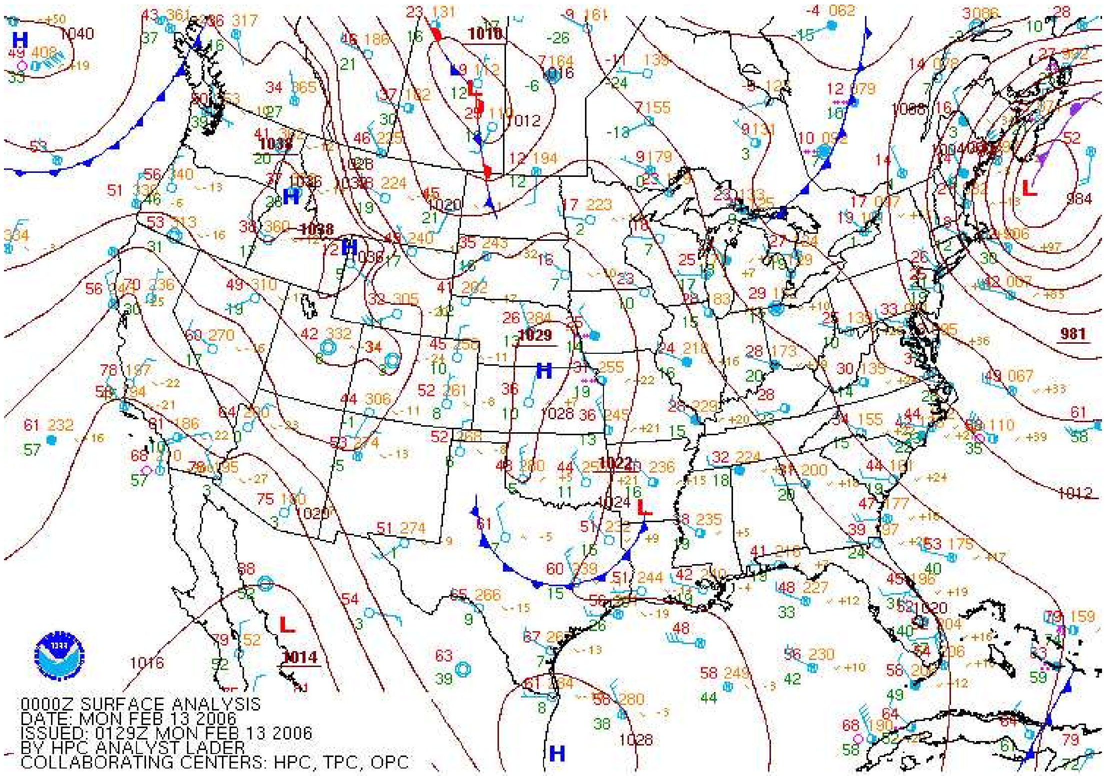
Trailers

I’ve worked in several areas:

- Atmospheric Science
- Porous Media
- Multiphysics
- Nonlinear Waves

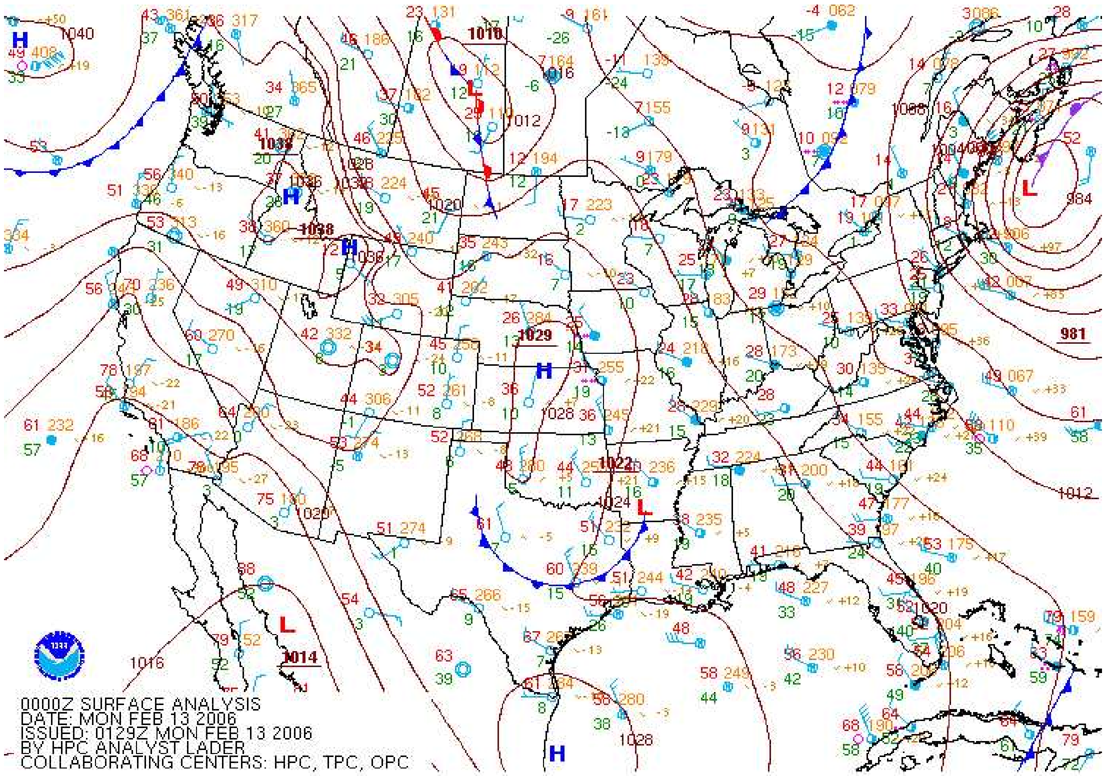
Atmospheric Science

(Cyclonic) Flows



Atmospheric Science

(Cyclonic) Flows

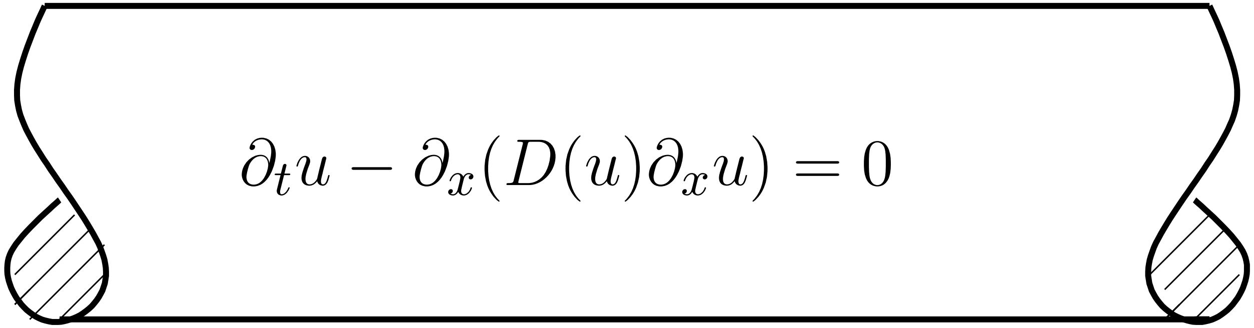


GOAL:

Recover wind velocity from pressure gradient.

