Adjoint based parameter recovery

or: the beauty of Integration by Parts

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Acknowledgments

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Outline

- Introduction
- Why?
- Adjoint Theory
- Some numerics
- Conclusions

Apology

We'd like to study the Richards Equation:

$$C(h)\partial_t h = \partial_x(K(h)(\partial_x h - 1))$$

where $C(h)$ is soil capacity and $K(h)$ is hydraulic conductivity. But...

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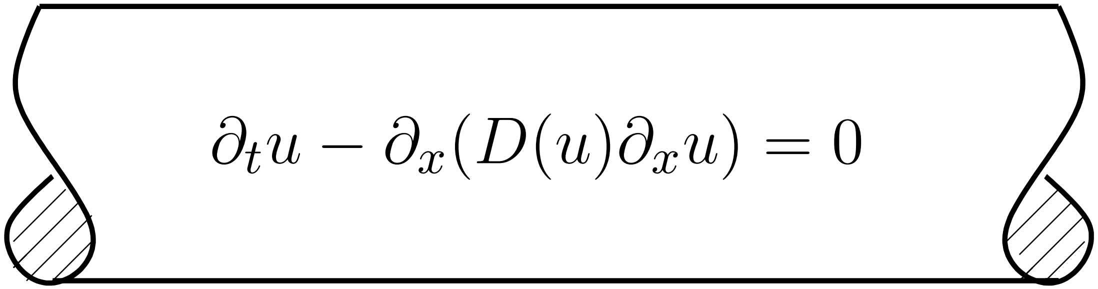
where $C(h)$ is soil capacity and $K(h)$ is hydraulic conductivity. But...
we'll consider the simpler problem

$$\partial_t u = \partial_x(D(u)\partial_x u),$$

which captures many details of the method.

Introduction

The physical set-up



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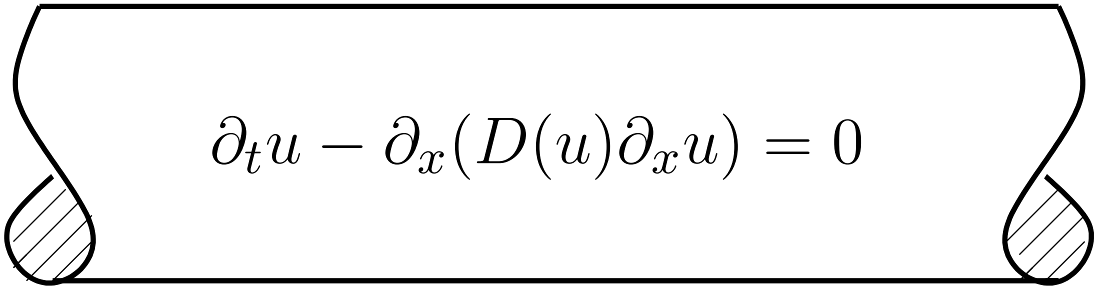

$$\partial_t u - \partial_x (D(u)\partial_x u) = 0$$

$$u(0,t) = f(t) \qquad\qquad \partial_x u(1,t) = 0$$

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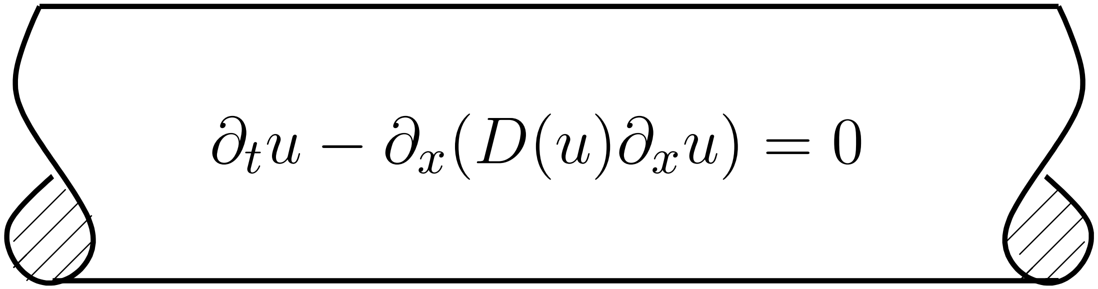

$u(0,t) = f(t)$ $\qquad\qquad$ $\partial_x u(1,t) = 0$

GOAL:

Given some output measurements of this system, recover $D(u)$.

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Let W represent this class of functions.

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- Why the toy problem?
- Why adjoint methods?

