A Smorgasbord of Inverse Problems

a feast of applications

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Definitions

Merriam-Webster Online Dictionary

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smor · gas · bord:

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Pronunciation: 'smor-g&s-"bOrd
Function: noun
Etymology: from Swedish
smö rgås (open sandwich) + bord (table)

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Pronunciation: 'smor-g&s-"bOrd
Function: noun
Etymology: from Swedish
smö rgås (open sandwich) + bord (table)

- a luncheon or supper buffet offering a variety of foods and dishes (as hors d'oeuvres, hot and cold meats, smoked and pickled fish, cheeses, salads, and relishes)
- a heterogeneous mixture

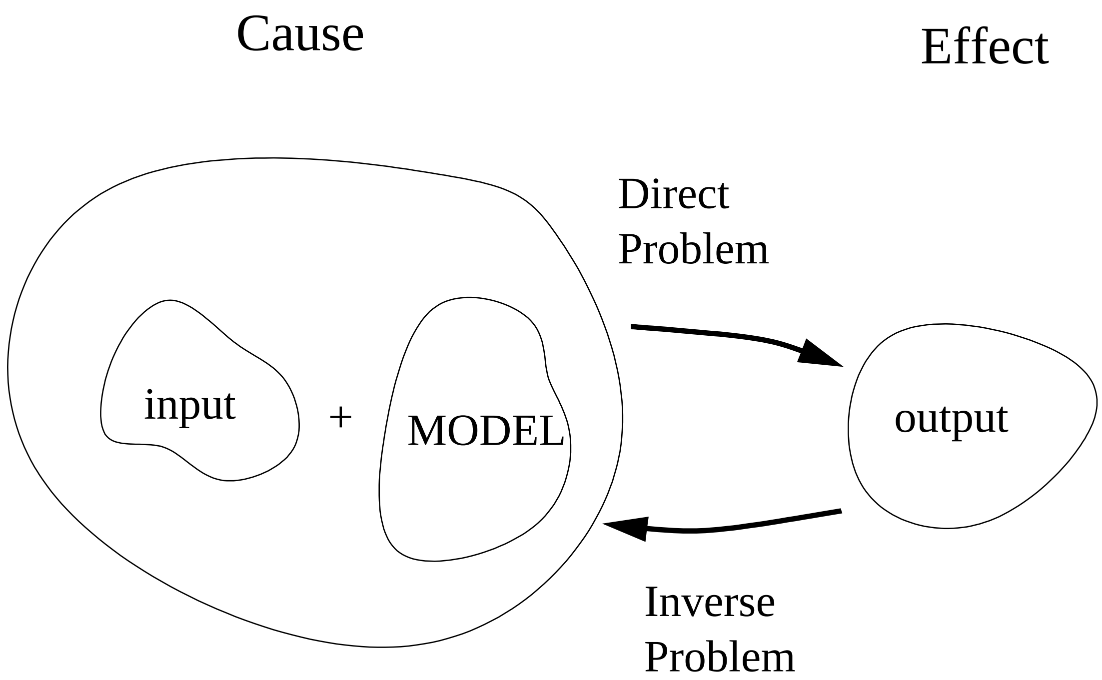
Merriam-Webster Online Dictionary

Introduction

We'll talk about two very basic inverse problems.

While the examples aren't hard, they make us aware of some common difficulties.

Examples



Inverse problems ask the following:
Given some output, can we determine properties of the model and/or of the input?

A simple idea

Given $f(x) = x^3 + 3x - 1$.

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Inverse Problems:

- If $f(x) = 3$, what is x?
- If $f(x) = 10$, what is x?

Definition

Well-Posedness:
Most inverse problems share the feature being not well-posed. Well-posedness is a concept developed by Hadamard (in the early 1900's).

A well-posed problem in one in which:

there **exists** *a* **unique** *solution that* **depends continuously** *on the data.*

Now for (just a little) math!

Pressure

Suppose: rate of change of pressure with respect to depth in a column of fluid is constant
Let $P(z)$ represent pressure at depth z.

$$\frac{dP}{dz} = \alpha, \qquad P(0) = \beta.$$

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In the direct problem, α and β are known, and we want to find $P(z)$. Then

$$P(z) = \alpha z + \beta.$$

Pressure

In the inverse problem, we have $\{(z_1, P_1), ..., (z_n, P_n)\}$ and want to find α and β.