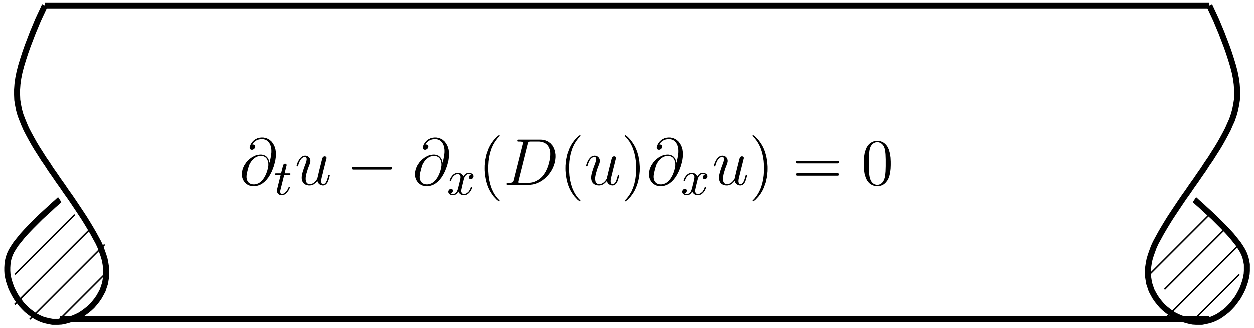
Porous Media

Parameter discovery



Porous Media

Parameter discovery

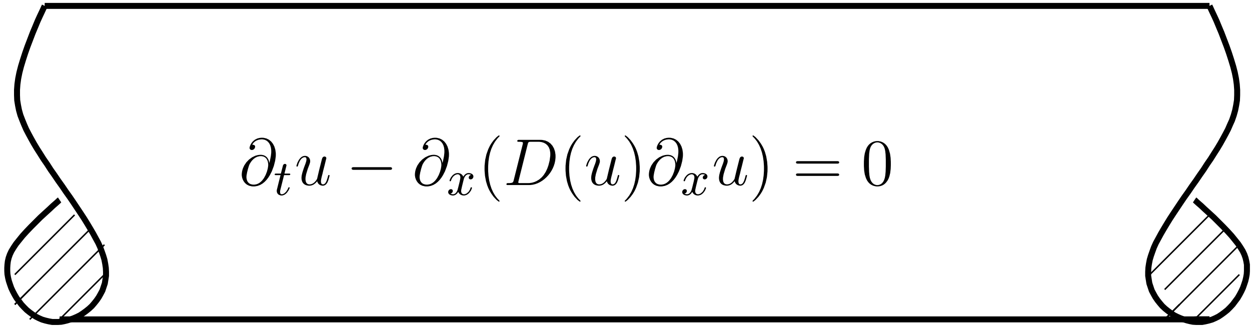


$$\partial_t u - \partial_x (D(u)\partial_x u) = 0$$

$$u(0,t) = f(t) \qquad \qquad \partial_x u(1,t) = 0$$

Porous Media

Parameter discovery



$$\partial_t u - \partial_x (D(u) \partial_x u) = 0$$

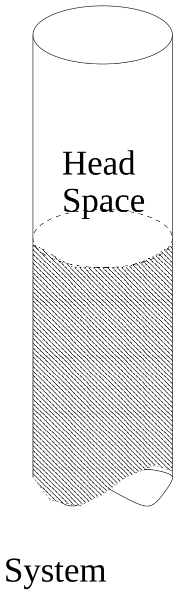
$$u(0,t) = f(t) \qquad \qquad \partial_x u(1,t) = 0$$

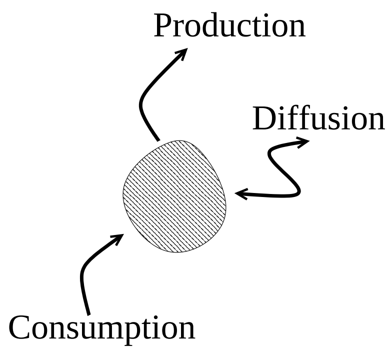
GOAL:

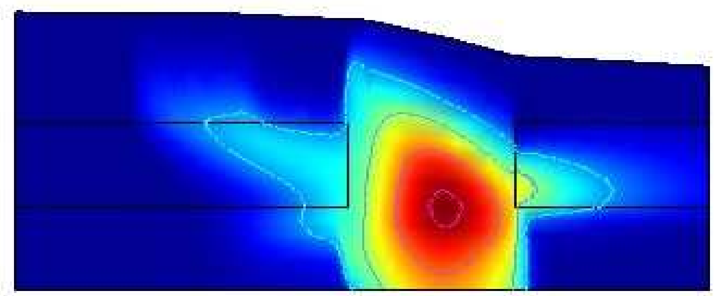
Given some output measurements of this system, recover $D(u)$.