Forward

The (Forward) Problem: Given $D(u)$ and $f(t)$, find u which satisfies:

$$\begin{aligned}
\partial_t u - \partial_x(D(u)\partial_x u) &= 0 & (x,t) \in U_T \\
u(0,t) &= f(t) & t \in (0,T) \\
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Why?

- Unique weak (forward) solution if D and f from W.
- Model of a simple physical process.
- Measurements of flux at $x = 0$ and state at $x = 1$ possible.

Inverse

The (Inverse) Problem: Given boundary measurements $g(t)$ and $h(t)$, recover $D(u)$ which satisfies:

$$\begin{aligned} \partial_t u - \partial_x(D(u)\partial_x u) &= 0 && (x,t) \in U_T \\ -D(u)\partial_x u(0,t) &= g(t) && t \in (0,T) \\ u(1,t) &= h(t) && t \in (0,T) \\ u(x,0) &= f(0) && x \in (0,1) \end{aligned}$$

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Why?

- Experimental interest.
- Simple measurements.
- If we can recover a D from W, it should be unique.

Adjoint

Max Min Statement

- for each $t, f(0) < u(x,t) < f(t), 0 < x < 1$
- $\partial_x u(x,t) < 0$ a.e. on $(0,1) \times (0,T)$

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Comments:

If we let $B(u) = \int_0^u D(s)\ ds$, then $D(u)\partial_x u = \partial_x B(u)$.

Also, $B(u) = \int_0^u D(s)\ ds = (u - 0)\ D(\tilde{u})$ for some $\tilde{u}$ between u and 0.

Adjoint

Again, let $g(t) := -D(u(0,t))\partial_x u(0,t)$ and $\Phi[D,f] = g(t)$.

We can show that if $D_1(u) > D_2(u)$ then the map Φ is:

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- Continuous: $||g_1 - g_2||_{L^2(0,T)} \leq C||D_1 - D_2||_\infty$
- Injective : $\Phi[D_1,f] = \Phi[D_2,f]$ implies $D_1 = D_2$

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Adjoint

Integral Identity:
Let u, v represent solutions to direct problem with $D = D_1, D_2$, respectively. Subtract, multiply by smooth ϕ and integrate by parts to yield:

$$\iint_{U_T} (u-v)(\phi_t + k\phi_{xx}) - \int_0^1 (u-v)\phi\Big|_{t=0}^{t=T} +$$
$$+ \int_0^T (D_1(u)u_x - D_2(v)v_x)\phi\Big|_{x=0}^{x=1} + \int_0^T (D_1(u) - D_1(v))(k\phi_x)\Big|_{x=0}^{x=1}$$
$$= \iint_{U_T} (D_1(v) - D_2(v)v_x\phi_x$$

Here $U_T = (0,1) \times (0,T)$ and $k := D(\tilde{u})$ for some $\tilde{u}$ in the interval (v, u)

Adjoint

The general adjoint problem:

$$
\begin{aligned}
\partial_t \phi + k \partial_{xx} \phi = &\quad F^*(x,t) && (x,t) \in U_T \\
\phi(x,T) = &\quad p^*(x) && x \in (0,1) \\
k\ \phi_x = &\quad g^*(t) && x = 0, t \in (0,T). \\
\phi = &\quad h^*(t) && x = 1, t \in (0,T)
\end{aligned}
$$

Here $k := D(\tilde{u})$ for some $\tilde{u}$ in the interval (v, u)

Adjoint

The dual data acts like switches. Data appears in a duality pairing with terms of the direct problem.

Taking $F^* = 0, p^* = 0, h^* = 0$, the identity collapses to

$$\int_0^T g^*(g_1 - g_2) = \iint_{U_T} (D_1(v) - D_2(v)) v_x \phi_x$$

Adjoint

Letting $\Phi[D, f] = g(t)$, we could cast this formally as:

$$
\begin{aligned}
(\Phi[D_1, f] - \Phi[D_2, f], \theta)_{L^2} &:= (\delta\Phi[D_1, D_2]\Delta D, \theta)_{L^2} \\
&= \langle \Delta D, {}^t\delta\Phi[D_1, D_2]\ \theta \rangle_{W \times W^*}
\end{aligned}
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The integral identity provides a realization of this expression.

$$\int_0^T g^*(g_1 - g_2) = \iint_{U_T} (D_1(v) - D_2(v))v_x\phi_x$$

Adjoint

Now some quick proofs:

$$\int_0^T g^*(g_1 - g_2)dt = \iint_{U_T} (D_1(v) - D_2(v))v_x \phi_x dx\ dt$$

- Monotone: Assume $D_1 > D_2$. We know $v_x < 0$. Taking $g^* > 0$ implies $\phi_x < 0$. So $g_1(t) > g_2(t)$.

