How I spent my Educational Leave

JMU Seminar

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Outline

1. The idea
   - The Parker-Sochacki Method
   - PSM: two quick examples
   - The proposal
2. What really happened
   - Oct 7 conference: SUMS JMU
   - Oct 25 deadline: Missile Defense Agency RFP JMU
   - Oct 12-20 trip: ODEPSM Davidson College
   - Nov 17 deadline: Spring Mass JMU
   - Jan 9: Coefficient recovery JMM 18
   - Feb and May: Smoke rings UCSD
   - June-July: Seattle
3. Future

PSM: what is it?

I proposed three projects, and all were applications of the PSM method.

What is the PSM method, you ask? Here is what Wikipedia has to say...

PSM: what is it?

I proposed three projects, and all were applications of the PSM method. What is the PSM method, you ask? Here is what Wikipedia has to say...

PSM : $y' = \alpha y$

Consider

$$\frac{d}{dt}y(t) = \alpha y(t)$$

Let

$$y(t) = \sum_{n=0}^{\infty} y_n t^n,$$

plug it into the ODE and get

$$\sum_{n=0}^{\infty}(n+1)y_{n+1}t^n = \alpha \sum_{n=0}^{\infty} y_n t^n$$

and compare coefficients for each power of t, we see

$$y_{n+1} = \frac{\alpha}{n+1}y_n$$

PSM : $y' = \alpha y$

Since

$$y_{n+1} = \frac{\alpha}{n+1} y_n$$

we have

$$y_1 = \frac{\alpha}{1} y_0 = \alpha y_0$$

$$y_2 = \frac{\alpha}{2} y_1 = \frac{\alpha}{2}(\alpha y_0) = \frac{\alpha^2 y_0}{2!}$$

and ...

$$y_n = \frac{\alpha^n}{n!} y_0$$

Noting that y_0 is the IV, we have $y(t) = y_0 exp(\alpha t)$.

PSM : $y' = \alpha(t)y$

Consider

$$\frac{d}{dt}y = \alpha(t)y \quad y(0) = y_0$$

a non-autonomous IVODE, with solution

$$y(t) = y_0\, exp\left(\int_0^t \alpha(\tau)d\tau\right).$$

Assume

$$\alpha(t) = \sum_{n=0}^{\infty} a_n t^n \quad \text{and} \quad y(t) = \sum_{n=0}^{\infty} y_n t^n$$

and substitute:

$$\sum_{n=0}^{\infty}(n+1)y_{n+1}t^n = \left(\sum_{n=0}^{\infty} a_n t^n\right) \cdot \left(\sum_{n=0}^{\infty} y_n t^n\right)$$

aside: Products

Power series are easy to add, subtract, differentiate and integrate - do it term by term.

If $A = \sum_{n=0} a_n t^n$ and $B = \sum_{n=0} b_n t^n$, what is $A \cdot B$?

$$\begin{aligned}
(a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \dots) \cdot (b_0 + b_1 t + b2 t^2 + b3 t^3 + \dots) \\
= (a_0 + b_0) + (a_0 b_1 + a_1 b_0) t + (a_0 b_2 + a_1 b_1 + a_2 b_0) t^2 \dots \\
+ (a_0 b_3 + a_1 b_2 + a_2 b_1 + a_3 b_0) t^3 + \dots \\
= \sum_{n=0} \left[\sum_{i=0}^{n} a_i b_{n-i} \right] t^n
\end{aligned}$$

We'll call this the Cauchy Product

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$$\begin{aligned}(a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \ldots) \cdot (b_0 + b_1 t + b2t^2 + b3t^3 + \ldots) \\ = (a_0 + b_0) + (a_0 b_1 + a_1 b_0)t + (a_0 b_2 + a_1 b_1 + a_2 b_0)t^2 \ldots \\ + (a_0 b_3 + a_1 b_2 + a_2 b_1 + a_3 b_0)t^3 + \ldots \\ = \sum_{n=0} \left[\sum_{i=0}^{n} a_i b_{n-i} \right] t^n\end{aligned}$$