Porous Media

Methane

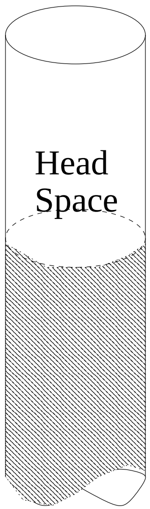




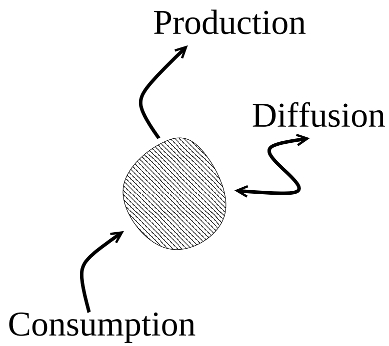


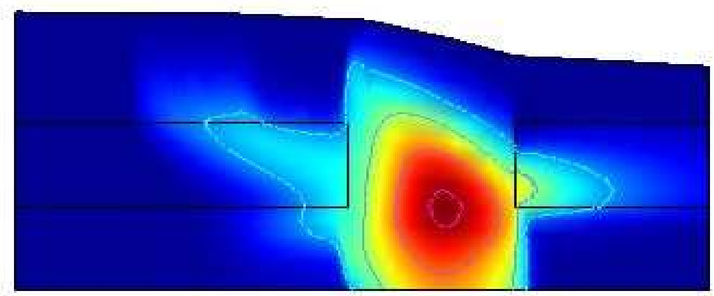
Porous Media

Methane



System



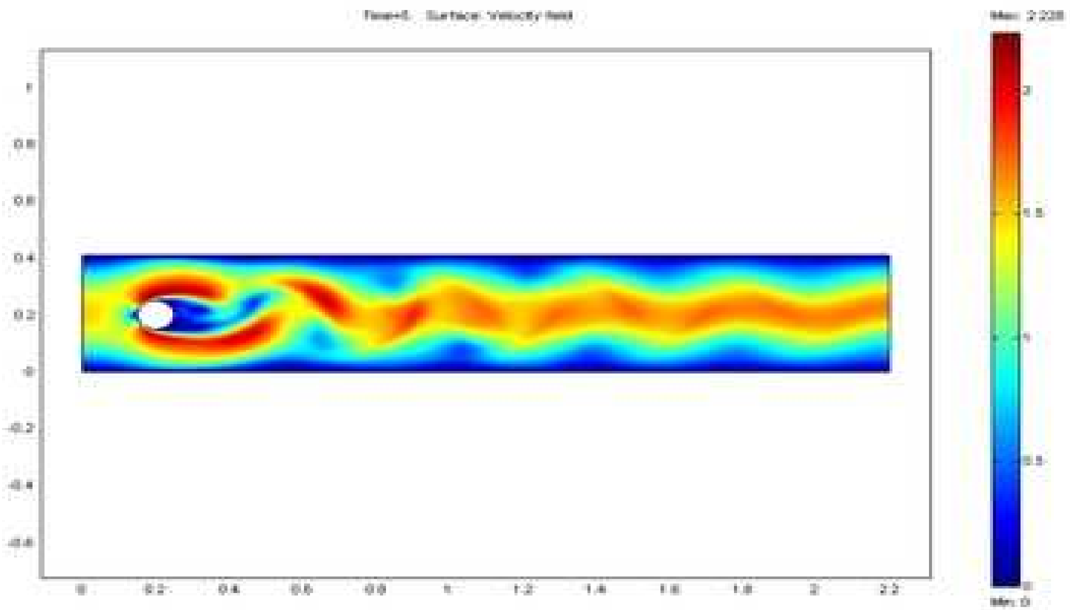


GOAL:

Understand methane dynamics over a wide range of ecological setting.

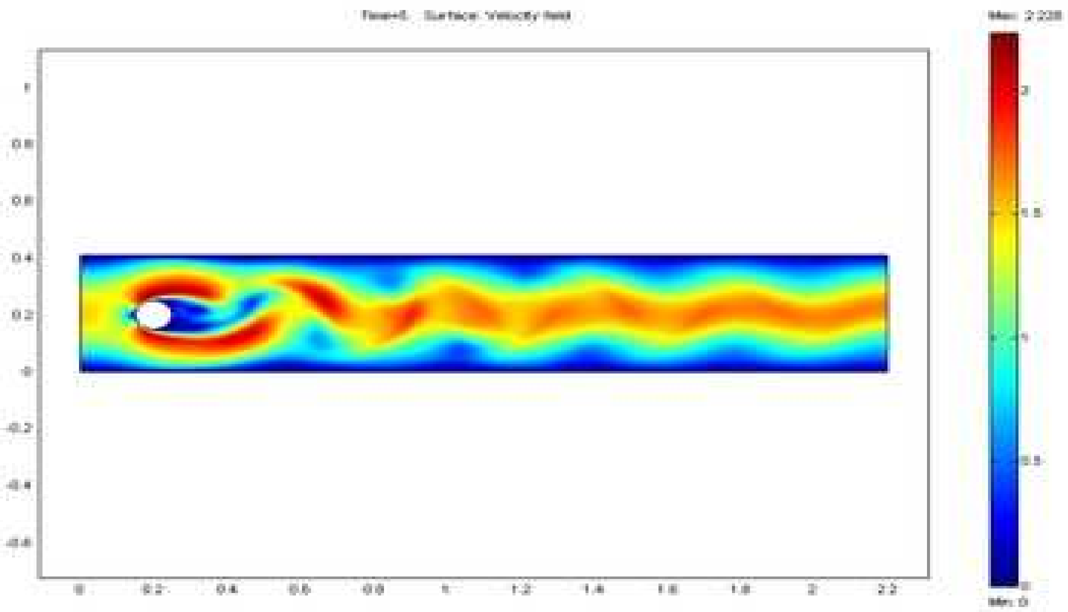
Multiphysics

Fluid flow with heating



Multiphysics

Fluid flow with heating



GOAL:

How is stability of flow affected by temperature dependent viscosity?

Nonlinear waves

Domain Recovery

Nonlinear waves

Domain Recovery

GOAL:

What does the bottom topography look like?

Main Event

Now, sit back and enjoy the main event!

Spectral Stability
or: Can we find stable wave forms?

Roger Thelwell

Department of Applied Math
University of Washington

thelwell@amath.washington.edu

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- *Bernard Deconinck* (UW) and

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Introduction

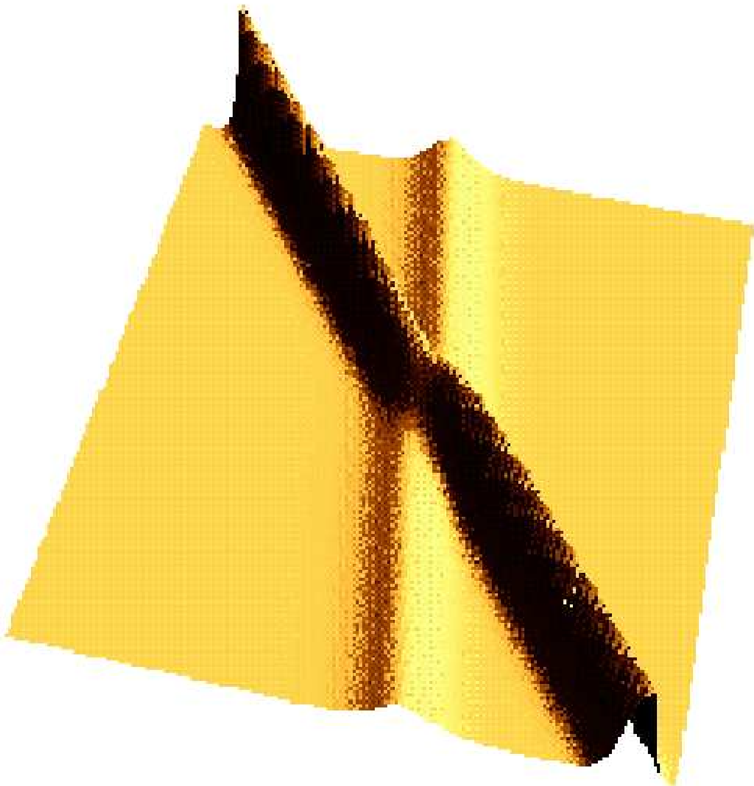
We'll talk about computing the spectra of linear operators, including the associated eigenfunctions.
Why is this important?

