Stability Analysis via Hill's method

or: Can we find stable wave forms?

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- *Bernard Deconinck* (UW)
- and soon ________________???

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Introduction

We'll talk about computing the spectra of linear operators, including the associated eigenfunctions.

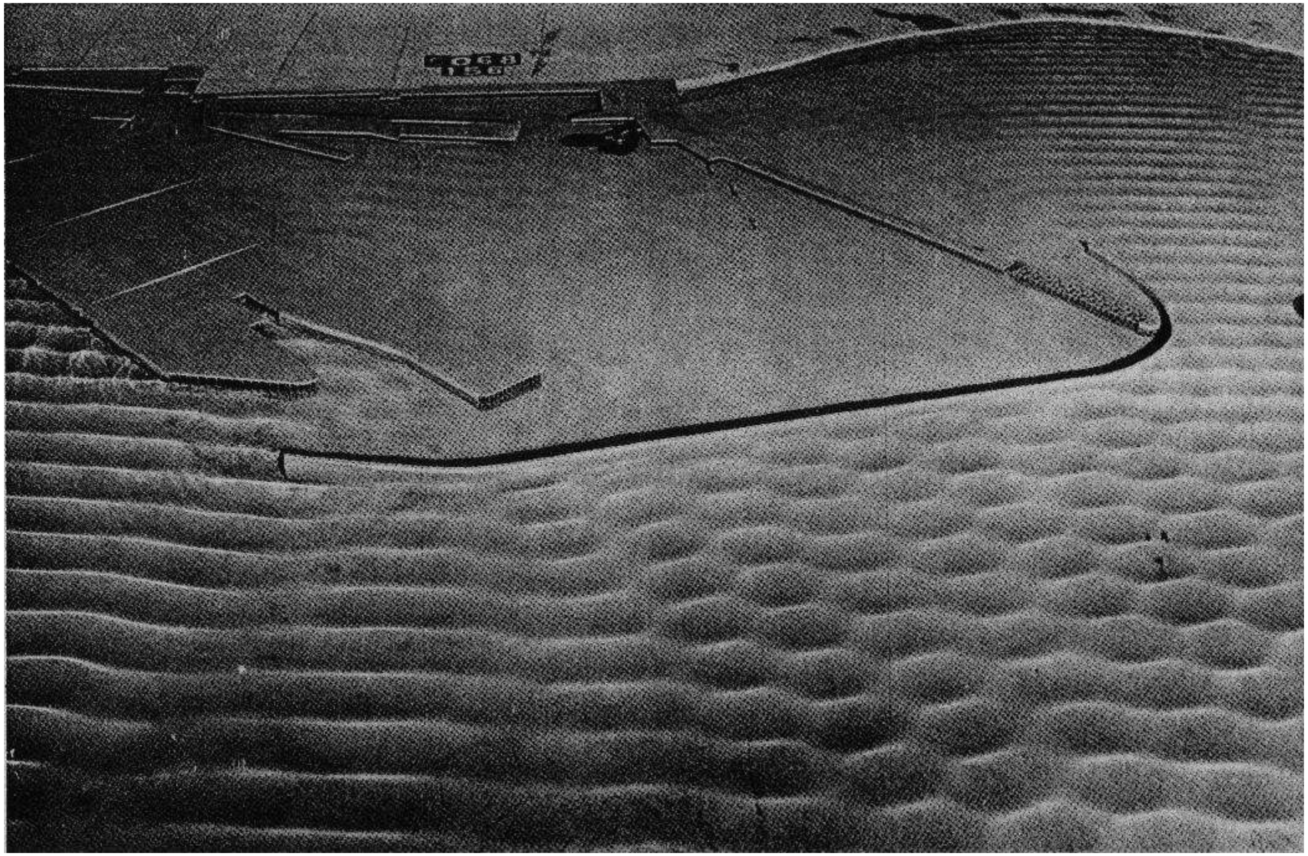
Why is this important?

Spectral Stability: Given an equilibrium solution, is it stable under perturbation? If so, it is a candidate for a solution of permanent form.

Patterns in waves

February, 2003 at Maalea Bay on Maui. Courtesy of Robert I. Odom, Applied Physics Laboratory, University of Washinton

More patterns



CERC harbor model (Source unknown)

Spectral Stability

Consider the evolution system

$$u_t = N(u)$$

with an equilibrium solution u_e:

$$N(u_e) = 0.$$

Is this solution *stable* or *unstable*?
Linear analysis: let

$$u = u_e + \epsilon\psi.$$

Substitute in and retain first-order terms in ϵ:

$$\psi_t = \mathcal{L}[u_e(x)]\psi.$$

Eigenfunction expansion

Separation of variables: $\psi(x,t) = e^{\lambda t} z(x)$:

$$\mathcal{L}[u_e(x)]z = \lambda z.$$

- This is a spectral problem.
- If $\Re(\lambda) \leq 0$ for all bounded $z(x)$, then u_e is spectrally stable.