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PSM : $y' = \alpha(t)y$

With MAPLE

```
>> restart:
>> Order := 4:
>> alpha := t -> sum(a[k]*t^k,k=0..Order):
>> GROWTH := diff(y(t),t) = alpha(t)*y(t):
>> Yseries := dsolve({GROWTH,y(0)=y[0]},y(t),type='series');
```

$$y(t) = y_0 + a_0\, y_0 t + \left(1/2\, a_0^{\,2}\, y_0 + 1/2\, a_1\, y_0\right) t^2 + \left(1/6\, a_0^{\,3}\, y_0 + 1/2\, a_1\, a_0\, y_0 + 1/3\, a_2\, y_0\right) t^3 + O\left(t^4\right)$$

which we can check

```
>> SOLN1 := y[0] * exp(int(alpha(tau),tau=0..t));
>> taylor(SOLN1,t=0);
```

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$$\begin{aligned} y(t) = & y_0 + a_0\, y_0 t + \left(1/2\, a_0{}^2\, y_0 + 1/2\, a_1\, y_0\right) t^2 + \\ & \left(1/6\, a_0{}^3\, y_0 + 1/2\, a_1\, a_0\, y_0 + 1/3\, a_2\, y_0\right) t^3 + O\left(t^4\right) \end{aligned}$$