- Spectral Stability Given an equilibrium solution, we can see if it is stable under perturbation

Patterns in waves

www.amath.washington.edu/ bernard

Soliton interaction



www.math.h.kyoto-u.ac.jp/images/soliton-big.jpg

More patterns

www.math.psu.edu/dmh/FRG

What I do

Use a simple numerical method to examine the spectral stability of solutions of various models:

- NLS (deep water,....)
- KP (shallow water)
- Euler

Spectral Stability

Consider the evolution system

$$u_t = N(u)$$

with an equilibrium solution u_e:

$$N(u_e) = 0.$$

Is this solution *stable* or *unstable*?
Linear analysis: let

$$u = u_e + \epsilon\psi.$$

Substitute in and retain first-order terms in ϵ:

$$\psi_t = \mathcal{L}[u_e(x)]\psi.$$

Eigenfunction expansion

Separation of variables: $\psi(x,t) = e^{\lambda t}\phi(x)$:

$$\mathcal{L}[u_e(x)]\phi = \lambda\phi.$$

- This is a spectral problem.
- If $\Re(\lambda) \leq 0$ for all bounded $\phi(x)$, then u_e is spectrally stable.

Application

Our starting point is

$$\mathcal{L}\phi = \lambda\phi,$$

with

$$\mathcal{L} = \sum_{k=0}^{M} f_k(x)\partial_x^k, \qquad f_k(x+L) = f_k(x).$$

We want to find

- Spectrum $\sigma(\mathcal{L}) = \{\lambda \in \mathbb{C} : ||\phi|| < \infty\}$.
- Corresponding eigenfunctions $\phi(\lambda, x)$?

NLS

I've been looking at solutions of the 2-D cubic nonlinear Schrödinger (NLS) equation, given by

$$i\phi_t + \alpha\phi_{xx} + \beta\phi_{yy} + \gamma\left|\phi\right|^2\phi = 0.$$

The NLS equation arises in many models:

- Bose-Einstein condensates ($\alpha\beta > 0$)
- Deep water models ($\alpha\beta < 0$)
- Optics ($\alpha\beta > 0$)

NLS

Consider

$$i\psi_t + \alpha\psi_{xx} + \beta\psi_{yy} + |\psi|^2\psi = 0.$$

This equation has exact 1-D traveling wave solutions of the form

$$\psi(x,t) = \phi(x)e^{i\omega t + i\theta(x)},$$

where ϕ and θ are real-valued functions and ω is a real constant.

- If $\theta(x) =$ constant then the solution has *trivial phase* (TP).
- if $\theta(x) \neq$ constant the solution has *nontrivial phase* (NTP).

NLS

More exactly,

$$\psi(x,t) = \phi(x) e^{i\omega t + i\theta(x)},$$

where

$$\phi^2(x) = \alpha\left(-2k^2 \text{sn}^2(x,k) + B\right),$$

$$\theta(x) = c\int_0^x \phi^{-2}(\tau)d\tau,$$

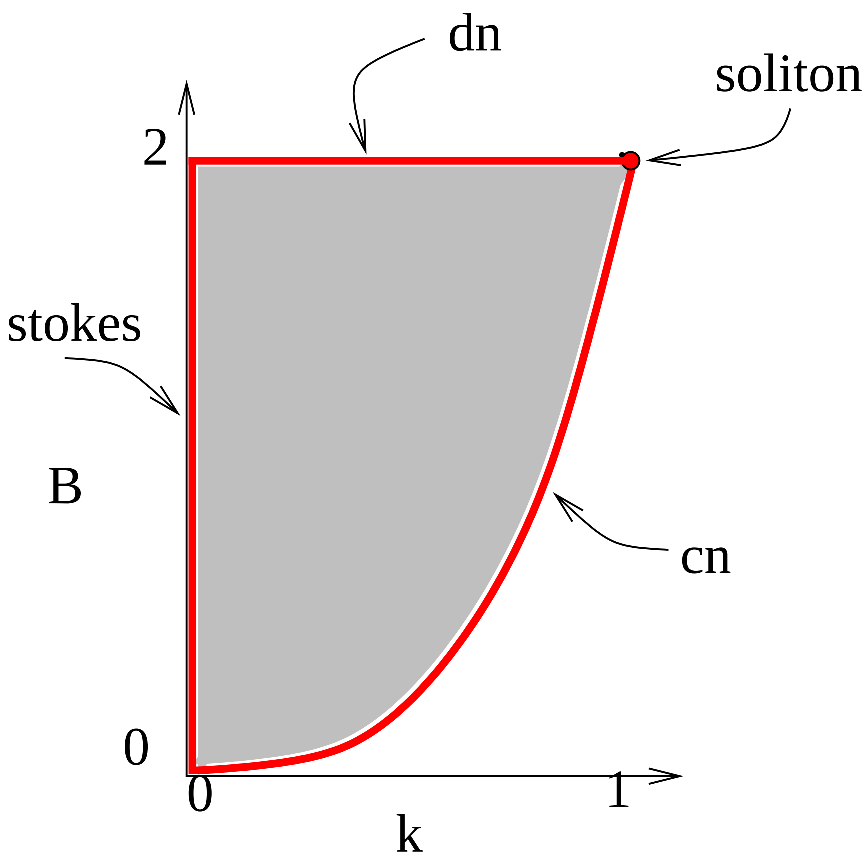
$$\omega = \frac{1}{2}\alpha(3B - 2(1+k^2)), \qquad \text{and}$$

$$c^2 = -\frac{\alpha^2}{2}B(B-2k^2)(B-2).$$

k and B are free parameters and sn(x,k) is the Jacobi elliptic sine function.