Application

Our starting point is

$$\mathcal{L}z = \lambda z,$$

with

$$\mathcal{L} = \sum_{k=0}^{M} f_k(x)\partial_x^k, \qquad f_k(x+L) = f_k(x).$$

We want to find

- Spectrum $\sigma(\mathcal{L}) = \{\lambda \in \mathbb{C} : ||z||_\infty < \infty\}$.
- Corresponding eigenfunctions $z(\lambda, x)$?

Floquet's Theorem

Consider

$$\varphi_x = A(x)\varphi, \quad A(x+L) = A(x). \qquad (*)$$

Floquet's theorem states that the fundamental matrix Φ for this system has the decomposition

$$\Phi(x) = P(x)e^{Rx},$$

with $P(x+L) = P(x)$ and R constant.

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$$\Phi(x) = P(x)e^{Rx},$$

with $P(x+L) = P(x)$ and R constant.
Conclusion: **All bounded solutions of ($*$) are of the form**

$$\varphi = e^{i\mu x} \sum_{n=-\infty}^{\infty} \hat{\varphi}_n \, e^{i2\pi n x/L},$$

with $\mu \in [0, 2\pi/L)$.

Eigenfunctions

The periodic eigenfunctions can be expanded as

$$\varphi = e^{i\mu x} \sum_{n=-\infty}^{\infty} \hat{\varphi}_n \, e^{i\pi n x/L},$$

with $\mu \in [0, \pi/L)$

Substitute in the equation and cancel $e^{i\mu x}$.

The Floquet parameter μ only appears in derivative terms.

Hill's method

- Find Fourier coefficients of all functions
- Choose a number of μ values $\mu_1, \mu_2, \ldots$
- For all chosen μ values, construct $\hat{\mathcal{L}}_N(\mu)$
- Use favorite eigenvalue/vector solver
- Reconstruct eigenfunctions corresponding to eigenvalues

NLS

I've been looking at solutions of the 2-D cubic nonlinear Schrödinger (NLS) equation, given by

$$i\phi_t + \alpha\phi_{xx} + \beta\phi_{yy} + \gamma\left|\phi\right|^2\phi = 0.$$

The NLS equation arises in many models:

- Bose-Einstein condensates ($\alpha\beta > 0$)
- Deep water models ($\alpha\beta < 0$)
- Optics ($\alpha\beta > 0$)

NLS

Consider

$$i\psi_t + \alpha\psi_{xx} + \beta\psi_{yy} + |\psi|^2\psi = 0.$$

This equation has exact 1-D traveling wave solutions of the form

$$\psi(x,t) = \phi(x)e^{i\omega t + i\theta(x)},$$

where ϕ and θ are real-valued functions and ω is a real constant.

- If $\theta(x) =$ constant then the solution has *trivial phase* (TP).
- if $\theta(x) \neq$ constant the solution has *nontrivial phase* (NTP).

NLS

More precisely,

$$\psi(x,t) = \phi(x)e^{i\omega t + i\theta(x)},$$

where

$$\phi^2(x) = \alpha\left(-2k^2\text{sn}^2(x,k) + B\right),$$

$$\theta(x) = c\int_0^x \phi^{-2}(\tau)d\tau,$$

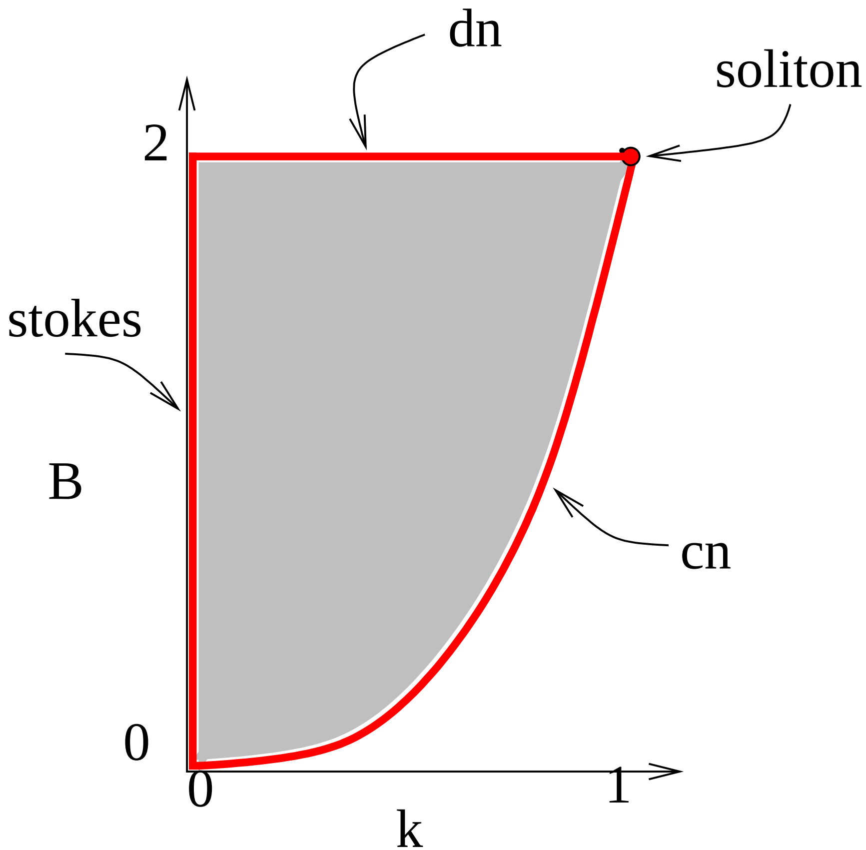
$$\omega = \frac{1}{2}\alpha(3B - 2(1+k^2)), \qquad \text{and}$$

$$c^2 = -\frac{\alpha^2}{2}B(B-2k^2)(B-2).$$

k and B are free parameters and sn(x,k) is the Jacobi elliptic sine function.