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```
>> SOLN1 := y[0] * exp(int(alpha(tau),tau=0..t));
>> taylor(SOLN1,t=0);
```

The proposal: my pitch

I pitched three projects:

1. Lyapunov exponents calculation
2. Sensitivity and Control
3. Nonlinear error estimates

My intention was to finish at least one, but I hoped to do more.

Intention

"There is always a gap between intention and action" — Paulo Coelho

SUMS

Professor Kohn and I organized SUMS 2017, which took place on October 7, 2017. The planning started in July.

- 290 registered
- 46 different schools
- 29 talks
- 13 posters

<http://www.jmu.edu/mathstat/sums/>

MDA: Triton Systems and JMU

Professor Jim Sochacki was first contacted by Triton Systems (a small defense contractor) in late August. A recent Missile Defense Agency RFP had mentioned the Parker-Sochacki method, and Triton was interested in working with him.

I helped worked on two classic problems for this:

- Hill problem (a standard and simple test case for two body gravitational dynamics) and
- Flame equation (a classic stiff equation),

and suggested applications of symbolic computation.

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MDA HIll's problem

The first problem we explored was the Hill system. It models the motion of the moon under the influence of the earth and an infinitely remote sun on a circular orbit.

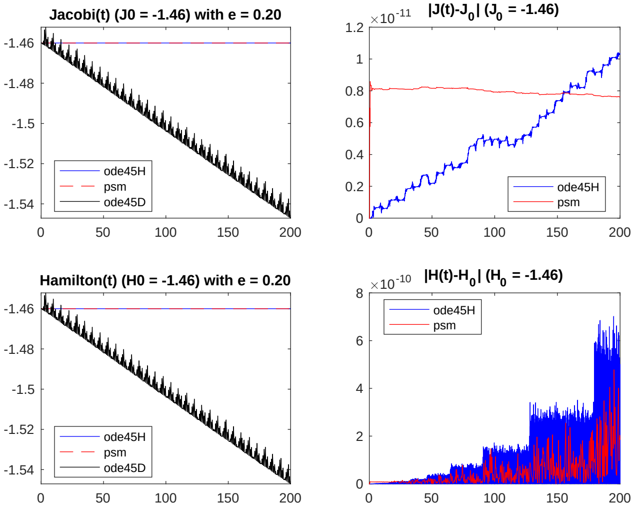
$$\ddot{x} - 2\dot{y} - 3x + xr^{-3} = 0 \tag{1}$$
$$\ddot{y} + 2\dot{x} \qquad + yr^{-3} = 0 \tag{2}$$

with $r^2 = x^2 + y^2$. Notice that $\frac{1}{2}(\dot{x}^2 + \dot{y}^2) - \frac{3}{2}x^2 - \frac{1}{r} = h$. The constant h is called the Jacobi constant. Rewritten as a first order system, we have

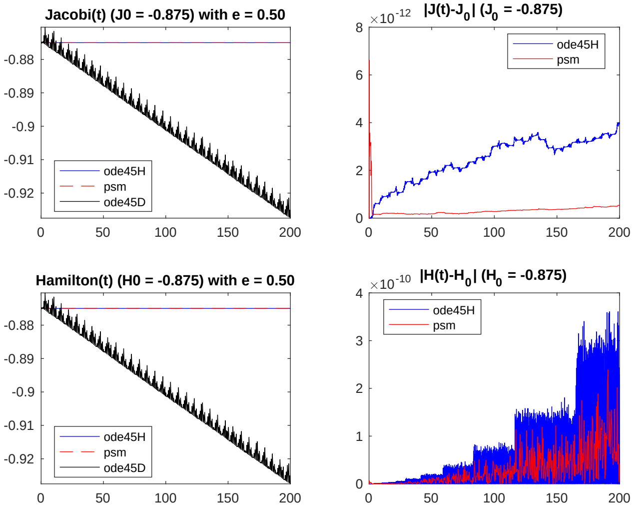
$$\dot{x} = v \tag{3}$$
$$\dot{v} - 2\dot{y} - 3x + xr^{-3} = 0 \tag{4}$$
$$\dot{y} = w \tag{5}$$
$$\dot{w} + 2\dot{x} \qquad + yr^{-3} = 0 \tag{6}$$

We take ICs to be [1-e,0,0,sqrt((1+e)/(1-e)] for e = 0.2, 0.5, and 0.8. and timespan was [0,200].

MDA: Hill $e = 0.2$

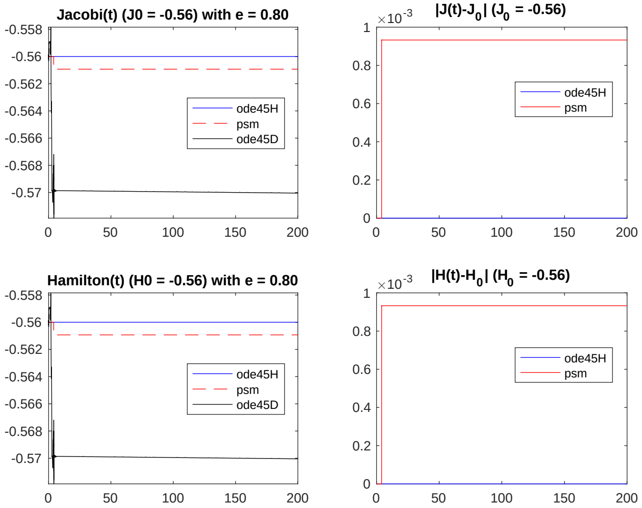

Jacobi(t) (J0 = -1.46) with e = 0.20
-1.46
-1.48
-1.5
-1.52
-1.54
0
50
100
150
200
ode45H
psm
ode45D
|J(t)-J0| (J0 = -1.46)
×10^-11
1.2
1
0.8
0.6
0.4
0.2
0
ode45H
psm
Hamilton(t) (H0 = -1.46) with e = 0.20
ode45H
psm
ode45D
|H(t)-H0| (H0 = -1.46)
×10^-10
8
6
4
2
0
ode45H
psm

MDA: Hill $e = 0.5$



Jacobi(t) (J0 = -0.875) with e = 0.50
ode45H
psm
ode45D
|J(t)-J0| (J0 = -0.875)
×10^-12
ode45H
psm
Hamilton(t) (H0 = -0.875) with e = 0.50
ode45H
psm
ode45D
|H(t)-H0| (H0 = -0.875)
×10^-10
ode45H
psm

MDA: Hill $e = 0.8$



Jacobi(t) (J0 = -0.56) with e = 0.80
-0.558
-0.56
-0.562
-0.564
-0.566
-0.568
-0.57
0
50
100
150
200
ode45H
psm
ode45D
|J(t)-J0| (J0 = -0.56)
×10-3
1
0.8
0.6
0.4
0.2
0
ode45H
psm
Hamilton(t) (H0 = -0.56) with e = 0.80
ode45H
psm
ode45D
|H(t)-H0| (H0 = -0.56)
×10-3
ode45H
psm

MDA: Smoke

The smoke equation is given by ...

MDA: Symbolic applications

This fit well with my ed leave proposal. I proposed to study sensitivity of ODEs, and used two examples Hill system and the smoke equation using symbolic PSM solutions. I could consider perturbations in both ICs and in parameters.

ODEPSM: FLOPS

