Differential Equations

through the lens of power series

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Outline

What are series?

If you have a sequence $\{a_n\}$, then add the terms:

$$a_0 + a_1 + a_2 + a_3 + \ldots = \sum_{n=0}^{\infty} a_n$$

Sometimes it converges, sometimes it doesn't.
We are usually interested in the series that coverge.

Examples

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \dots$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \dots$$

$$\sum_{n=0}^{\infty} x^n = 1 + x^1 + x^2 + x^3 + \dots$$

$$\sum_{n=0} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \frac{f^{(3)}(0)}{3!} x^3 + \dots$$

And more examples

And series for some common functions:

$$\exp(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{(2n+1)}$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

Example 1

Consider

$$\frac{d}{dx}y(x) = y(x) \quad y(0) = y_0$$

Phase Portrait

What do you think the solutions of

$$\frac{d}{dx}y(x) = y(x) \quad y(0) = y_0$$

to look like?

Let's use a phase portrait to get some intuition about typical solutions.

An exact solution

We've got

$$\frac{dy}{dx} = y$$

Let's try to integrate it.

An exact solution

We've got

$$\frac{dy}{dx} = y$$

$$\frac{1}{y}\frac{dy}{dx} = 1$$

$$\int \frac{1}{y}\frac{dy}{dx}\,dx = \int 1\,dx$$

$$\int \frac{1}{y}\,dy = \int 1\,dx$$

$$\ln(y) = x + C \quad \textit{careful!}$$

$$\text{So } y(x) = \exp(x + C) = K\exp(x).$$

Using the initial condition $y(x = 0) = y_0$,
we have

$$y(x) = y_0 \exp(x) = y_0 e^x$$

A power series solution?

How do we solve using power series?
Find coefficients a_n so that

$$y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$$

to satisfy

$$\frac{d}{dx} y(x) = y(x)$$

Power series

If

$$y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots = \sum_{n=0}^{\infty} a_n x^n,$$

then

$$\frac{d}{dx} y = \quad a_1 + 2a_2 x + 3a_3 x^2 + \ldots = \sum_{n=0}^{\infty} ???x^n$$

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And substitute into

$$\frac{d}{dx} y(x) = y(x)$$

Power series

We get

$$\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n = \sum_{n=0}^{\infty} a_n x^n$$

Power series

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$$\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n = \sum_{n=0}^{\infty} a_n x^n$$

If we compare coefficients for each power of x, we see

$$(n+1)a_{n+1} = a_n \implies a_{n+1} = \frac{1}{n+1}a_n$$

So,

$$a_1 = \frac{1}{1}a_0 = a_0$$
$$a_2 = \frac{1}{2}a_1 = \frac{1}{2}(a_0) = \frac{a_0}{2!}$$
$$a_3 = \frac{1}{3}a_2 = \frac{1}{3}\left(\frac{a_0}{2!}\right) = \frac{a_0}{3!}$$

and ...

Power series

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and ...

$$a_n = \frac{1}{n!}a_0$$

Power series

Since

$$a_n = \frac{1}{n!}a_0,$$

then

$$y(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_0 \frac{1}{n!} x^n = ???$$

Power series

Since

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then

$$y(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_0 \frac{1}{n!} x^n = a_0 \exp(x)$$

Again using $y(x = 0) = y_0$, we see that $a_0 = y_0$, and recover the exact solution

$$y(x) = y_0 \exp(x)$$

Usually, though, the function is harder identify.

Example 2:

Now consider

$$\frac{d}{dx}y = y(1-y),$$

the logistic equation.

Phase Portrait

What do you think solutions of

$$\frac{d}{dx}y = y(1-y) \quad y(0) = y_0$$

to look like?

Let's use a phase portrait again to get some intuition about typical solutions.

