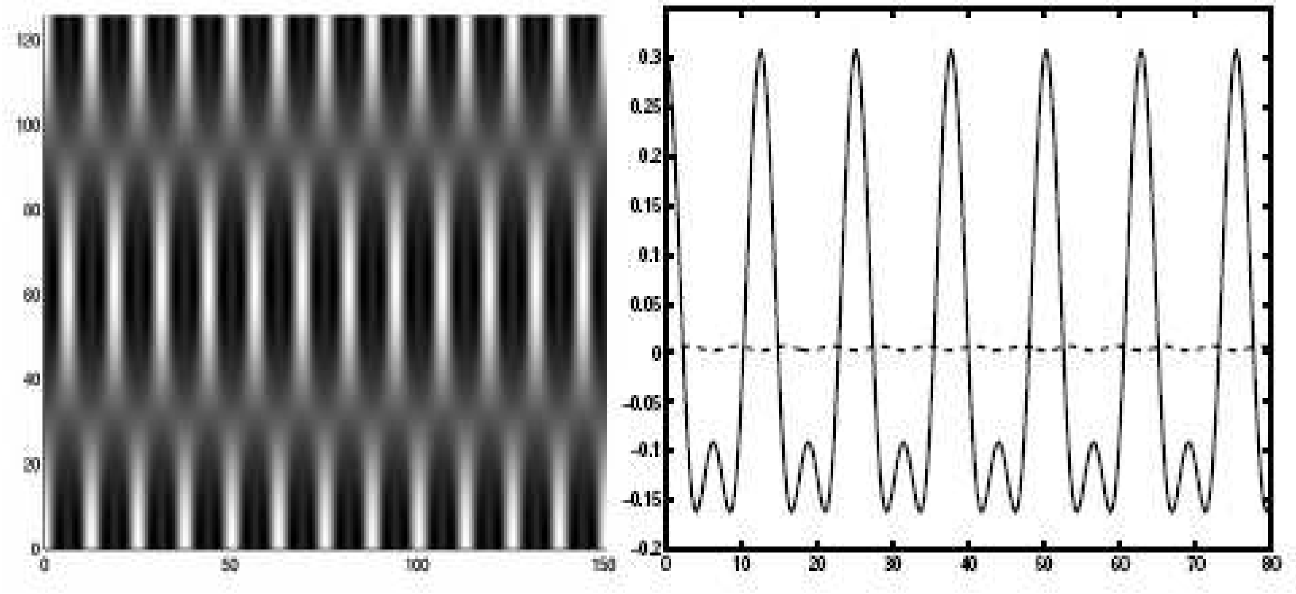
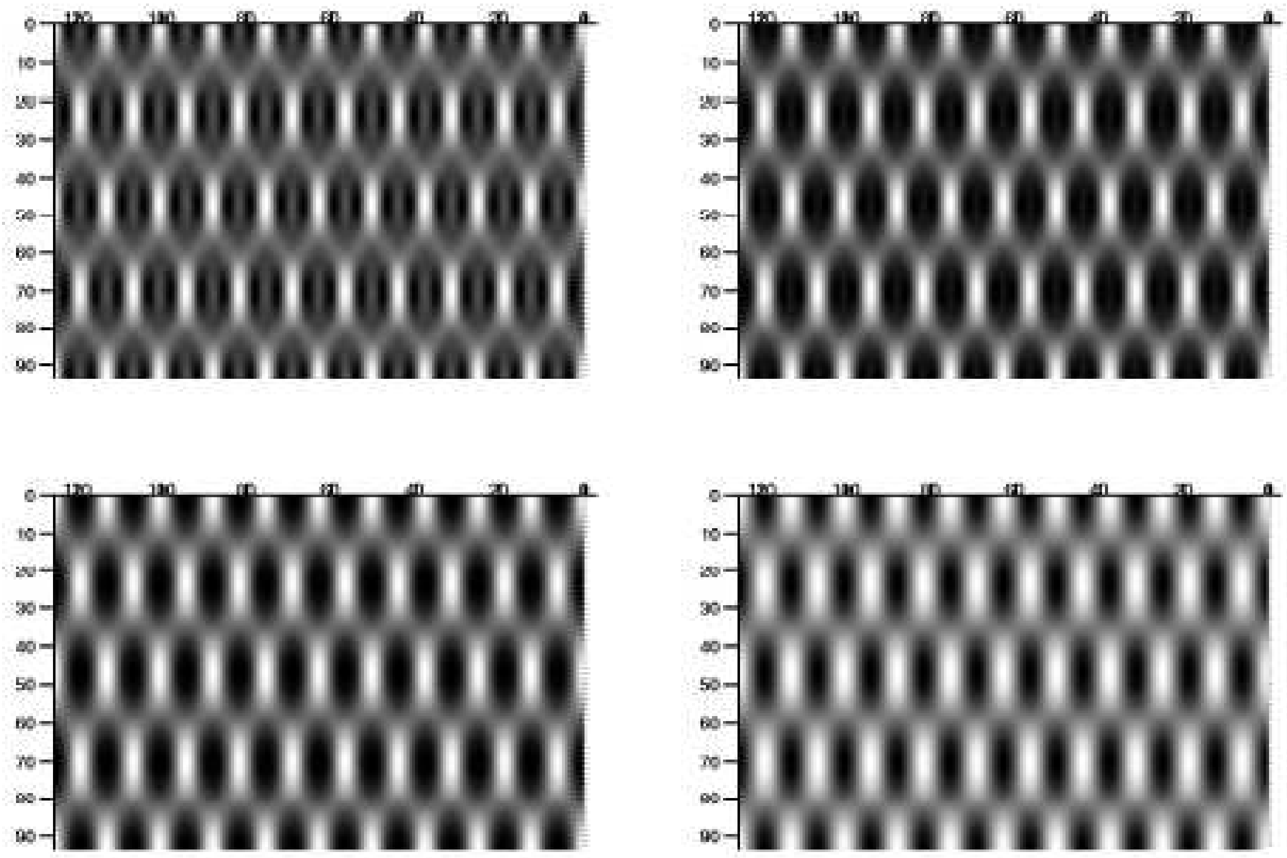