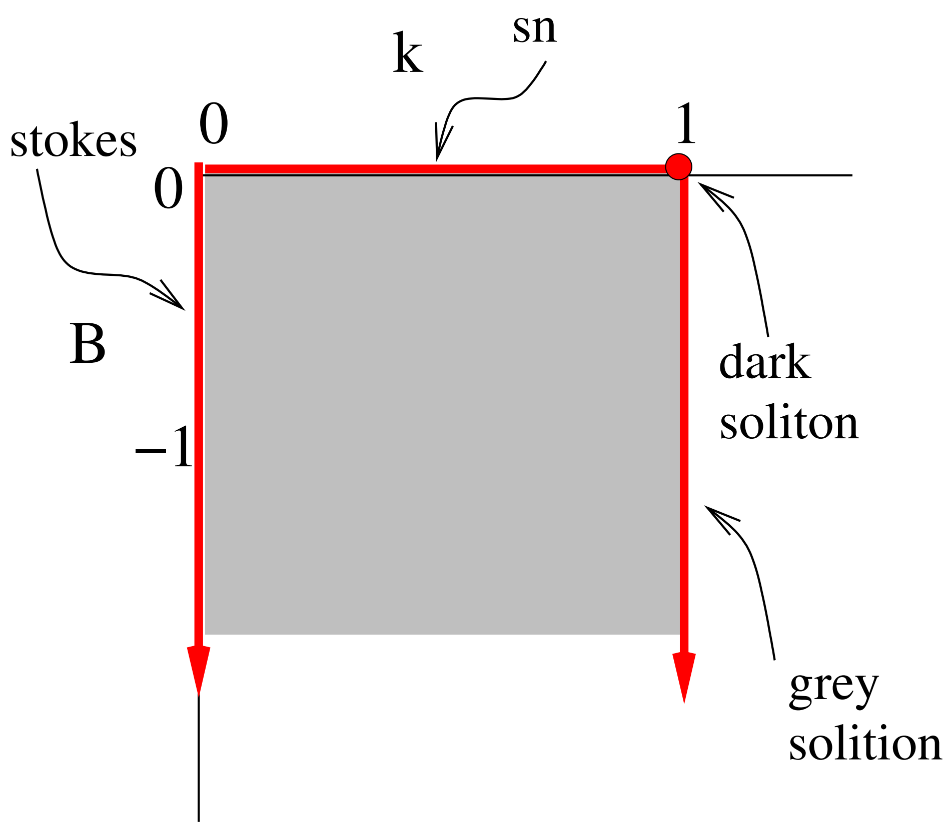
Focusing

NLS is *focusing* or *attractive* in the x dimension if $\alpha > 0$. To make ϕ real in this case, we choose B in $[2k^2, 2]$.



Defocusing

NLS is *defocusing* or *repulsive* if $\alpha < 0$. To make ϕ real in this case, we choose $B \leq 0$.



Linearized TP spectral problem

Now consider the (modulus and phase) perturbed TP solution of the form

$$\psi_p = (\phi + \epsilon u + i\epsilon v)e^{i\omega t}$$

Linearizing and considering real and imaginary contributions yields the system

$$\begin{aligned} \omega u - 3\phi^2 u - \beta u_{yy} - \alpha u_{xx} &= -v_t \\ \omega v - \phi^2 v - \beta v_{yy} - \alpha v_{xx} &= u_t \end{aligned}$$

Linearized TP spectral problem

Let $u(x, y, t) = U(x)e^{i\rho y + \lambda t}$
and $v(x, y, t) = V(x)e^{i\rho y + \lambda t}$.

Then

$$\begin{aligned} \omega u - 3\phi^2 u - \beta u_{yy} - \alpha u_{xx} &= -v_t \\ \omega v - \phi^2 v - \beta v_{yy} - \alpha v_{xx} &= u_t \end{aligned}$$

becomes

$$\begin{aligned} \omega U - 3\phi^2 U + \beta\rho^2 U - \alpha U_{xx} &= -\lambda V \\ \omega V - \phi^2 V + \beta\rho^2 V - \alpha V_{xx} &= \lambda U \end{aligned}$$

Linearized TP spectral problem

We write

$$\begin{aligned} \omega U - 3\phi^2 U + \beta\rho^2 U - \alpha U_{xx} &= -\lambda V \\ \omega V - \phi^2 V + \beta\rho^2 V - \alpha V_{xx} &= \lambda U \end{aligned}$$

as

$$\mathcal{L}\begin{bmatrix} U \\ V \end{bmatrix} := \begin{bmatrix} 0 & L_- \\ -L_+ & 0 \end{bmatrix}\begin{bmatrix} U \\ V \end{bmatrix} = \lambda \begin{bmatrix} U \\ V \end{bmatrix}$$

where

$$L_+ = \omega - 3\phi^2 + \beta\rho^2 - \partial_{xx}$$

and

$$L_- = \omega - \phi^2 + \beta\rho^2 - \partial_{xx}$$

This is our linearized TP spectral problem. The coefficients are periodic.