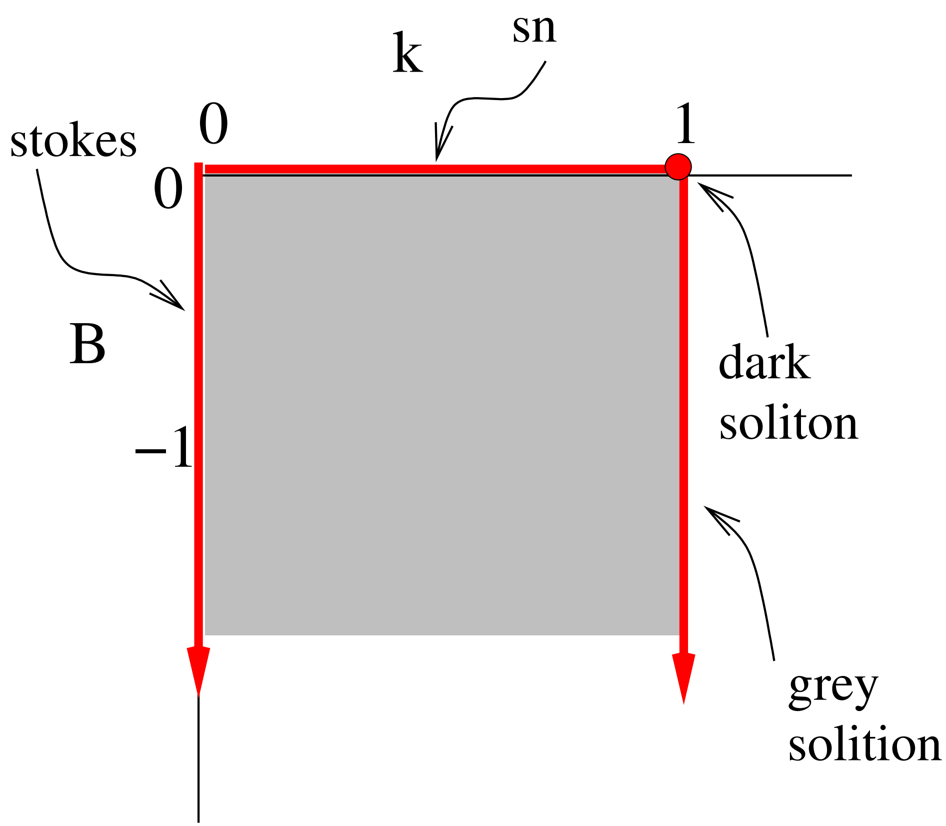
Focusing

NLS is *focusing* or *attractive* in the x dimension if $\alpha > 0$. To make ϕ real in this case, we choose B in $[2k^2, 2]$.



Defocusing

NLS is *defocusing* or *repulsive* if $\alpha < 0$. To make ϕ real in this case, we choose $B \leq 0$.



Linearized TP spectral problem

Now consider the (modulus and phase) perturbed TP solution of the form

$$\psi_p = (\phi + \epsilon u + i\epsilon v)e^{i\omega t}$$

Linearizing and considering real and imaginary contributions yields the system

$$\begin{aligned} \omega u - 3\phi^2 u - \beta u_{yy} - \alpha u_{xx} &= -v_t \\ \omega v - \phi^2 v - \beta v_{yy} - \alpha v_{xx} &= u_t \end{aligned}$$

Linearized TP spectral problem

Let $u(x, y, t) = U(x)e^{i\rho y+\lambda t}$
and $v(x, y, t) = V(x)e^{i\rho y+\lambda t}$.
Then

$$\begin{aligned} \omega u - 3\phi^2 u - \beta u_{yy} - \alpha u_{xx} &= -v_t \\ \omega v - \phi^2 v - \beta v_{yy} - \alpha v_{xx} &= u_t \end{aligned}$$

becomes

$$\begin{aligned} \omega U - 3\phi^2 U + \beta\rho^2 U - \alpha U_{xx} &= -\lambda V \\ \omega V - \phi^2 V + \beta\rho^2 V - \alpha V_{xx} &= \lambda U \end{aligned}$$

Linearized TP spectral problem

We write

$$\begin{aligned} \omega U - 3\phi^2 U + \beta\rho^2 U - \alpha U_{xx} &= -\lambda V \\ \omega V - \phi^2 V + \beta\rho^2 V - \alpha V_{xx} &= \lambda U \end{aligned}$$

as

$$\mathcal{L}\begin{bmatrix} U \\ V \end{bmatrix} := \begin{bmatrix} 0 & L_- \\ -L_+ & 0 \end{bmatrix}\begin{bmatrix} U \\ V \end{bmatrix} = \lambda \begin{bmatrix} U \\ V \end{bmatrix}$$

where

$$L_+ = \omega - 3\phi^2 + \beta\rho^2 - \partial_{xx}$$

and

$$L_- = \omega - \phi^2 + \beta\rho^2 - \partial_{xx}$$

This is our linearized TP spectral problem. The coefficients are periodic.

Floquet's Theorem

Consider

$$\varphi_x = A(x)\varphi, \quad A(x+L) = A(x). \qquad (*)$$

Floquet's theorem states that the fundamental matrix Φ for this system has the decomposition

$$\Phi(x) = P(x)e^{Rx},$$

with $P(x+L) = P(x)$ and R constant.

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with $P(x+L) = P(x)$ and R constant.
Conclusion: **All bounded solutions of ($*$) are of the form**

$$\varphi = e^{i\mu x} \sum_{n=-\infty}^{\infty} \hat{\varphi}_n \, e^{i2\pi n x/L},$$

with $\mu \in [0, 2\pi/L)$**.**

Eigenfunctions

The periodic eigenfunctions can be expanded as

$$\varphi = e^{i\mu x} \sum_{n=-\infty}^{\infty} \hat{\varphi}_n \, e^{i\pi n x/L},$$

with $\mu \in [0, \pi/L)$

Substitute in the equation and cancel $e^{i\mu x}$.

The Floquet parameter μ only appears in derivative terms.

Hill's method

- Find Fourier coefficients of all functions
- Choose a number of μ values $\mu_1, \mu_2, \ldots$
- For all chosen μ values, construct $\hat{\mathcal{L}}_N(\mu)$
- Use favorite eigenvalue/vector solver

Recall

We can finally consider the spectral stability of the periodic coefficient linear problem

$$\mathcal{L}\begin{bmatrix} U \\ V \end{bmatrix} := \begin{bmatrix} 0 & L_- \\ -L_+ & 0 \end{bmatrix}\begin{bmatrix} U \\ V \end{bmatrix} = \lambda \begin{bmatrix} U \\ V \end{bmatrix}$$

where

$$L_+ = \omega - 3\phi^2 + \beta\rho^2 - \partial_{xx}$$

and

$$L_- = \omega - \phi^2 + \beta\rho^2 - \partial_{xx}$$

We now build the matrix $\hat{L}_-(\mu)$. The same method generates $\hat{L}_+(\mu)$.