Exact solution

Let's solve $\frac{d}{dx}y = y(1-y)$ by integrating directly.

$$\begin{aligned} \frac{d}{dx}y &= y(1-y) \\ \frac{1}{y(1-y)}\frac{d}{dx}y &= 1 \\ \int \frac{1}{y(1-y)}\,dy &= \int 1\,dy \end{aligned}$$

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$$\int \frac{A}{y} + \frac{B}{1-y}\,dy = \int 1\,dy$$

Exact solution

Lets solve $\frac{d}{dx}y = y(1-y)$ by integrating directly.

$$
\begin{aligned}
\frac{d}{dx}y &= y(1-y) \\
\frac{1}{y(1-y)}\frac{d}{dx}y &= 1 \\
\int \frac{1}{y(1-y)}\,dy &= \int 1\,dy \\
\int \frac{1}{y} + \frac{1}{1-y}\,dy &= \int 1\,dy \\
\ln(y) - \ln(1-y) &= x + C, \\
\text{so } \frac{y}{1-y} &= \exp(x+C) = K\exp(x) \\
y(x) = \frac{1}{1+K\exp(-x)} &\text{ with } K = \frac{y_0}{1-y_0}
\end{aligned}
$$

Power series

Let's now solve $\frac{d}{dx}y = y(1-y)$ using power series. Again, assume

$$y(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Substituting,

$$\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n = \left(\sum_{n=0}^{\infty} a_n x^n\right)\cdot\left(1-\sum_{n=0}^{\infty} a_n x^n\right)$$

aside: Products

Power series are easy to add, subtract, differentiate and integrate - do it term by term.

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If $A = \sum_{n=0} a_n x^n$ and $B = \sum_{n=0} b_n x^n$, what is $A \cdot B$?

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If $A = \sum_{n=0} a_n x^n$ and $B = \sum_{n=0} b_n x^n$, what is $A \cdot B$?

$$
\begin{aligned}
(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots) \cdot (b_0 + b_1 x + b2x^2 + b3x^3 + \ldots) \\
= (a_0 + b_0) + (a_0 b_1 + a_1 b_0)x + (a_0 b_2 + a_1 b_1 + a_2 b_0)x^2 \ldots \\
+ (a_0 b_3 + a_1 b_2 + a_2 b_1 + a_3 b_0)x^3 + \ldots \\
= \sum_{n=0} \left[\sum_{i=0}^{n} a_i b_{n-i} \right] x^n
\end{aligned}
$$

We'll call this the Cauchy Product

aside: MATLAB

In MATLAB,

$$\sum_{n=0}^{degree} \sum_{i=0}^{n} a_i b_{n-i}$$

is

```
function cvec = cauchy_product(avec,bvec,degree)

for n = 1:degree + 1
    for i = 1:n
        j = n-i+1;
        cvec(n) = avec(i)*bvec(j)+cvec(n);
    end
end
```

Power series

Back to it:

$$\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n = \left(\sum_{n=0}^{\infty} a_n x^n\right) \cdot \left(1 - \sum_{n=0}^{\infty} a_n x^n\right)$$

$$= \sum_{n=0}^{\infty} a_n x^n - \sum_{n=0}^{\infty}\sum_{i=0}^{n} a_i a_{n-i} x^n)$$

$$= \sum_{n=0}^{\infty} (a_n - c_n)\, x^n$$

where c_n is the result of the Cauchy product.

Power series

Back to it:

$$\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n = \left(\sum_{n=0}^{\infty} a_n x^n\right)\cdot\left(1-\sum_{n=0}^{\infty} a_n x^n\right)$$
$$= \sum_{n=0}^{\infty} a_n x^n - \sum_{n=0}^{\infty}\sum_{i=0}^{n} a_i a_{n-i} x^n)$$
$$= \sum_{n=0}^{\infty} (a_n - c_n)\, x^n$$

where c_n is the result of the Cauchy product. We can again write a recursion relation:

$$(n+1)a_{n+1} = a_n - \sum_{i=0}^{n} a_i a_{n-i}$$

or

$$a_{n+1} = \frac{1}{n+1}\left(a_n - \sum_{i=0}^{n} a_i a_{n-i}\right)$$

MATLAB solution

MATLAB makes it easy. Since $a_0 = y_0$, we need

$$a_{n+1} = \frac{1}{n+1}\left(a_n - \sum_{i=0}^{n} a_i a_{n-i}\right)$$

```
function a = solve_logistic(a0,degree)

for n =1:degree+1
   c(n) = cauchy_product(a,a,n);
   a(n+1) = 1/(n+1) * ( a(n) - c(n) );
 end
```