Spectral Stability

We can finally consider the spectral stability of the periodic coefficient linear problem

$$\mathcal{L}\begin{bmatrix} U \\ V \end{bmatrix} := \begin{bmatrix} 0 & L_- \\ -L_+ & 0 \end{bmatrix}\begin{bmatrix} U \\ V \end{bmatrix} = \lambda \begin{bmatrix} U \\ V \end{bmatrix}$$

where

$$L_+ = \omega - 3\phi^2 + \beta\rho^2 - \partial_{xx}$$

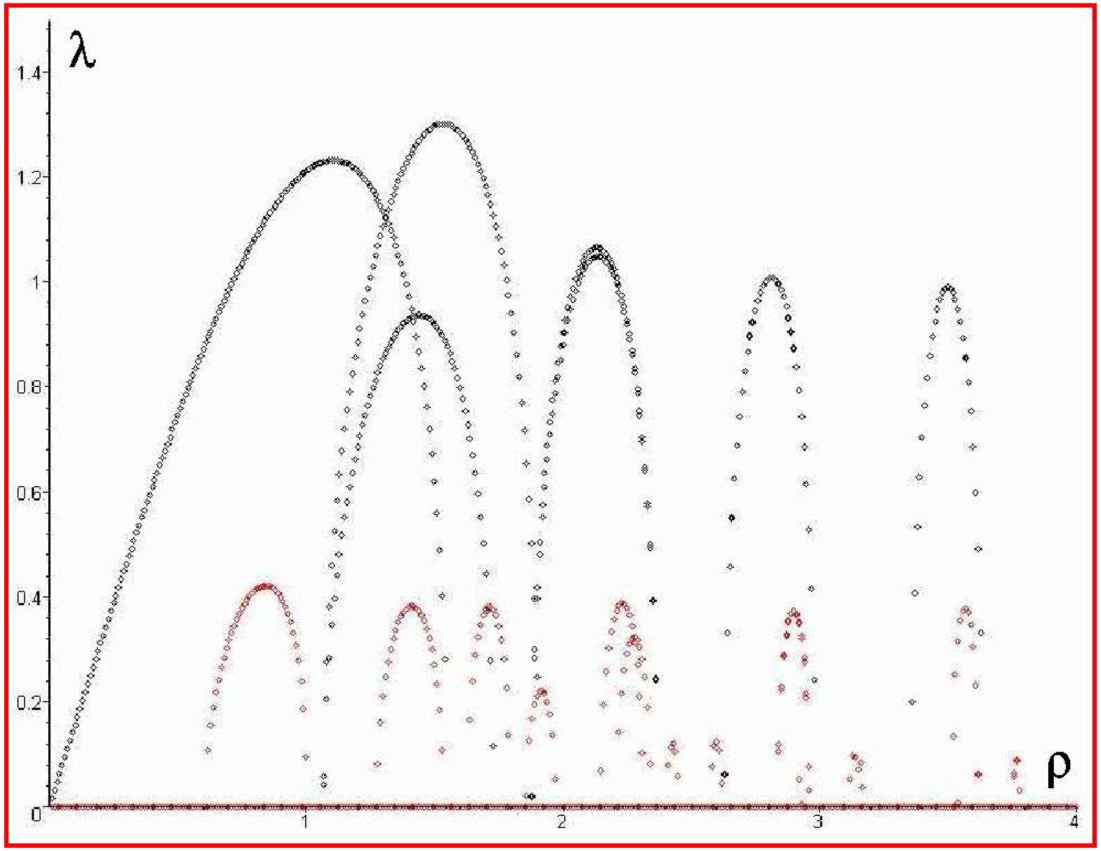
and

$$L_- = \omega - \phi^2 + \beta\rho^2 - \partial_{xx}$$

We now build the matrix $\hat{L}_-(\mu)$. The same method generates $\hat{L}_+(\mu)$.