Fourier coefficients

We could compute Fourier coefficients, but ...

Fourier coefficients

We could compute Fourier coefficients, but ...
thanks to Jacobi, we have an exact form:

$$\mathrm{sn}^2(x,k) = \frac{1}{k^2}\left(1 - \frac{E}{K}\right) - \frac{2\pi^2}{k^2K^2}\sum_{n=1}^{\infty}\frac{nq^n}{1-q^{2n}}\cos\left(\frac{n\pi x}{K}\right),$$

with

$$\begin{aligned} k' &= \sqrt{1-k^2}, \\ K(k) &= \int_0^{\pi/2}\left(1-k^2\sin^2 x\right)^{-1/2} dx, \\ E(k) &= \int_0^{\pi/2}\left(1-k^2\sin^2 x\right)^{1/2} dx, \\ q &= e^{-\pi K(k')/K(k)}. \end{aligned}$$

Construct $\hat{L}_-(\mu)$

Since

$$\hat{\text{sn}}^2(x) = \left(\ldots, -\frac{\pi^2}{k^2K^2}\frac{q}{1-q^2}, \frac{1}{k^2}\left(1-\frac{E}{K}\right), -\frac{\pi^2}{k^2K^2}\frac{q}{1-q^2}, \ldots\right)$$

and

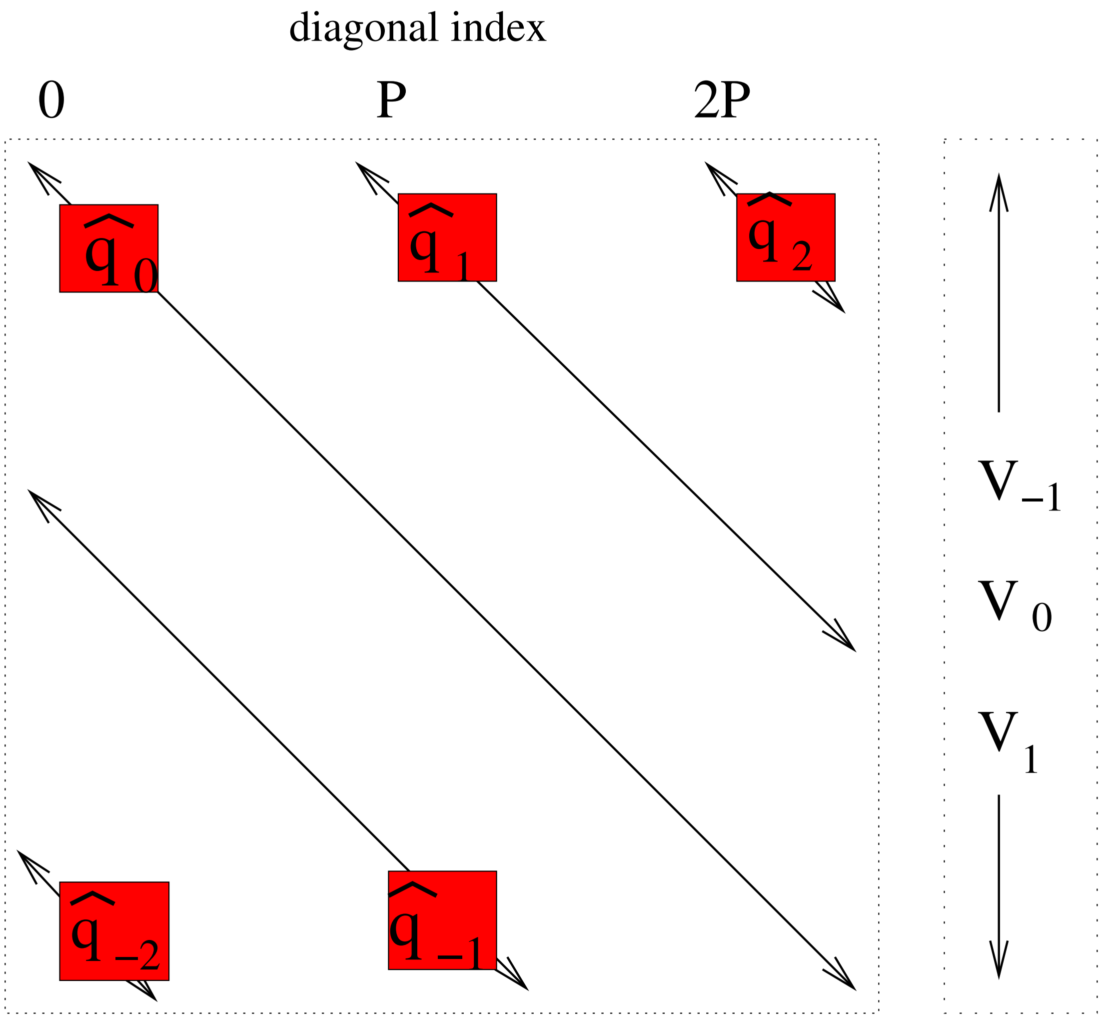
$$\hat{\phi}^2(k,B) \quad = \alpha\left(-2k^2\hat{\text{sn}}^2(k) + B\right),$$

we write

$$\hat{L}_- \quad = \underbrace{\omega - \hat{\phi}^2 + \beta\rho^2}_{(\ldots,\hat{q}_{-1},\hat{q}_0,\hat{q}_1,\ldots)} - \left(i\mu + \frac{i2\pi n}{PL}\right)^2$$