Example 3

Let's explore

$$\frac{d}{dx}y = \sin(y)$$

Intuition

Consider

$$\frac{d}{dx}y = \sin(y) \quad y(0) = y_0$$

What do we expect?

Phase Portrait

Exact solution

$$\frac{d}{dx}y = sin(y)$$

$$\int \frac{1}{\sin(y)}\,dy = \int 1\,dx$$

Exact solution

$$\frac{d}{dx}y = sin(y)$$
$$\int \frac{1}{\sin(y)}\,dy = \int 1\,dx$$

These are getting a little tedious:
Wolfram Alpha says: `y(x) = 2 arccot(exp(C-x))`

ok...

$$\frac{d}{dx}y = \sin(y)$$

$$\int \frac{1}{\sin(y)}\,dy = \int 1\,dx$$

$$\int \frac{\sin(y)}{1-\cos^2(y)}\,dy = \int 1\,dx$$

$$\int \frac{-1}{1-u^2}\,du = \int 1\,dx$$

$$\int \frac{A}{1-u} + \frac{B}{1+u}\,du = \int 1\,dx$$

$$1/2\ln((1-u)/(1+u)) = x + C$$

$$\frac{1-\cos(y)}{1+\cos(y)} = K\exp(2x)$$

Power series

We have

$$\frac{d}{dx}y = \sin(y) \quad y(0) = y_0$$

We do have a series for $\sin(\bullet)$, but ...
Let's turn this into a polynomial system,
Let

$$v_1(x) = y(x), \quad v_2(x) = \sin(y(x)), \quad \text{and} \quad v_3(x) = \cos(y(x))$$

The polynomial system

Taking $v_1(x) = y, \quad v_2(x) = \sin(y) \quad$ and $\quad v_3(x) = \cos(y)$, then

$$\begin{aligned} v_1' &= 1 \cdot y' \\ &= v_2 \qquad\qquad v_1(0) = y_0 \\ v_2' &= v_3 \cdot y' \\ &= v_2 \; v_3 \qquad v_2(0) = \sin(y_0) \\ v_3' &= -v_2 \cdot y' \\ &= -v_2^2 \qquad v_3(0) = \cos(y_0) \end{aligned}$$

We can solve this system with series recursion, just as before.

The polynomial system

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&= -v_2^2 \qquad v_3(0) = \cos(y_0)
\end{aligned}$$

We can solve this system with series recursion, just as before.

Wwe can also consider the geometry....

Geometry

MATLAB allows us to see the structure.

Decoupling

Since $v_1(x) = y, \quad v_2(x) = \sin(y) \quad \text{and} \quad v_3(x) = \cos(y),$

$$\begin{aligned} v_1'' = y' &= v_2 \; v_3 \\ v_1''' = (v_2 \; v_3)' &= v_2 v_3' + v_2' v_3 \\ &= v_2(-v_2)^2 + (v_2 v_3) v_3 \\ &= -v_2^3 + v_2 v_3^2 \\ &= -v_2^3 + v_2(1 - v_2^2) \\ &= -2(v_2)^3 + v_2 \end{aligned}$$

but $v_2 = v_1' = y'$, and we've recast

$$y' = \sin(y) \quad \text{as} \quad y''' - y' + 2(y')^3 = 0$$

Gröbner basis theory says this can always be done.