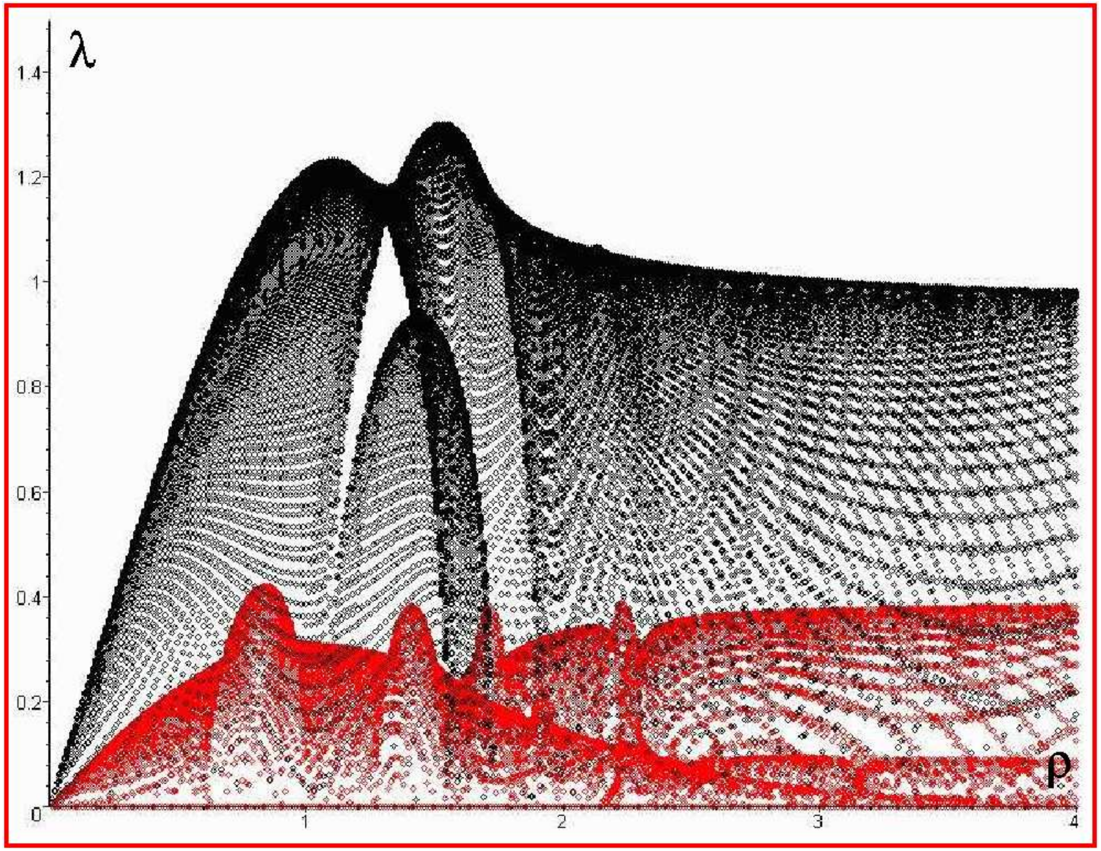
SN plus

In the literature, you *might* find graphs for spectra associated to periodically perturbed TP solutions.



SN plus

But now we can compute “all” unstable modes.



Linearized NTP spectral problem

Now consider the (modulus and phase) perturbed NTP solution of the form

$$\psi_p = (\phi + \epsilon u + i\epsilon v)e^{i\lambda t + i\theta}$$

As in the TP case, we linearize, separate variables and introduce transverse perturbation to generate the system

$$\begin{aligned}(\omega - 3\gamma\phi^2 + \beta\rho^2)U \quad - \alpha U_{xx} &= -\lambda V \\ (\omega - \gamma\phi^2 + \beta\rho^2)V \quad - \alpha V_{xx} &= \lambda U\end{aligned}$$

What does this spectrum look like?

Linearized NTP spectral problem

Now consider the (modulus and phase) perturbed NTP solution of the form

$$\psi_p = (\phi + \epsilon u + i\epsilon v)e^{i\lambda t + i\theta}$$

As in the TP case, we linearize, separate variables and introduce transverse perturbation to generate the system

$$(\omega - 3\gamma\phi^2 + \beta\rho^2)U + \alpha c^2\phi^{-4}U + \alpha c\phi_x^{-2}V + 2\alpha c\phi^{-2}V_x - \alpha U_{xx} = -\lambda V$$
$$(\omega - \gamma\phi^2 + \beta\rho^2)V + \alpha c^2\phi^{-4}V - \alpha c\phi_x^{-2}U - 2\alpha c\phi^{-2}U_x - \alpha V_{xx} = \lambda U$$

What does this spectrum look like?

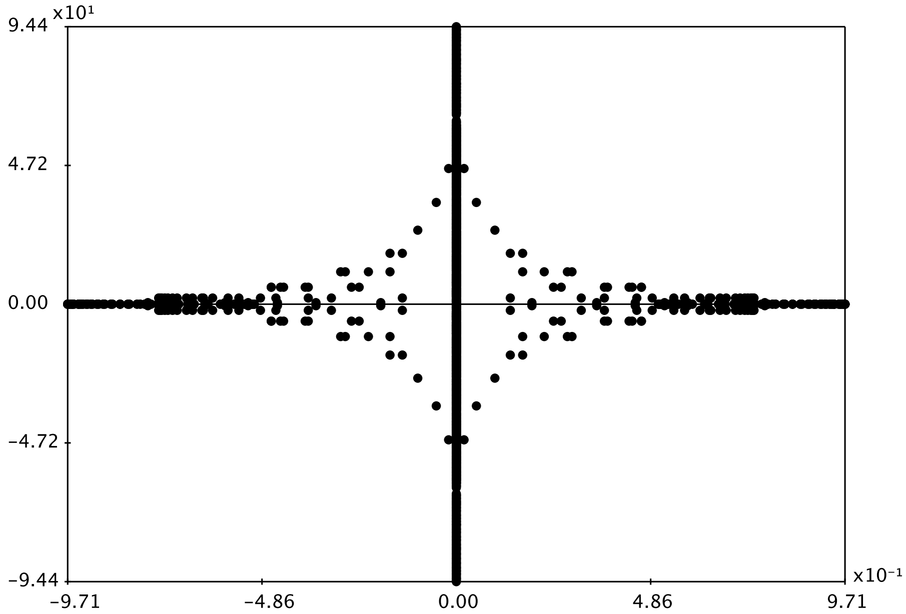
Linearized NTP spectral problem

There are a lot of parameters: for each pair $(\pm\alpha, \pm\beta)$, we can pick