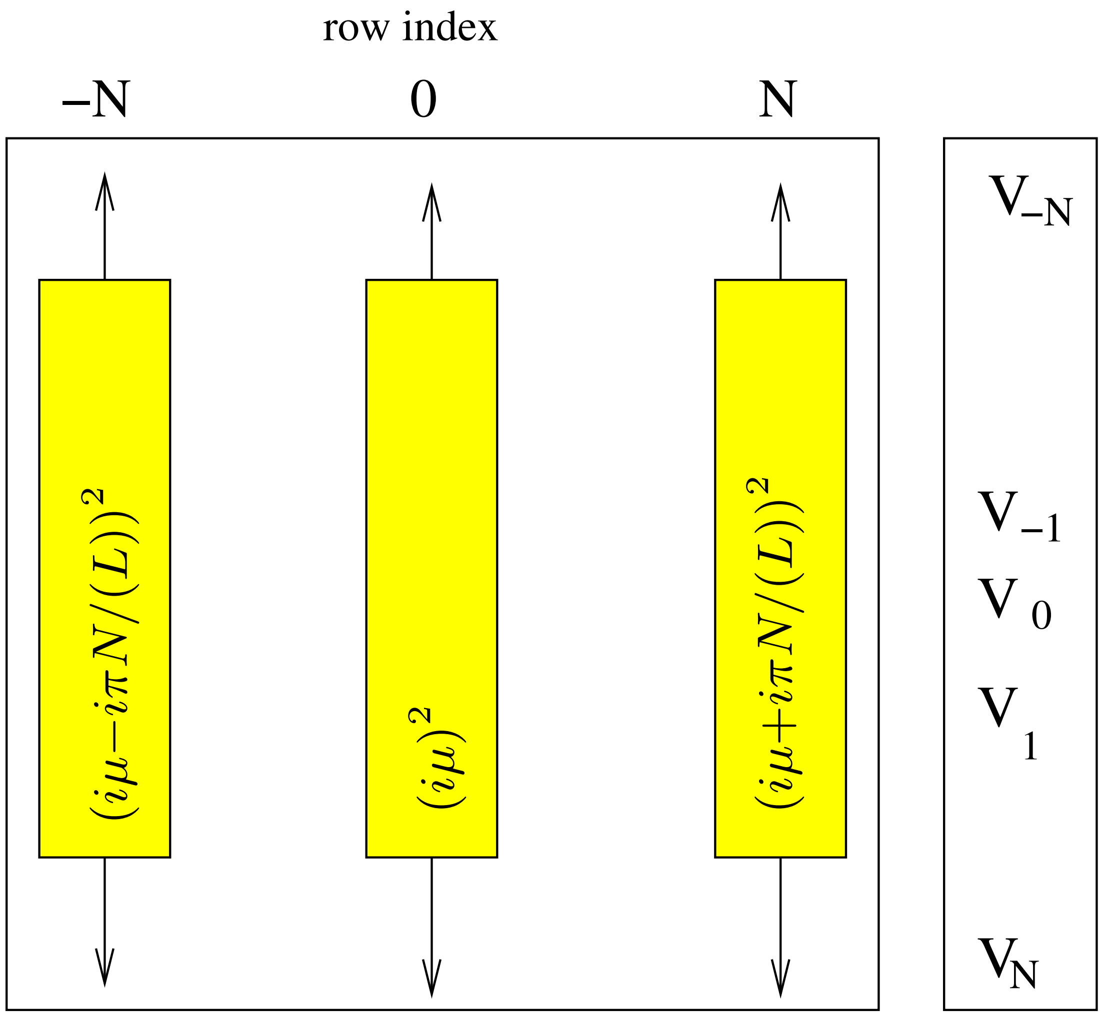
Construct $\hat{L}_-(\mu)$

The Fourier coefficients ...



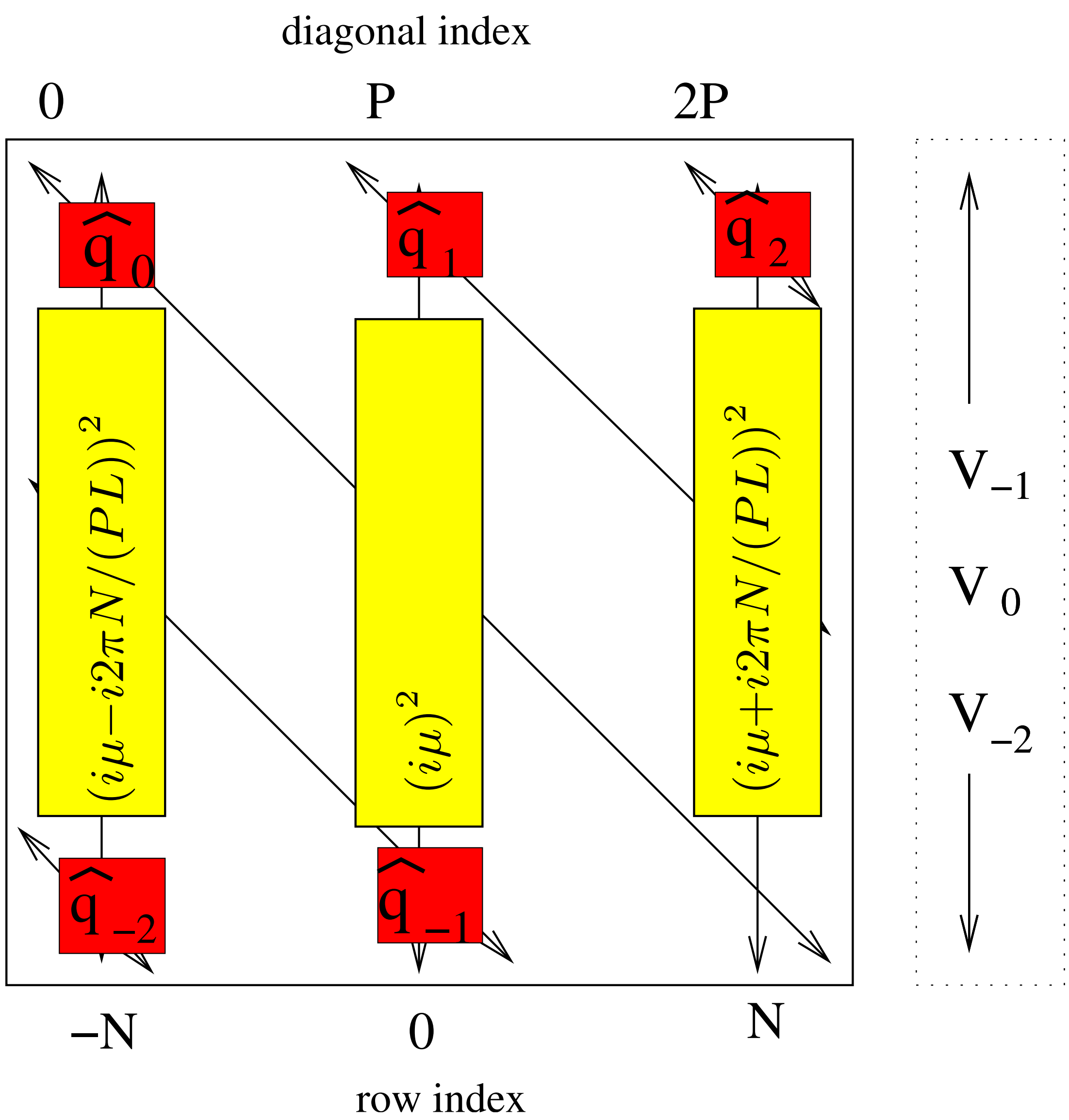
Construct $\hat{L}_-(\mu)$

The partial operator ...