Example 4

And one more to think about:

$$\frac{d}{dx}y = y\sin(y)$$

Intuition?

$$\frac{d}{dx}y = y\sin(y)$$

Analytic soln?

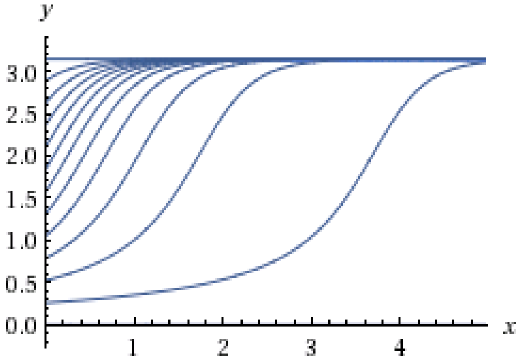
Maple

```
> dsolve(diff(y(x),x) = y(x)*sin(y(x)));
           y(x)
         /
         |          1
    x -  |      ---------- d_a + _C1 = 0
         |      _a sin(_a)
         /
```

Analytic soln?

Wolfram Alpha

Sample solution family:



(sampling $y(0)$)

Series?

Taking $v_1 = y, v_2 = y\sin(y), \quad v_3 =????, v_4 =????,$
then

Series?

Taking $v_1 = y, v_2 = y\sin(y), \quad v_3 = \sin(y), v_4 = \cos(y),$ then

Series?

Taking $v_1 = y, v_2 = y\sin(y), \quad v_3 = \sin(y), v_4 = \cos(y),$ then

$$\begin{aligned}
v_1' &= 1 \cdot y' \\
&= v_2 & v_1(0) &= y_0 \\
v_2' &= (y\cos(y) + \sin(y)) \cdot y' \\
&= (v_1\ v_4 + v_3)\ v_2 & v_2(0) &= y_0 \sin(y_0) \\
v_3' &= \cos(y) \cdot y' \\
&= v_2 v_3 & v_3(0) &= \sin(y_0) \\
v_4' &= -\sin(y) \cdot y' \\
&= -v_2^2 & v_4(0) &= \cos(y_0)
\end{aligned}$$

Let $v_5 = v_1 v_4$ to reduce to a quadratic system.

THEORY

We've been able to recast EVERY ODE that we've considered as a polynomial system.

U$D 50 CASH PRIZE to the first person to send me an ODE with analytic solution that **CAN'T** be recast as a polynomial system.

THEORY

Carothers et. al. 2005 [2]

Theorem

A function v is the solution to an arbitrary component of a polynomial system of differential equations if and only if for some n there is a polynomial Q in $n+1$ variables so that $Q(v, v', \cdots, v^{(n)}) = 0$.

THEORY

Carothers et. al. 2005 [2]

Theorem

A function v is the solution to an arbitrary component of a polynomial system of differential equations if and only if for some n there is a polynomial Q in $n+1$ variables so that $Q(v, v', \cdots, v^{(n)}) = 0$.

This implies that the motion of our moon may be described
without
reference to the earth, sun, or any other planets!

Warne et. al. 2006 [3]

If we have (at $a = 0$) a system $\mathbf{x}'(t) = \mathbf{h}(\mathbf{x}(t))$, $\mathbf{x}(0) = \mathbf{b}$, then

$$\left\| \mathbf{x}(t) - \sum_{k=0}^{n} \mathbf{x}_k t^k \right\|_\infty \leq \frac{\|\mathbf{b}\|_\infty |Kt|^{n+1}}{1 - |Mt|} \quad \text{for} \quad m \geq 2 \tag{1}$$

Where the parameters K and M depend on immediately observable quantities of the original system;

M is the largest row sum of coefficients, and $K = (m-1)c^{m-1}$, where $c = \max\{1, ||\mathbf{b}||_\infty\}$ and $m = deg(\mathbf{h})$.

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- Easily compute *arbitrarily high order Taylor coefficients*

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- *interpolation free* to machine capability (error and calculation)

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References

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For more information

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