- k elliptic modulus)
- B offset, (constrained by k)
- ρ wavenumber of perturbation
- μ Floquet parameter

Linearized NTP spectral problem

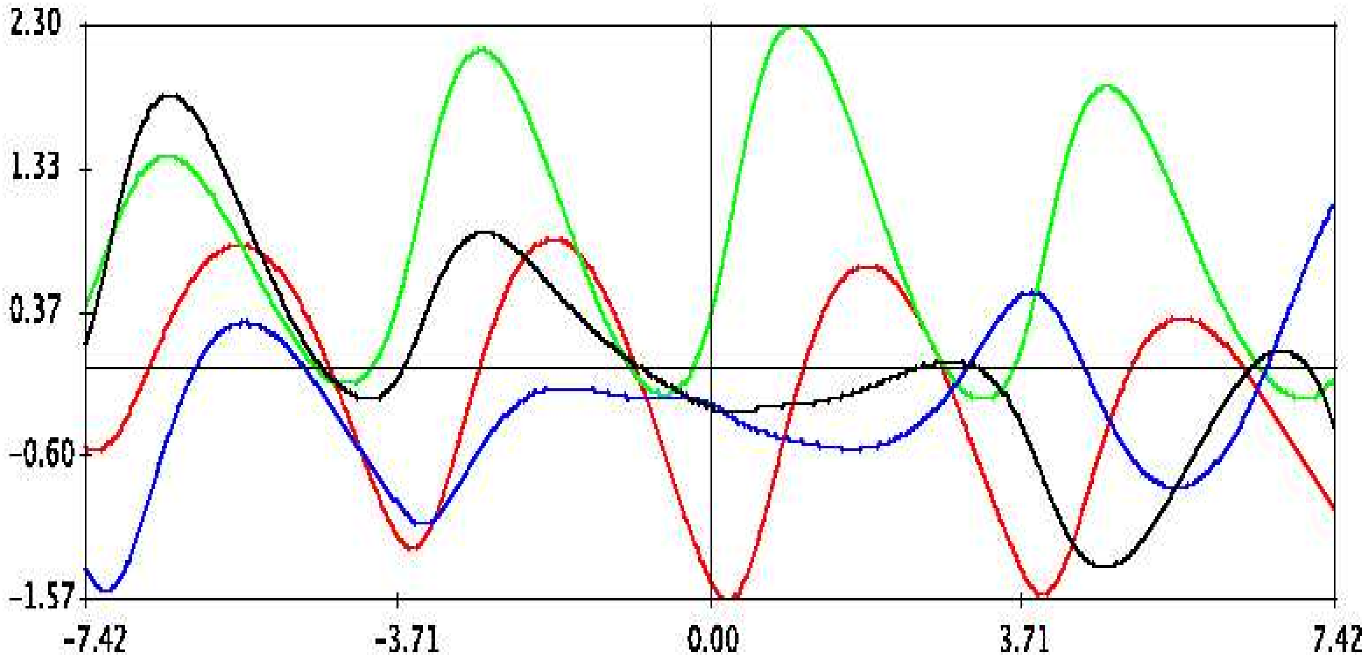
For $\alpha = -\beta = 1$ the spectrum might look like:



for $k = 0.5, B = 1$ and $\rho = 0$.