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- Continuous: Take $g^* = (g_1 - g_2)/||g_1 - g_2||_{L^2}$.

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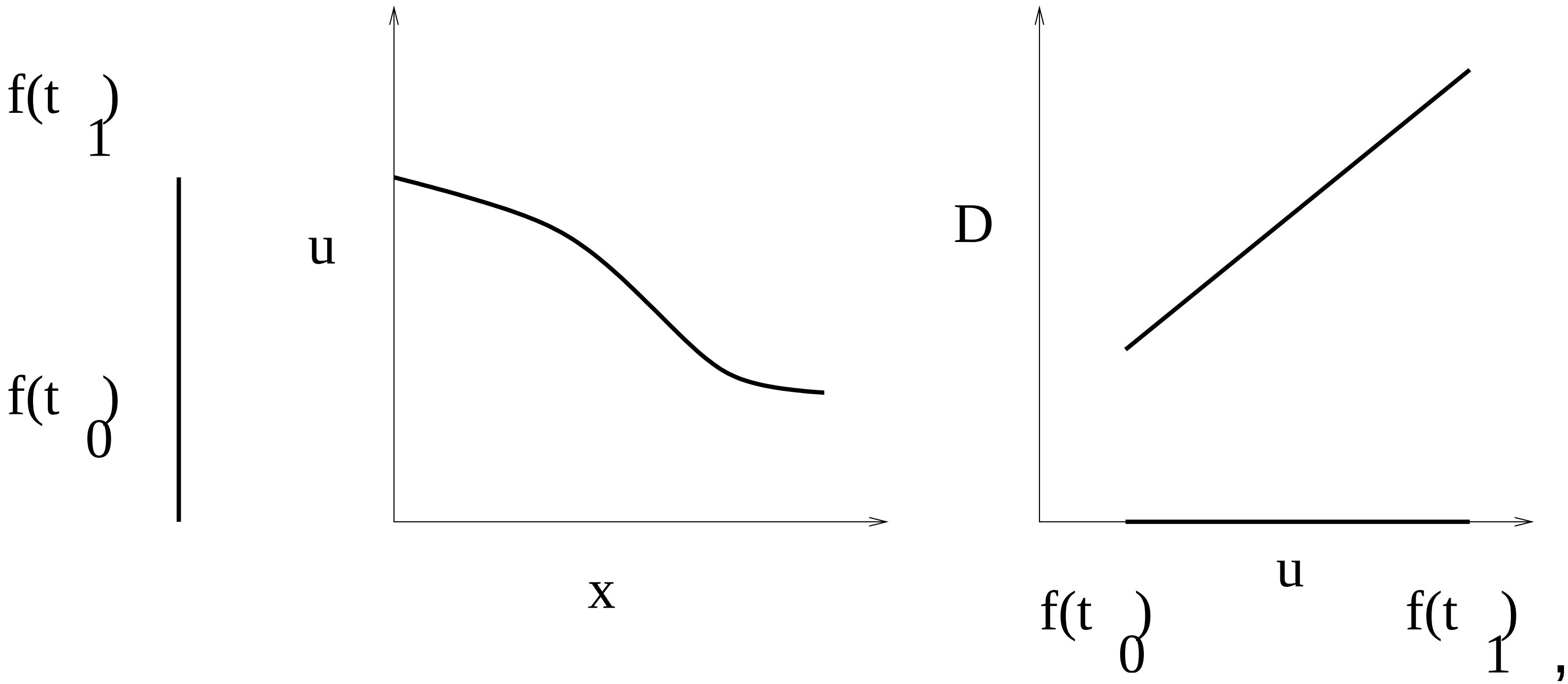
- Monotone: Assume $D_1 > D_2$. We know $v_x < 0$. Taking $g^* > 0$ implies $\phi_x < 0$. So $g_1(t) > g_2(t)$.
- Continuous: Take $g^* = (g_1 - g_2)/||g_1 - g_2||_{L^2}$.
- Injective: Assume $D_1 - D_2 \neq 0$. This implies that $g_1 - g_2 \neq 0$.

