Stability of Differential Equations

through the lens of power series

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Outline

Series - a quick introduction

Power series show up first in Calculus, and then again in ODE courses. Usually, they are used to solve linear ODEs like this one from Stewart:

$$y'' - 2ty' + y = 0 \quad y(0) = 0, y'(0) = 1$$

with solution

$$y(t) = t + \sum_{n=1}^{\infty} \frac{1 \cdot 5 \cdot 9 \cdot \ldots \cdot (4n-3)}{(2n+1)!} t^{2n+1}$$

Ex 0: $y' = \alpha y$

Consider

$$\frac{d}{dt}y(t) = \alpha y(t)$$

Let

$$y(t) = \sum_{n=0}^{\infty} y_n t^n,$$

plug it into the ODE and get

$$\sum_{n=0}^{\infty}(n+1)y_{n+1}t^n = \alpha \sum_{n=0}^{\infty} y_n t^n$$

and compare coefficients for each power of t, we see

$$y_{n+1} = \frac{\alpha}{n+1} y_n$$

Ex 0: $y' = \alpha y$

Since

$$y_{n+1} = \frac{\alpha}{n+1} y_n$$

we have

$$y_1 = \frac{\alpha}{1} y_0 = \alpha y_0$$
$$y_2 = \frac{\alpha}{2} y_1 = \frac{\alpha}{2}(\alpha y_0) = \frac{\alpha^2 y_0}{2!}$$

and ...

$$y_n = \frac{\alpha^n}{n!} y_0$$

Noting that y_0 is the IV, we have $y(t) = y_0 exp(\alpha t)$.

Ex 1: $y' = \alpha(t)y$

Now consider

$$\frac{d}{dt}y = \alpha(t)y \quad y(0) = y_0$$

a non-autonomous IVODE, with solution

$$y(t) = y_0\, exp\left(\int_0^t \alpha(\tau)d\tau\right).$$

Assume

$$\alpha(t) = \sum_{n=0}^{\infty} a_n t^n \quad \text{and} \quad y(t) = \sum_{n=0}^{\infty} y_n t^n$$

and substitute:

$$\sum_{n=0}^{\infty}(n+1)y_{n+1}t^n = \left(\sum_{n=0}^{\infty} a_n t^n\right)\cdot\left(\sum_{n=0}^{\infty} y_n t^n\right)$$

aside: Products

Power series are easy to add, subtract, differentiate and integrate - do it term by term.

If $A = \sum_{n=0} a_n t^n$ and $B = \sum_{n=0} b_n t^n$, what is $A \cdot B$?

$$\begin{aligned}(a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \dots) \cdot (b_0 + b_1 t + b2t^2 + b3t^3 + \dots)\\ = (a_0 + b_0) + (a_0 b_1 + a_1 b_0)t + (a_0 b_2 + a_1 b_1 + a_2 b_0)t^2 \dots\\ + (a_0 b_3 + a_1 b_2 + a_2 b_1 + a_3 b_0)t^3 + \dots\\ = \sum_{n=0} \left[\sum_{i=0}^{n} a_i b_{n-i} \right] t^n\end{aligned}$$

We'll call this the Cauchy Product

Ex 1: $y' = \alpha(t)y$

With MAPLE

```
>> restart:
>> Order := 4:
>> alpha := t -> sum(a[k]*t^k,k=0..Order):
>> GROWTH := diff(y(t),t) = alpha(t)*y(t):
>> Yseries := dsolve({GROWTH,y(0)=y[0]},y(t),type='series');
```

$$y(t) = y_0 + a_0 y_0 t + \left(1/2\, a_0{}^2 y_0 + 1/2\, a_1 y_0\right) t^2 + \left(1/6\, a_0{}^3 y_0 + 1/2\, a_1 a_0 y_0 + 1/3\, a_2 y_0\right) t^3 + O\left(t^4\right)$$

which we can check

```
>> SOLN1 := y[0] * exp(int(alpha(tau),tau=0..t));
>> taylor(SOLN1,t=0);
```

But why?

From

$$y(t) = y_0 + a_0\, y_0 t + \left(1/2\, {a_0}^2\, y_0 + 1/2\, a_1\, y_0\right) t^2 + \\ \left(1/6\, {a_0}^3\, y_0 + 1/2\, a_1\, a_0\, y_0 + 1/3\, a_2\, y_0\right) t^3 + O\left(t^4\right)$$

we can find

$$\partial_{y_0} y(t) = 1 + a_0 t + \left(1/2\, {a_0}^2 + 1/2\, a_1\right) t^2 + \\ \left(1/6\, {a_0}^3 + 1/2\, a_1\, a_0 + 1/3\, a_2\right) t^3 + O\left(t^4\right)$$

Sensitivity to initial conditions! Which we can verify...

```
>> Yp := taylor(diff(Yseries,y_0),t=0);
```

$$= 1 + a_0 t + \left(1/2\, {a_0}^2 + 1/2\, a_1\right) t^2 + \\ \left(1/6\, {a_0}^3 + 1/2\, a_1 a_0 + 1/3\, a_2\right) t^3 + O\left(t^4\right)$$

The Lyapunov exponent

The *Lyapunov* exponent, $\lambda(t)$, is the exponential growth rate measuring sensitivity to initial conditions. It is classically computed as:

$$||\delta\mathbf{x}(t)|| \approx exp(\lambda(t) \cdot t)||\delta\mathbf{x_0}||,$$

where $\delta\mathbf{x}$ is the (first) variation of trajectory $\mathbf{x}$.