Linearized NTP spectral problem

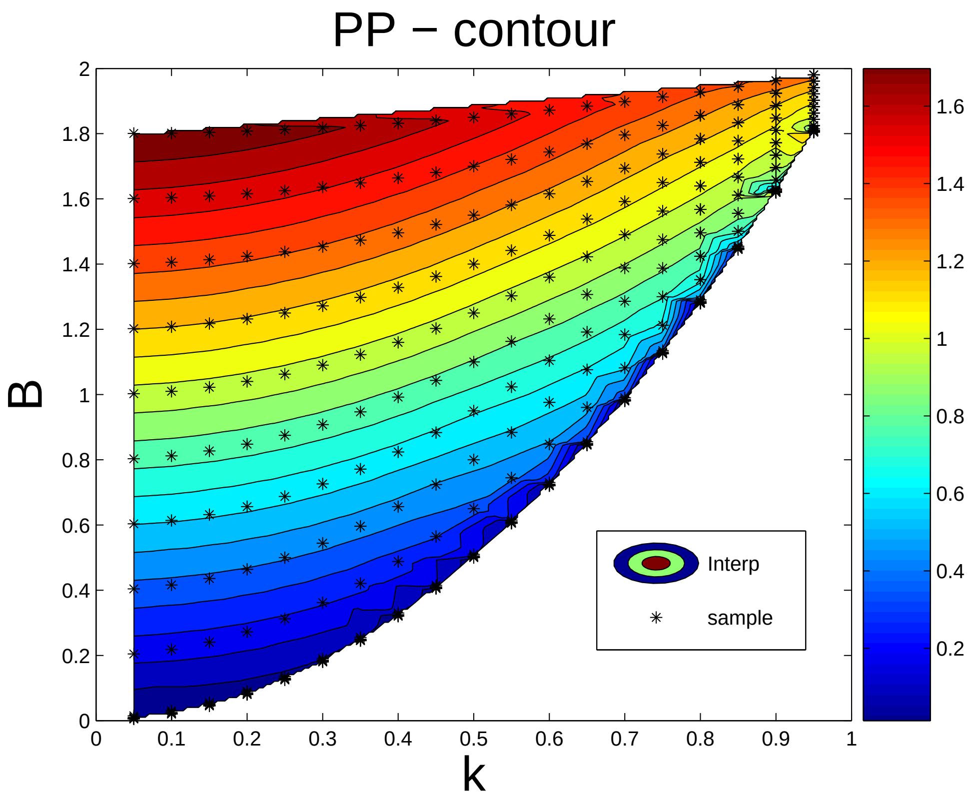
and the eigenfunction corresponding to dominate eigenvalue:



where red = $\Re(U)$, blue = $\Im(U)$, green = $\Re(V)$ and black = $\Im(V)$.

Linearized NTP spectral problem

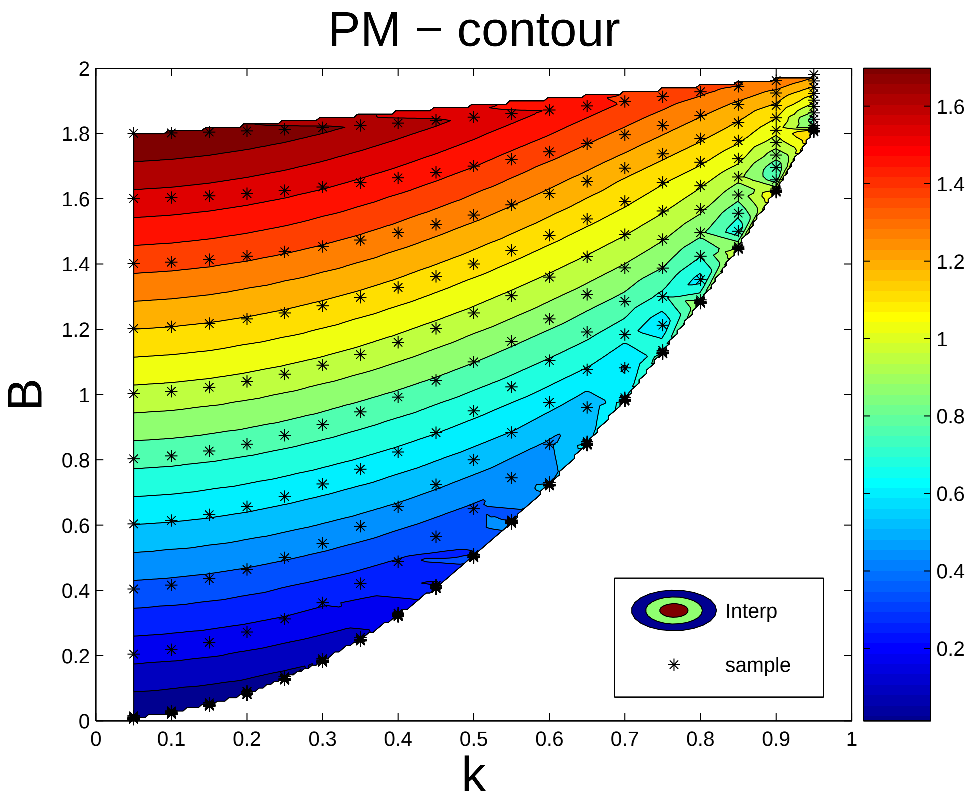
$\alpha = \beta = 1$



$(k, B) \in (0, 1) \times (2k^2, 2)$ for $\rho \in [0, 3]$.