Numerics

Since



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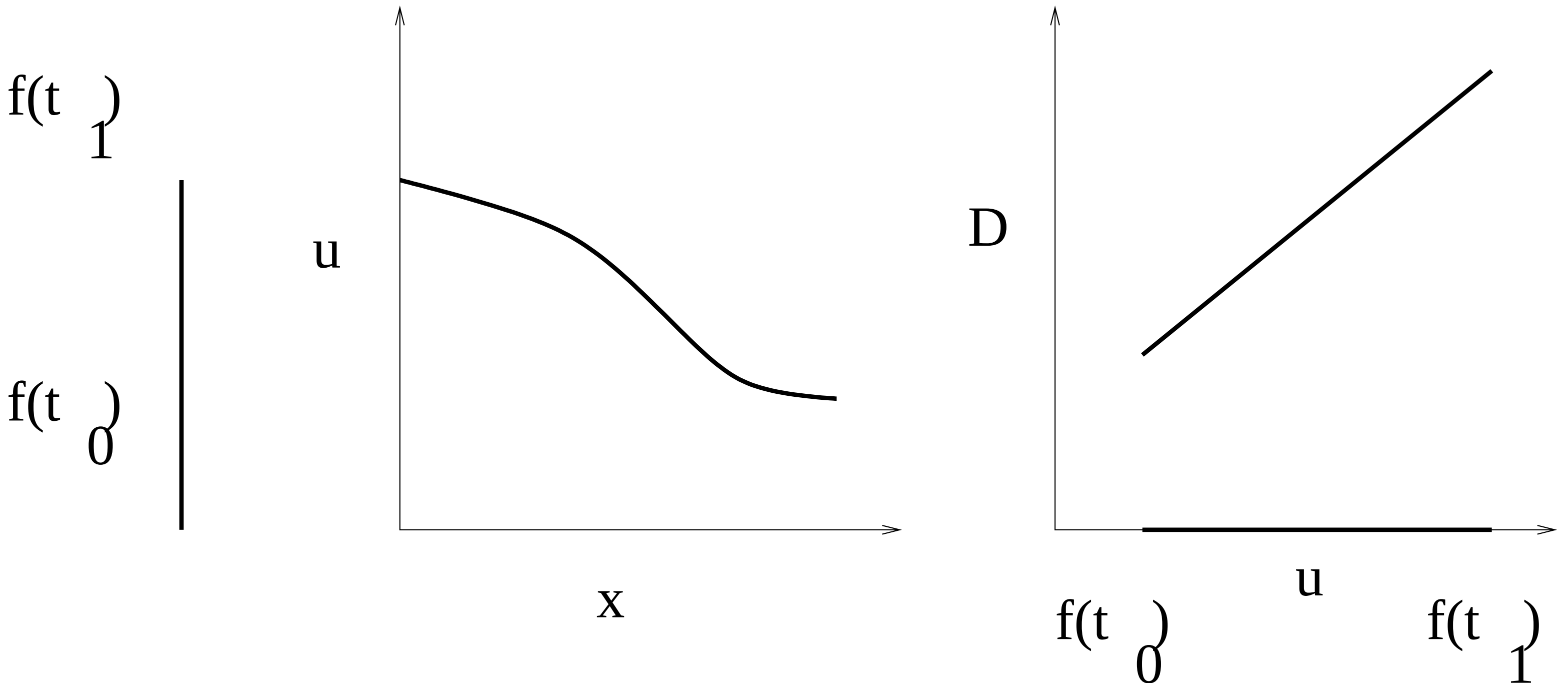

,

and so $[t_0, t_1] \implies [u_0, u_1]$. If

$$\hat{D}(u) = \sum_{k=0}^{N} d_k \Lambda_k,$$

Numerics

The integral ID,

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$$\int_0^T g^*(g_1 - g_2)dt = (d_1 - \delta_1) \iint_{U_t} \Lambda u_x\phi_x dx\ dt$$

or

Numerics

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becomes

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or

$$(d_1 - \delta_1) = \frac{\int_0^T g^*(g_1 - g_2)dt}{\iint_{U_t} \Lambda u_x \phi_x dx\ dt}$$