We can instead calculate $\lambda(t)$ directly via:

$$\partial_{y_0} y(t) \approx \exp(\lambda(t) \cdot t)$$

The Lyapunov exponent

We have

$$\partial_{y_0} y\,(t) = \sum_{n=0}^{\infty} f_n(y_0) t^n$$

and so $\lambda(t)$ is easy to compute:

```
>> simplify(taylor(ln(Yp))/t);
```

$$\lambda(t) = a_0 + 1/2 a_1 t + 1/3 a_2 t^2 + O(t^3)$$

For our problem, a direct calculation verifies this:

$$\frac{1}{t}\int_0^t \alpha(\tau) d\tau).$$

This time average is the mean coefficient.

Ex 2: $\frac{d}{dt}y = y^3$

Consider

$$\frac{d}{dt}y(t) = y^3(t).$$

Now let $v(t) = y^2$ and notice that

$$\begin{aligned} y' &= y^3 = v \cdot y \\ v' &= 2y \cdot y' = 2v^2 \end{aligned}$$

The variational problem for this would be

$$\delta \mathbf{w}' = \begin{bmatrix} v & y \\ 0 & 4v \end{bmatrix} \delta \mathbf{w}$$

from which we find the exponents to be $\lambda_1 = \frac{1}{t}\int_0^t v(\tau), \lambda_2 = \frac{1}{t}\int_0^t 4v(\tau)$.

Stability

If $y(0) = y_0$, then $v(0) = y_0^2$, and our series solution is (from MAPLE)

$$y(t) = y_0 + {y_0}^3 t + 3/2\, {y_0}^5 t^2 + 5/2\, {y_0}^7 t^3 + O\left(t^4\right)$$
$$v(t) = y_0^2 + 2\, {y_0}^4 t + 4\, {y_0}^6 t^2 + 8\, {y_0}^8 t^3 + O\left(t^4\right),$$

and we have complete stability information.

$$\partial_{y_0} y(t) = 1 + 3y_0^2 t + (15/2) y_0^4 t^2 + (35/2) y_0^6 t^3 + O(t^4)$$

with the corresponding exponent $(\lambda(t) = \frac{1}{t} \ln(\partial_{y_0} y(t)))$,

$$\lambda(t) = y_0^2 + y_0^4 t + (4/3) y_0^6 t^2 + O(t^3)$$

at any point in the flow!

Ex 3: $y' = \sin(y)$

Consider

$$\frac{d}{dt}y(t) = \sin(y)$$

and the auxiliary variables $u = \sin(y)$ and $v = \cos(y)$. Then

$$\begin{aligned} y' &= 1 \cdot u \\ u' &= v \cdot u \\ v' &= -u \cdot u \end{aligned}$$

We can build a series solution of $y(t)$ and compute stability as before.

$$i.e. \quad y(t) = \sum_{n=0}^{\infty} f_n(y_0)t^n$$

The PROCESS

Generate a polynomial (quadratic) system.(Reverse Polish)

Construct series solution $Y(t; y_0)$ using (y_0, t).

Construct sensitivity to ICS $M(t; y_0) := \partial_{y_0} Y(t; y_0)$

$t =$ large value

20 WHILE T$<$ Tmax

  Evaluate $Y(T)$ to generate IC

  Evaluate $M(T; IC)$

  Compute $\lambda(T)$ (or mean) for local time interval

  (compute local radius of convergence, r?)

  Advance: $T = T + r$

GOTO 20

THEORY

Carothers et. al. 2005 [2]

Theorem

A function v is the solution to an arbitrary component of a polynomial system of differential equations if and only if for some n there is a polynomial Q in $n+1$ variables so that $Q(v, v', \cdots, v^{(n)}) = 0$.

This implies that the motion of our moon may be described
without
reference to the earth, sun, or any other planets!

THEORY

Error Bound

Warne et. al. 2006 [3]

If we have (at $a = 0$) a system $\mathbf{y}'(t) = \mathbf{f}(\mathbf{y}(t))$, $\mathbf{y}(0) = \mathbf{b}$, then

$$\left\| \mathbf{y}(t) - \sum_{k=0}^{n} \mathbf{y}_k t^k \right\|_\infty \leq \frac{\|\mathbf{b}\|_\infty |Kt|^{n+1}}{1 - |Mt|} \quad \text{for} \quad m \geq 2 \tag{1}$$

Where the parameters K and M depend on immediately observable quantities of the original system;

M is the largest row sum of coefficients, and $K = (m-1)c^{m-1}$, where $c = \max\{1, ||\mathbf{b}||_\infty\}$ and $m = deg(\mathbf{f})$.

Conclusions

- Easy to find approximate solution operator as a function of IV.
- Easy to compute stability and exponents (and spectrum?).
- No need to evolve tangent space (a la Wolf).
- Non-autonomous? No fear!
- Non-linear? No problem!

Questions??

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References

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For more information

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