Linearized NTP spectral problem

$\alpha = -\beta = 1$



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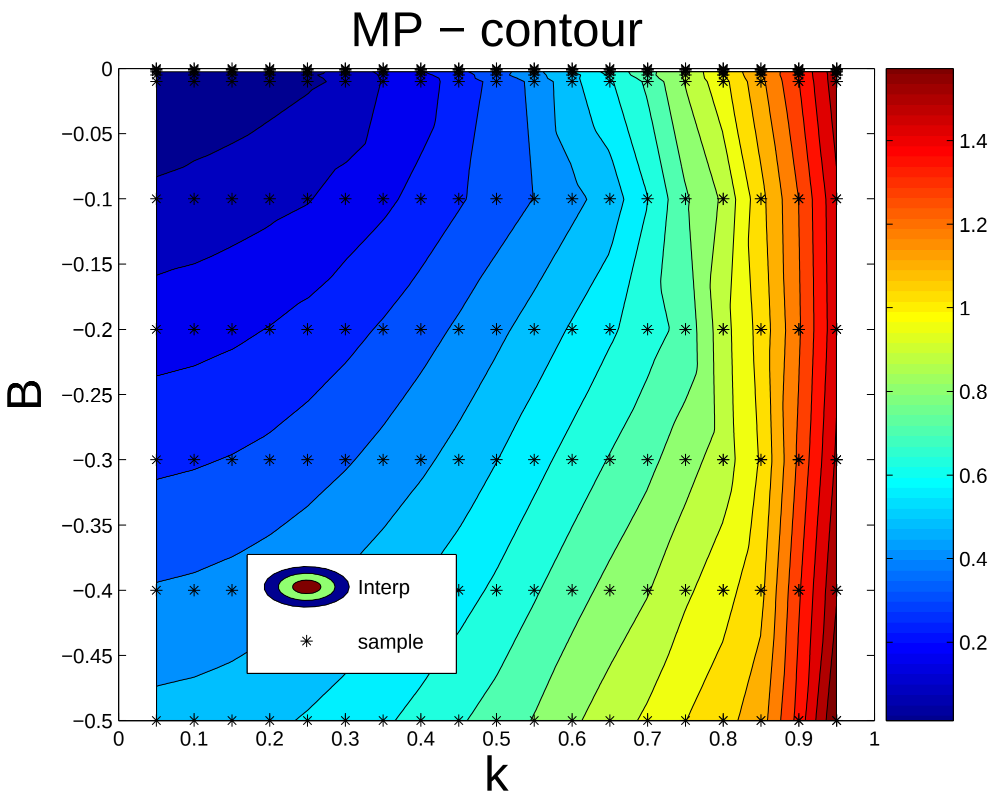
Linearized NTP spectral problem

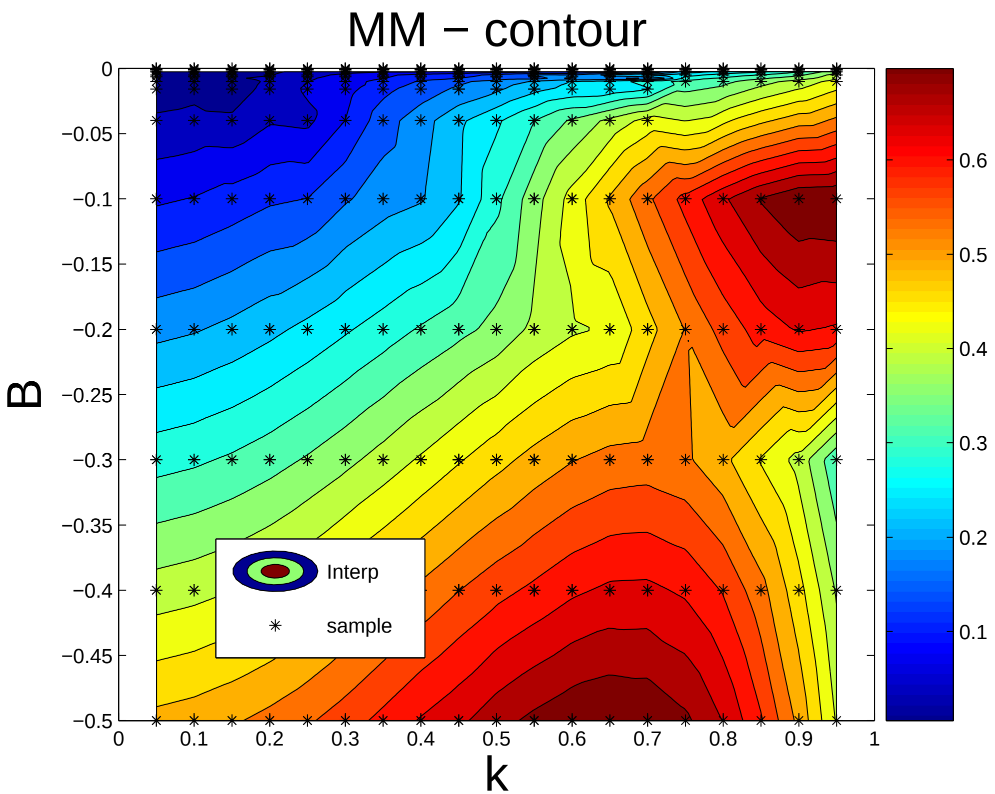
$-\alpha = \beta = 1$



$(k, B) \in (0, 1) \times [-0.5, 0]$ for $\rho \in [0, 3]$.

Linearized NTP spectral problem

$-\alpha = -\beta = 1$



$(k, B) \in (0, 1) \times [-0.5, 0]$ for $\rho \in [0, 3]$.

Conclusions

Samples suggest that:

All NTP solutions to the cubic NLS equation are unstable with respect to perturbation.

Hill's method:

- Allows non-periodic eigenfunctions
- Simple to implement

Thanks!!!