Figure 1: $\tau = 1/10$, $L_{x_1} = 12.6$, $L_{x_2} = 126$, $c_0 = 0.96$, $c \approx 1.03$

Chen & Iooss: Periodic Wave Patterns 2D Boussinesq

Why??

Changing water depth.



Boussinesq

Recall the governing system for the simulations:

$$\eta_t + \nabla \cdot \vec{v} + \nabla \cdot \eta \vec{v} - \frac{1}{6} \nabla^2 \eta_t = 0$$

$$\vec{v}_t + \nabla \eta + \frac{1}{2} \nabla |\vec{v}|^2 - \frac{1}{6} \nabla^2 \vec{v}_t = 0.$$

Boussinesq

Abstractly, this is

$$L_1 \begin{bmatrix} \eta \\ \vec{v} \end{bmatrix}_t = L_2 \begin{bmatrix} \eta \\ \vec{v} \end{bmatrix}$$

or

$$\begin{bmatrix} \eta \\ \vec{v} \end{bmatrix}_t = \underbrace{L_1^{-1} L_2}_{N} \begin{bmatrix} \eta \\ \vec{v} \end{bmatrix}$$

where $L_1 = (-\mathbb{I} + \frac{1}{6}\nabla^2)$ and

$$L_2 \begin{bmatrix} \eta \\ \vec{v} \end{bmatrix} = \begin{bmatrix} \nabla \cdot \vec{v} + \nabla \cdot \eta \vec{v} \\ \nabla \eta + \frac{1}{2} \nabla |\vec{v}|^2 \end{bmatrix}$$

Boussinesq

Problem: This is a nonlinear and nonlocal operator.
Some possibilities:

- Time evolution and lagging nonlinearities.
- Linearize around perturbed amplitude η and $\vec{v}$.
- Linearize around perturbed $\vec{c}$ velocity of travelling wave.
- Write as an integral equation?

If we can form a local evolution problem, then stability should be possible to compute.

Spectral Stability

Consider the evolution system

$$u_t = N(u)$$

with an equilibrium solution u_e:

$$N(u_e) = 0.$$

Is this solution *stable* or *unstable*?
Linearize: let

$$u = u_e + \epsilon\psi.$$

Substitute in and retain first-order terms in ϵ:

$$\psi_t = \mathcal{L}[u_e(x)]\psi.$$

Eigenfunction expansion

Separation of variables: $\psi(x,t) = e^{\lambda t} z(x)$:

$$\mathcal{L}[u_e(x)]z = \lambda z.$$

- This is a spectral problem.
- If $\Re(\lambda) \leq 0$ for all bounded $z(x)$, then u_e is spectrally stable.