```
>> analyze(@fhill_lunar,0,[1;0;0;1])
max m = 6; max C = 10.000000, and flops(degree) = 6 d^2 + 30 d

    'Definition             Series recur *=CP
    'u1 = y1 * y1           u1 = y1 * y1
    'u2 = y3 * y3           u2 = y3 * y3
    'u3 = u1 + u2
    'u4 = u3^1.5            u3 * u4' = 1.5 u4 * u3' solve for u4(
    'u5 = 1 / u4            1 = u5 * u4 solve for u5(k)
    'u6 = 2 * y4
    'u7 = 3 * y1
    'u8 = u6 + u7
    'u9 = y1 * u5           u9 = y1 * u5
    'u10 = u8 - u9
    'u11 = -2 * y2
    'u12 = y3 * u5          u12 = y3 * u5
    'u13 = u11 - u12
```

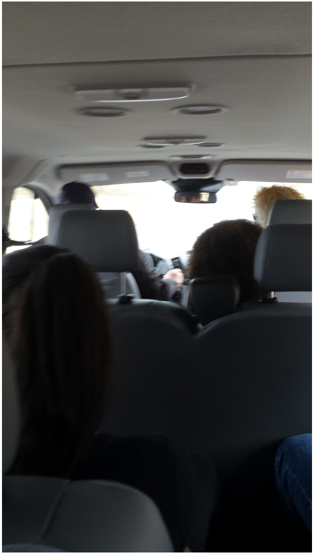
Spring Mass: PRIMUS

How can you think about

$$y''(t) + y(t) = tan(t)?$$

Professors Tongen, Sochacki, and I tried to explain how we do, in:
`PRIMUS_final.pdf`,
and with a MAPLE worksheet.
Submitted in November, accepted in April, and online in June.

JMM: talks



JMM: talks

and a talk:
2ptalk.pdf

JMM: and some social commentary

I also gave a talk about the state of our republic.

Smoke rings: motivation

- https://phys.org/news/2017-12-ocean-space.html
- https://www.youtube.com/watch?v=pnbJEg9r1o8

Smoke rings: getting there

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Smoke rings: getting there

Smoke rings: a little math

Thanks to Shigeo Kida (in 1981), we have analytic solutions that represent some solutions to a smoke ring equation. Its easier to visualize them:

`circular.jpg` `helicoidal.jpg` `double.jpg`

Others filaments include:

- straight line (Kelvin 1880)
- Solitary wave type ($k = 1, r, z \tanh$)
- plane curves

Smoke rings: solution from curvature

A differential equation prescribing the (square) of the curvature of the filament, $R'^2 = |x''|^2$, is found to be integrable. The complete equation is given by

$$R'^2 = F(r) \equiv -\left[R^3 + (V^2 - 2A)R^2 \right.$$
$$\left. + (A^2 - 4 - 2AV^2 + 4VC)R - (2C - AV)^2\right]$$

where the new parameter A arises as a constant of integration, C measures slippage along the filament, and V is velocity of translation. This has solution

$$R(\xi) = \alpha + (\beta - \alpha)\texttt{JacobiSN}(1/2\sqrt{\alpha + \gamma}\xi | m) \tag{7}$$

where $\alpha > \beta > 0 > \gamma$ are real roots of the cubic, and elliptic parameter m is given by $m = \frac{\alpha - \beta}{\alpha + \gamma}$.

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Smoke rings: Jacobi Elliptic functions

— 120 —

$$4\left\{3(3+2k^2+3k^4)\left(\frac{2K}{\pi}\right)^4 - 3^3.10(1+k^2)\left(\frac{2K}{\pi}\right)^2 + 3^4\right\}\frac{\sqrt{q^3}\sin 3x}{1-q^3} +$$