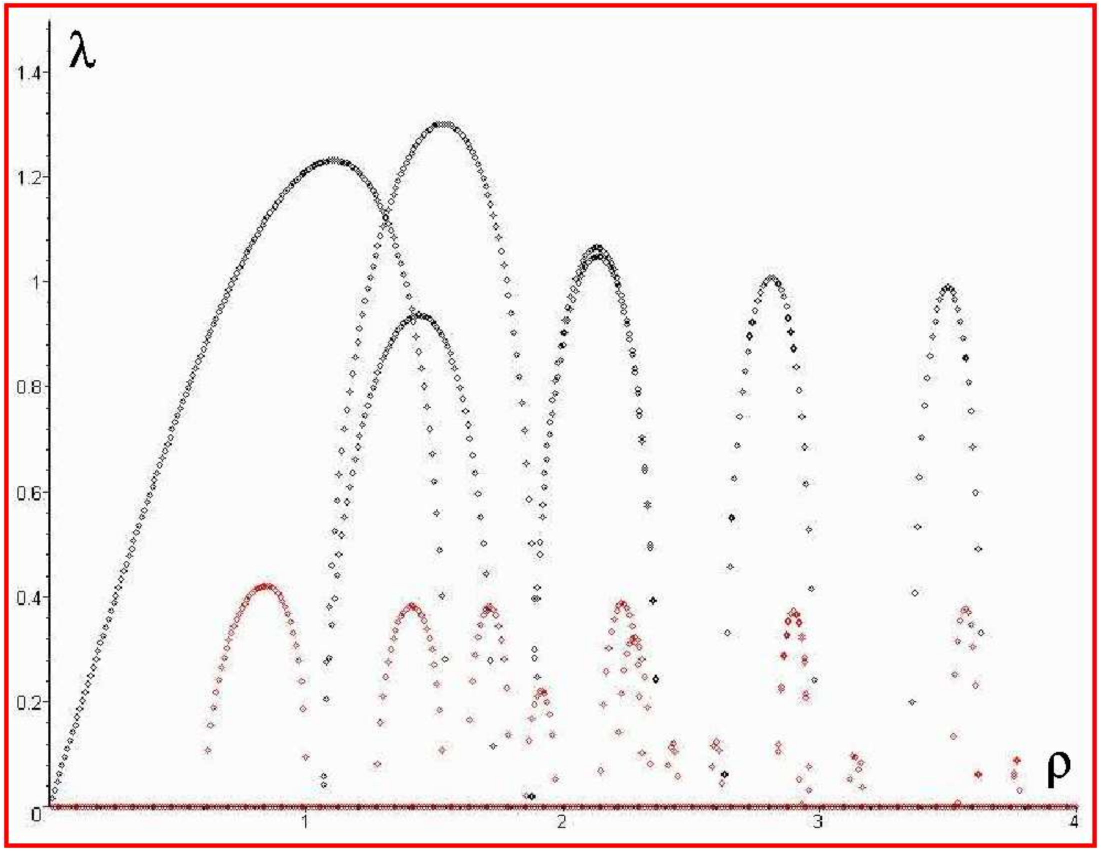
Construct $\hat{L}_-(\mu)$

Combining these, we get



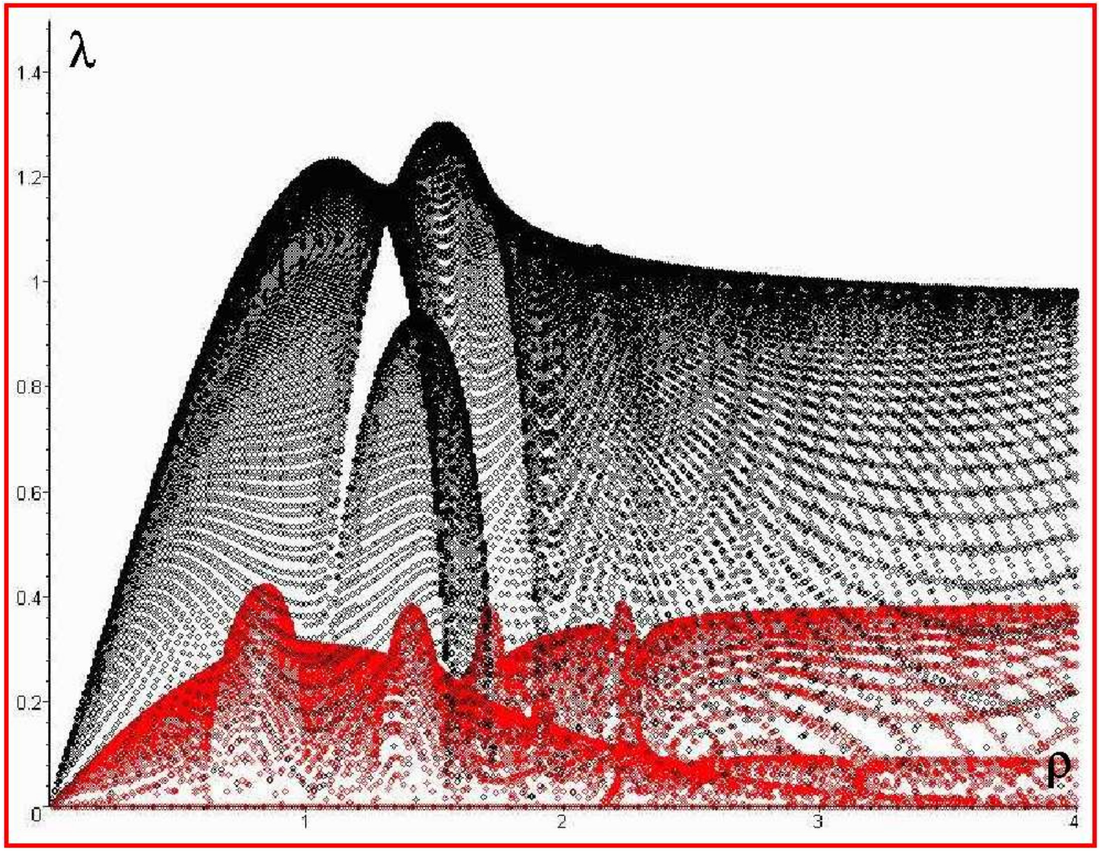
SN plus

In the literature, you *might* find graphs for spectra associated to periodically perturbed TP solutions.



SN plus

We can now compute “all” unstable modes.



Conclusions

We now have a simple method that we can use to understand spectral stability. The method is great. It is:

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- Simple to implement
- Faster than many methods
- Provides eigenfunction information

But it has some problems, too:

Operator is NOT COMPACT!!

Thanks!!!