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- n=1: Not enough info to find both α and β
- n=2: Maybe enough info find α and β
- n>2: Probably enough info to *approximate* α and β

Pressure

We find α, β by solving the system of n equations:

$$\begin{aligned} \alpha z_1 + \beta &= P_1 \\ \alpha z_2 + \beta &= P_2 \\ \vdots &= \vdots \\ \alpha z_n + \beta &= P_n \end{aligned}$$

Pressure

The accuracy of the inverse solution depends on several components.

- The data
- Sensitivity to the data
- (sometimes) the inversion method

Usually these are hard questions to answer.

Population

Consider the growth equation with constant rate r.

$$\frac{d}{dt}Q(t) = rQ(t)$$

Given initial population Q_0 and growth rate r, find $Q(t)$.
The solution:

$$Q(t) = Q_0 \exp(rt)$$

This process is relatively stable.

Population

Consider the growth equation with variable rate $r(t)$.

$$\frac{d}{dt}Q(t) = r(t)Q(t)$$

Given initial population Q_0 and growth rate $r(t)$, find $Q(t)$.
The solution:

$$Q(t) = Q_0 \exp\left(\int_0^t r(s)\ ds\right)$$

This process is relatively stable.

But ...

Inverse Problem - given

$$\frac{d}{dt}Q(t) = r(t)Q(t),$$

can we find $r(t)$ given some measured $Q(t)$?

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Inverse Problem - given

$$\frac{d}{dt}Q(t) = r(t)Q(t),$$

can we find $r(t)$ given some measured $Q(t)$?
Sure!

$$r(t) = \frac{1}{Q(t)}\frac{d}{dt}Q$$

But if there is even a small measurement error in Q, then $\frac{1}{Q}$ can change a lot! It's an *ill-posed* problem.

Examples

Inverse problems naturally occur in many areas. Many are related to *tomography*. Tomography uses external measurement to recover internal information.

Xray

a scattering problem!