$$4\left\{5(3+2k^2+3k^4)\left(\frac{2K}{\pi}\right)^4 - 5^3.10(1+k^2)\left(\frac{2K}{\pi}\right)^2 + 5^4\right\}\frac{\sqrt{q^5}\sin 5x}{1-q^5} +$$

. . . .

cet. cet. ;

II.

e formula:

$$\left(\frac{2kK}{\pi}\right)^2 \sin^2 \mathrm{am}\,\frac{2Kx}{\pi} =$$

$$\frac{2K}{\pi}\cdot\frac{2K}{\pi} - \frac{2K}{\pi}\cdot\frac{2E^{I}}{\pi} - 4\left\{\frac{2q\cos 2x}{1-q^2} + \frac{4q^2\cos 4x}{1-q^4} + \frac{6q^3\cos 6x}{1-q^6} + \cdots\right\}$$

sequentes:

$$2.3\left(\frac{2kK}{\pi}\right)^4 \sin^4 \mathrm{am}\,\frac{2Kx}{\pi} =$$

$$4(1+k^2)\left(\frac{2K}{\pi}\right)^3\left(\frac{2K}{\pi} - \frac{2E^{I}}{\pi}\right) - 2k^2\left(\frac{2K}{\pi}\right)^4$$

$$-4\left\{2.4(1+k^2)\left(\frac{2K}{\pi}\right)^2 - 2^3\right\}\frac{q\cos 2x}{1-q^2}$$

$$-4\left\{4.4(1+k^2)\left(\frac{2K}{\pi}\right)^2 - 4^3\right\}\frac{q^2\cos 4x}{1-q^4}$$

$$-4\left\{6.4(1+k^2)\left(\frac{2K}{\pi}\right)^2 - 6^3\right\}\frac{q^3\cos 6x}{1-q^6}$$

. . . .

$$2.3.4.5\left(\frac{2kK}{\pi}\right)^6 \sin^6 \mathrm{am}\,\frac{2Kx}{\pi} =$$

$$8(8+7k^2+8k^4)\left(\frac{2K}{\pi}\right)^5\left(\frac{2K}{\pi}\cdot\frac{2K}{\pi} - \frac{2K}{\pi}\cdot\frac{2E^{I}}{\pi}\right) - 32k^2(1+k^2)\Big($$

$$-4\left\{2.8(8+7k^2+8k^4)\left(\frac{2K}{\pi}\right)^4 - 2^3.20(1+k^2)\left(\frac{2K}{\pi}\right)^2 + 2^5\right\}\frac{q\cos 2x}{1-q^2}$$

$$-4\left\{4.8(8+7k^2+8k^4)\left(\frac{2K}{\pi}\right)^4 - 4^3.20(1+k^2)\left(\frac{2K}{\pi}\right)^2 + 4^5\right\}\frac{q^2\cos 4x}{1-q^4}$$

$$-4\left\{6.8(8+7k^2+8k^4)\left(\frac{2K}{\pi}\right)^4 - 6^3.20(1+k^2)\left(\frac{2K}{\pi}\right)^2 + 6^5\right\}\frac{q^3\cos 6x}{1-q^6}$$

Smoke rings: math

Kida's parameters (C, V) are intuitive, but he computes stability of a (to me, non-intuitive) non-dimensional form based the roots α, *beta*, γ arising from closure conditions on the vortex filament. We hope that compute stability in (A, C, V) space. But haven't yet.

roadrip

Gains?

I learned a bunch of new stuff on leave, and have some work ahead.
Three manuscripts are in progress:

- ODEPSM: a power series approach to ODE: with Rich Niedinger
- Fourier-Floquet-Hill Stability of Steady Vortex Filament: with Stefan Llewellyn Smith and C. Chang
- Two Coefficient recovery in a Quasilinear Parabolic PDE

and most importantly ...
I've returned to the classroom with a renewed sense of excitement and enjoyment. My patience and sense of goodwill towards my students is high.

Thanks!

Thanks!