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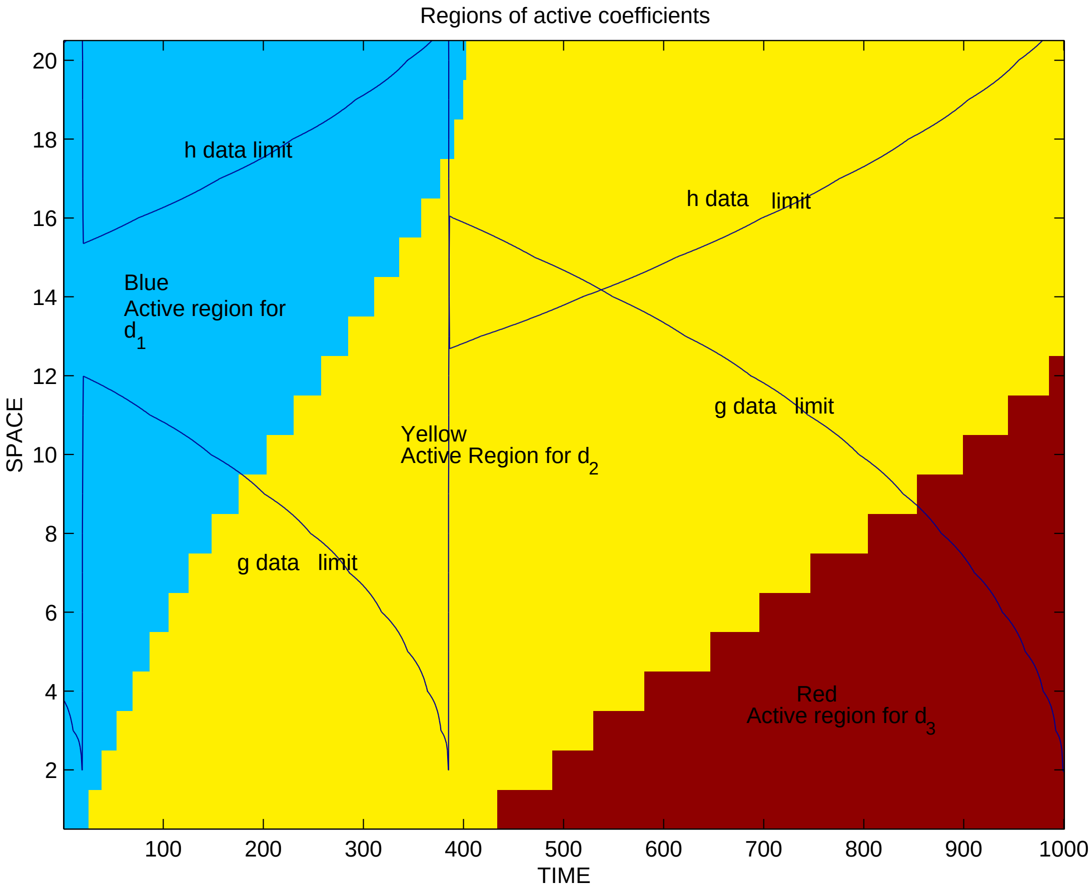
- Given $g^*(t)$ and k, solve adjoint problem to generate $\phi(x,t)$
- Apply integral ID to compute update for D.
- Repeat

Numerics

Regions of coefficient activity



Numerics

Apply identity to region where only d_1 is active.

$$b_1 = \int_0^{t_1} g^* \Delta g = (d_1 - \delta_1) \iint_{R_{11}} \Lambda_1(u) u_x \phi_x = A_{11} \Delta d_1$$

where $R_{11} = \{(x,t) \in U_{t_1} : u_0 < u(x,t) < u_1\}$

Numerics

Apply identity to the U_{t_2} time strip. Only d_1 and d_2 are active.

$$
\begin{aligned}
b_2 := \int_{t_1}^{t_2} g^* \Delta g = (d_2 - \delta_2) \iint_{R_{22}} \Lambda_2(u) u_x \phi_x + \\
+ (d_1 - \delta_1) \iint_{R_{21}} \Lambda_1(u) u_x \phi_x \\
= A_{22} \Delta d_2 + A_{21} \Delta d_1
\end{aligned}
$$

with

$$
\begin{aligned}
R_{21} &= \{(x,t) \in U_{t_2} : u_0 < u < u_1\} \\
R_{22} &= \{(x,t) \in U_{t_2} : u_1 < u < u_2\}
\end{aligned}
$$

Numerics

By the Q_{t_N} strip, a diagonal system forms.

$$\begin{bmatrix} A_{11} & 0 & \cdots & 0 \\ A_{21} & A_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ A_{N1} & \cdots & \cdots & A_{NN} \end{bmatrix} \begin{bmatrix} \Delta d_1 \\ \Delta d_2 \\ \vdots \\ \Delta d_N \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_N \end{bmatrix}$$

where $A_{ij} = \iint_{R_{ij}} \Lambda_j(u) u_x \phi_x$ and $b_i = \int_{t_{i-1}}^{t_i} g^* \Delta g$

Numerics

Now for some demos:

- Is some data better?
- Noise?
- ????

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