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info.med.yale.edu

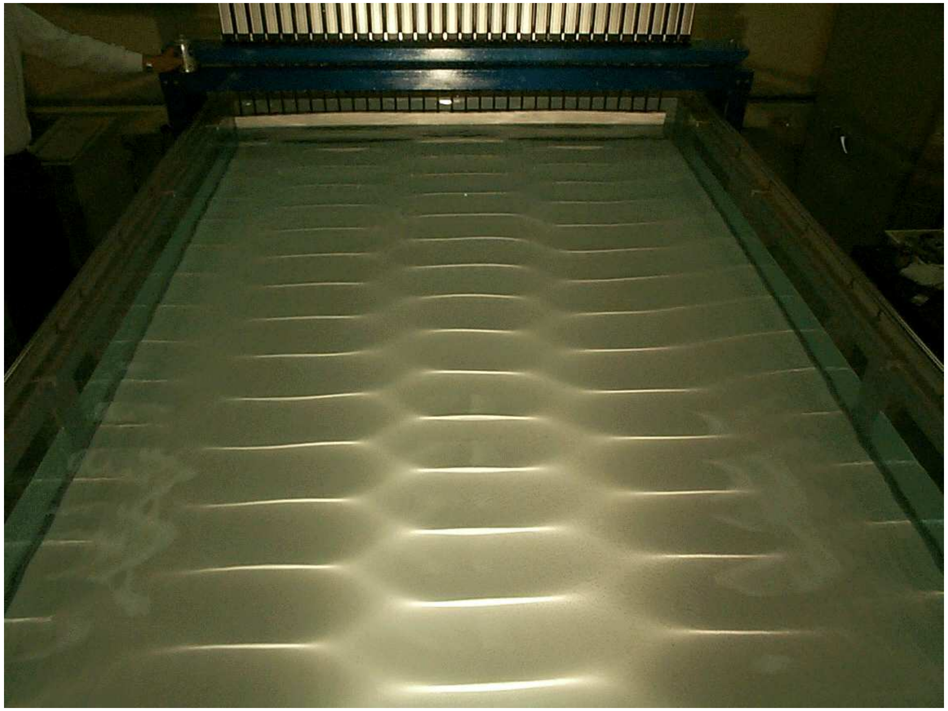
MRI

Magnetic Resonance Image

www.cis.rit.edu/htbooks/mri

Caustics

Ray Tracing



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www.math.psu.edu/dmh/FRG/

EIT

Electrical Impedence Tomography (EIT): part 1

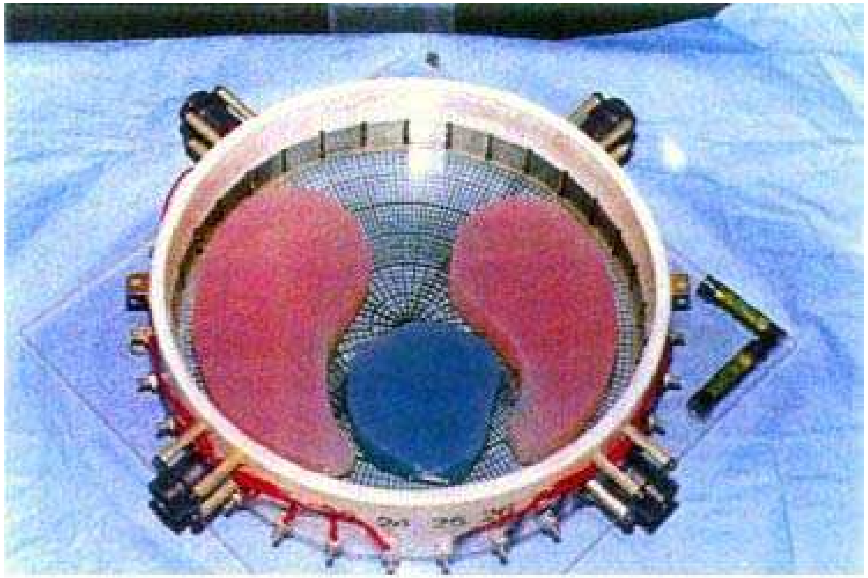

Figure A1. The two-dimensional phantom thorax with pink agar lungs, blue agar heart and black skin in saline. The electrodes are stainless steel, 2.54 x 2.54 cm. The resistivity of the heart is 150 ohm-cm, and that of the lungs is 1000 ohm-cm.

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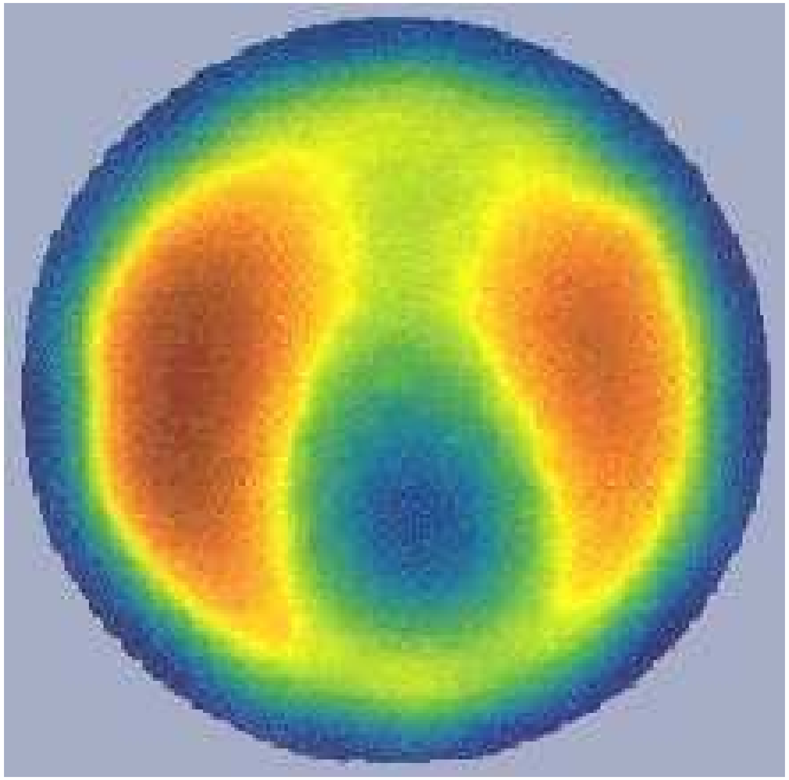
www.math.colostate.edu/˜mueller/index.html