Spectral Problem

So

$$\mathcal{L}z = \lambda z,$$

with

$$\mathcal{L} = \sum_{k=0}^{M} f_k(x)\partial_x^k, \qquad f_k(x+L) = f_k(x).$$

We want to find

- Spectrum $\sigma(\mathcal{L}) = \{\lambda \in \mathbb{C} : ||z||_\infty < \infty\}$.
- Corresponding eigenfunctions $z(\lambda, x)$?

Floquet's Theorem

Consider

$$\varphi_x = A(x)\varphi, \quad A(x+L) = A(x). \qquad (*)$$

Floquet's theorem states that the fundamental matrix Φ for this system has the decomposition

$$\Phi(x) = P(x)e^{Rx},$$

with $P(x+L) = P(x)$ and R constant.

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with $P(x+L) = P(x)$ and R constant.
Conclusion: **All bounded solutions of ($*$) are of the form**

$$\varphi = e^{i\mu x} \sum_{n=-\infty}^{\infty} \hat{\varphi}_n \, e^{i2\pi nx/L},$$

with $\mu \in [0, 2\pi/L)$**.**

Eigenfunctions

The periodic eigenfunctions can be expanded as

$$\varphi = e^{i\mu x} \sum_{n=-\infty}^{\infty} \hat{\varphi}_n \, e^{i\pi n x/L},$$

with $\mu \in [0, \pi/L)$

Substitute in the equation and cancel $e^{i\mu x}$.

The Floquet parameter μ only appears in derivative terms.

Hill's approach

- Write as spectral problem. (?????)
- Find Fourier coefficients of all functions
- Choose a number of μ values $\mu_1, \mu_2, \ldots$
- For all chosen μ values, construct $\hat{\mathcal{L}}_N(\mu)$
- Eigenvalue/vector solver

Any ideas?

$$\eta_t + \nabla\cdot\vec{v} + \nabla\cdot\eta\vec{v} - \frac{1}{6}\nabla^2\eta_t = 0$$

$$\vec{v}_t + \nabla\eta + \frac{1}{2}\nabla|\vec{v}|^2 - \frac{1}{6}\nabla^2\vec{v}_t = 0.$$

Moving frame ...

Min solves her problem in the moving reference frame $\vec{x} - \vec{c}t$.
In this frame, the system is

$$\nabla \cdot (\vec{v} + \eta\vec{v}) - \vec{c} \cdot \nabla(\eta - \frac{1}{6}\nabla^2\eta) = 0$$

$$\nabla \cdot (\eta + \frac{1}{2}|\vec{v}|^2) - \vec{c} \cdot \nabla(\vec{v} - \frac{1}{6}\nabla^2\vec{v}) = 0$$

where η and $\vec{v}$ are periodic functions of $\vec{x} - \vec{c}t$, where $\vec{x} = (x_1, x_2) \in \mathbb{R}$ and $\vec{c}$ is the travelling wave velocity.

Linearized problem

The linearized operator is then:

$$\mathcal{L}_c U = \mathcal{G} F$$

where

$$U = (\eta, \vec{v})^T, F = (g, \vec{f})^T$$

with

$$\mathcal{L}_c U = \begin{bmatrix} \nabla \cdot \vec{v} - \vec{c} \cdot \nabla(\eta - \frac{1}{6}\nabla^2 \eta \\ \nabla \eta - \vec{c} \cdot \nabla(\vec{v} - \frac{1}{6}\vec{v}) \end{bmatrix}$$

and

$$\mathcal{G} F = (\nabla \cdot F, \nabla g)$$

more detail

Since $\vec{f}$ and g are periodic on a lattice Γ prescribed by the angle of interaction, we write

$$\vec{f}(\vec{x},t) = \sum \vec{f}(t)_{\vec{k}} \exp(i\vec{k}^T\vec{x})$$

and

$$g(\vec{x},t) = \sum g(t)_{\vec{k}} \exp(i\vec{k}^T\vec{x})$$

She then finds solutions of the form

$$\eta(\vec{x},t) = \sum \eta(t)_{\vec{k}} \exp(i\vec{k}^T\vec{x})$$

and

$$\vec{v}(\vec{x},t) = \sum \vec{v}(t)_{\vec{k}} \exp(i\vec{k}^T\vec{x})$$