EIT

Electrical Impedence Tomography (EIT), part 2



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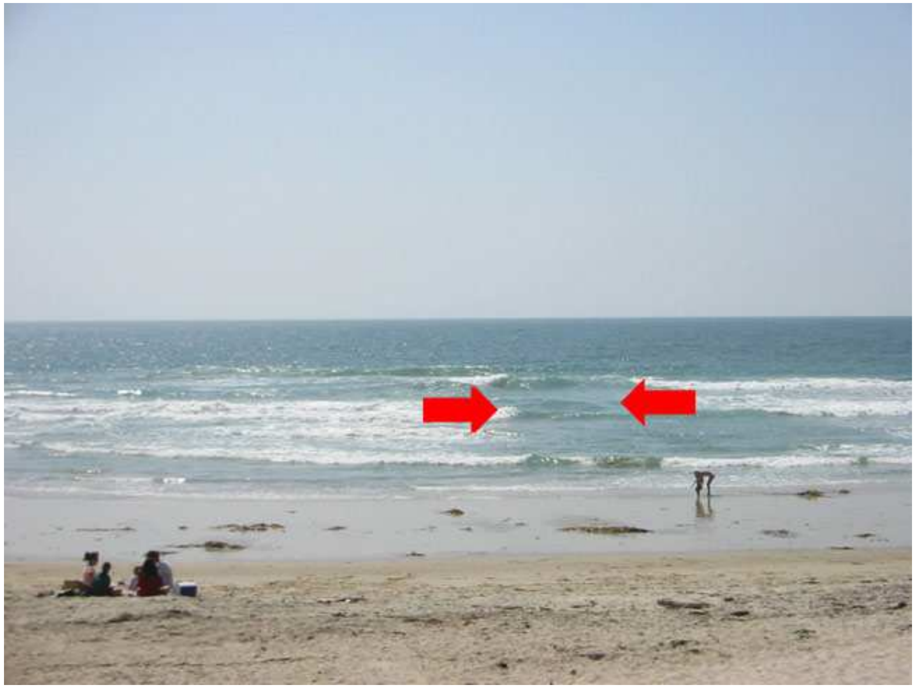
www.math.colostate.edu/˜mueller/index.html

Domain

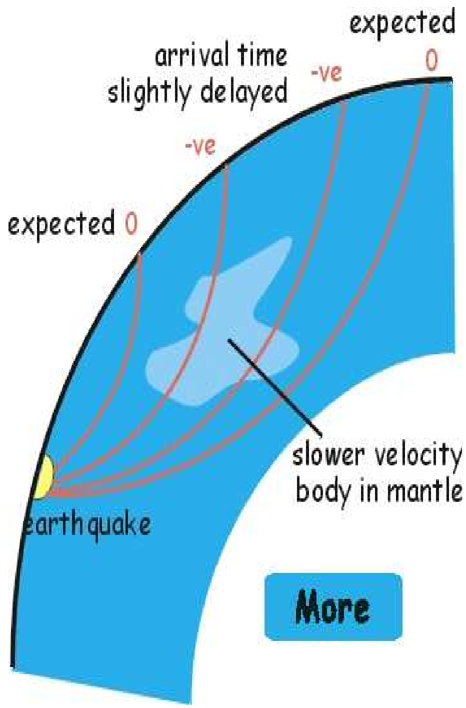
Domain Recovery



www.livescience.com/forcesofnature/050712 rip currents.html

Geophysics

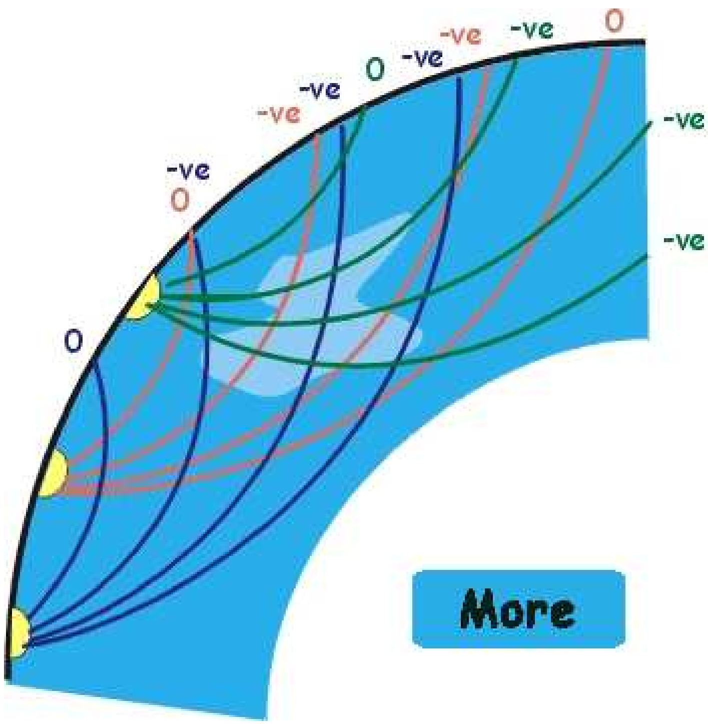
Earth Tomography



earth.leeds.ac.uk/dynamicearth/internal/tomography

Geophysics

Earth Tomography



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earth.leeds.ac.uk/dynamicearth/internal/tomography

Conclusions

- Many practical applications
- Even problems that seem simple can be pretty hard
- Existence is usually implied
- Uniqueness of solution harder
- Difficult to know even the dimension of the problem
- Or if it is well-posed

THANKS!!